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Obstacles and Affordances for Integer Reasoning: An Analysis of Children's Thinking and the History of Mathematics

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We identify and document 3 cognitive obstacles, 3 cognitive affordances, and 1 type of integer understanding that can function as either an obstacle or affordance for learners while they extend their numeric domains from whole numbers to include negative integers. In particular, we highlight 2 key subsets of integer reasoning: understanding or knowledge that may, initially, interfere with one's learning integers (which we call cognitive obstacles) and understanding or knowledge that may afford progress in understanding and operating with integers (which we call cognitive affordances). We analyzed historical mathematical writings related to integers as well as clinical interviews with children ages 6–10 to identify critical, persistent cognitive obstacles and powerful ways of thinking that may help learners to overcome obstacles.

Key words: Children's thinking; Cognitive affordance; Cognitive obstacle; Integers; Mathematical history; Negative numbers

As adults, we proficiently operate with numbers, specifically negative numbers, often without deep thought or reflection. The various metaphors, contexts, and understanding that we bring to problems enable us to think of and use numbers flexibly and in multiple ways. For example, consider the number -5. We can interpret -5 as any of the following:

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- An action of removing 5 from a set.
- The integer between -6 and -4.
- An action of moving 5 units left or down.
- The number that, when added to 5, yields 0.
- The location on a number line (coordinate plane, etc.) 5 units to the left of, or below, 0.
- Describing the equivalence class [(0, 5)] in which we define (a, b) to mean $a - b$. Members of [(0, 5)] include (1, 6), (2, 7), (100, 105), and all other ordered pairs (a, b) such that $a + 5 = 0 + b$ for $a, b \in \mathbb{N}$. [More formally, we can write $(0, 5) \sim (a, b)$.]
- A representation of a $5 debt.

In this article, we juxtapose the ways that children reason about integers with the historical development of integers and the collective understanding of the broader mathematical community. Our purpose is to document obstacles and affordances for learners within the domain of integers and to consider similarities and differences between and among children's conceptions and the historical acceptance of negative numbers. In the following sections, we synthesize previous research on integers and situate our study within the broader literature bases of historical analyses of mathematics, research on children's mathematical thinking, and the notion of cognitive obstacles and affordances.

Research on integers is relatively sparse compared to the literature on students' understanding of mathematical topics such as rational numbers or whole numbers (National Research Council, 2001). In fact, in our review of integer-related research, we identified only 47 publications in research journals between 1980 and 2012 that are focused on teaching, learning, and understanding integers (fewer than two publications per year during this 30-year span). We organized this literature into three broad categories: research documenting the historical development of integers and integer arithmetic; research on students' computational fluency and understanding of integers; and research related to teaching, instruction, or curriculum. Much of the integer research falls into the last category, in which researchers use teaching or design experiments to develop and test a variety of instructional models, tools, and contexts for teaching integers—e.g., scores and forfeits, debts and assets, number lines, and microworlds (see Freudenthal, 1983; Janvier, 1983; Liebeck, 1990; Linchevski & Williams, 1999; Moreno & Durán, 2004; Stephan & Akyuz, 2012; Streefland, 1996; Thompson & Dreyfus, 1988). In the following sections, we do not provide a summary of the entire literature base but instead synthesize literature within the first two categories of integer-related research—historical research and research on student understanding of integers—inasmuch as these pertain most clearly to the goal of the research we report in this article.

What Do We Know About Integers From History?

In critically analyzing other scholars' syntheses of the historical development of integers, we found that some authors focused squarely on the historical evolution of integers (Glaeser, 1981; Hefendehl-Hebeker, 1991; Henley, 1999) and others on the intersection between integers and algebra (Gallardo, 1995, 2002; Thomaidis & Tzanakis, 2007). Still, other authors focused on broader ideas such as the concept of number (Klein, 1968; Neal, 2002) and how that evolved over time while mathematics itself evolved (Schubring, 2005). Regardless of the subtle differences in research foci, we distilled three key findings from our analysis of these historical syntheses.

First, a formal algebraic approach to number in particular and mathematics more generally had an interdependent relationship with the status of negative numbers. All authors listed above alluded to this idea, but Glaeser (1981), Hefendehl-Hebeker (1991), Henley (1999), Klein (1968), Neal (2002), and Schubring (2005) did so explicitly. The question was whether mathematical concepts and ideas were justified because they existed in the real world or because they satisfied the conditions within a mathematical system (Glaeser, 1981; Hefendehl-Hebeker, 1991; Klein, 1968; Schubring, 2005). The underlying epistemology of justification by internal consistency (as opposed to justification by realism) was fully realized in a formal view of mathematics that ultimately extended our number system to include negatives (Hefendehl-Hebeker, 1991; Henley, 1999; Neal, 2002; Schubring, 2005). But an ongoing tension existed between this more formal, generalizable, and symbolic understanding of number and the notion that numbers were concrete, magnitude-based signifiers of reality—in fact, the second finding that emerged from this literature was this magnitude-based notion of number. And third, the historical syntheses we analyzed pointed toward an important milestone in the historical development of integers, what Henley (1999) called an *operational understanding* of integers. By this term, we mean mathematicians' use of computational procedures for integer arithmetic (e.g., the rule of signs) and the acceptance of these procedures as convention without providing explicit justification for them (see also Gallardo, 2002; Neal, 2002; Schubring, 2005).

Though the historical analyses sensitized us to major shifts and trends in the historical development of integers—the shift toward an algebraic and formal view of mathematics, the role of magnitude, and operational fluency with integers—our research focus was on the underlying ways of reasoning exemplified by mathematicians historically when they engaged with specific integer problems. Our goal was not to retell the story of the broader historical development of the field or to map the progression of integer acceptance (from intermediate solutions of equations, to final solutions of equations, to coefficients, etc.). Our goal was instead to document how specific problems were or were not solved and why, as a means to look for commonalities and patterns that would help us identify obstacles and affordances for integer reasoning.

In general, publications synthesizing the historical development of integers did not provide enough detail for us to understand how, for example, a formal algebraic approach enabled one to solve a particular integer-related problem. Moreover, it was not clear to us whether the three key findings from our analysis of these documents were more philosophical in nature and would have emerged had we focused on imaginary numbers or rational numbers as opposed to negative numbers or if these three findings pertained to specific problem-solving approaches related to integers themselves. Even in the two studies that explicitly identified integer-related obstacles, Glaeser (1981) and Hefendehl-Hebeker (1991), the obstacles were not always exemplified in particular mathematicians' responses, and the obstacles were of a grain size that was challenging to link to specific sense-making strategies or underlying ways of reasoning (e.g., obstacles included the desire for a single unifying model for addition and multiplication and the ambiguity of two zeros). Thus, we turned to the original historical writings of mathematicians to identify obstacles and affordances for integer reasoning as they were expressed in the details of solutions to specific integer problems.

Students' Integer Understanding

In much of the integer-related research pertaining to students' understanding, researchers have documented the abilities of students in Grades 5–12 to compute (e.g., Gallardo, 1995, 2002; Kloosterman, 2012, in press; Thomaidis & Tzanakis, 2007; Vlassis, 2002, 2008). For example, Gallardo (1995) documented that less than 40% of 12- to 13-year-olds she interviewed were able to subtract integers. Similarly, in his analyses of mathematics results of the National Assessment of Educational Progress in 2008, Kloosterman (2012) documented that more than one fourth of all 13-year-olds were unable to add a positive number and a negative number, and further, about half were unable to divide integers correctly.

Students also have difficulties solving algebraic equations (Vlassis, 2002, 2008), simplifying algebraic expressions (Christou & Vosniadou, 2012; Lamb et al., 2012), and comparing quantities (Vlassis, 2004) after negative integers have been introduced. For example, when simplifying $-2x + 7x$, students often detach, or ignore, the negative sign in the expression $-2x$, add $2x$ and $7x$ to get $9x$, and then reattach the negative sign to the expression to obtain $-9x$. Others have found that students tend to either assign only natural numbers to literal symbols or treat expressions such as $-x$ as if they represent solely negative quantities (Christou & Vosniadou, 2012; Lamb et al., 2012).

To date, few researchers have focused on students' ways of reasoning about integers. Notable exceptions include Chiu's (2001) study of the metaphors students used when reasoning about integer problems and Peled's (1991) theoretical taxonomy describing levels of integer knowledge along two dimensions, a number-line dimension and a quantitative dimension. Peled proposed a hierarchy of development for each dimension wherein students could successfully reason about integer arithmetic using progressively more generalizable strategies. For example, Peled and colleagues (Peled, Mukhopadhyay, & Resnick, 1989) identified two number-line models involving integers: a divided number line and a continuous number line, both of which influenced the kinds of problems one could successfully solve and, at times, the strategies themselves. Chiu drew from Lakoff's work on metaphor (see Lakoff & Johnson, 1980; Lakoff & Núñez, 2000) and embodied cognition to identify three categories of metaphorical reasoning that both middle school students and undergraduate mathematics and engineering majors used during problem-solving interviews. These categories were motion, manipulation of objects/opposing objects, and social transaction (grounded in experiences of giving and exchanging). Chiu found that metaphors were used during computations in only roughly one fourth of the problems posed but that middle school students were more accurate when using a metaphor to compute.

In summary, we know that integer arithmetic is an area of mathematics that is challenging for many, and from the work of Peled and colleagues (1989; Peled et al., 1991) and Chiu (2001), we know that students approach negative numbers in multiple and varied ways that include (a) using tools such as the number line or (b) using metaphors to reason about integers by relating them to other ideas such as oppositional quantities. Our work, then, grows out of a desire to build from and integrate two seemingly disparate research areas to identify obstacles and affordances for integer reasoning that will support the development of a framework for integer reasoning.

Mapping the Domain of Integer Reasoning: Identifying Obstacles and Affordances

Our purpose in this article is to identify and document cognitive obstacles and cognitive affordances within the domain of integers. Note that we are not presenting a comprehensive framework of integer reasoning in this article. Instead, we begin to develop a framework wherein we highlight two key aspects of integer reasoning: understanding or knowledge that may, initially, stand in the way of one's learning integers (which we call *cognitive obstacles*) and understanding or knowledge that may afford progress in understanding and operating with integers (which we call *cognitive affordances*).

To identify critical, persistent cognitive obstacles and powerful ways of thinking that may help students to overcome obstacles, we draw from an analysis of historical mathematical writings related to integers as well as analyses of clinical interviews with students. We look to the historical record to identify challenges and successes of mathematicians when they grappled with the idea of negative numbers to deepen our own understanding of integers and integer operations. However, we also take a children's mathematical thinking perspective (e.g., Carpenter, Fennema, Franke, Levi, & Empson, 1999; Carpenter, Franke, & Levi, 2003; Empson & Levi, 2011; Ginsburg, 1997), the purpose of which is to understand and describe children's thinking about a particular mathematical topic—in this case, integers. Given these goals, our first two research questions focus on the historical development of integers and children's understanding of integers, respectively. On the basis of our findings in answering the first two questions, we consider one more question. Thus, the research questions guiding our study are:

1. What does the historical record tell us about the difficulties mathematicians faced and what afforded progress in their ways of reasoning about negative numbers?
2. What are students' conceptions of integers and operations on integers? In particular, what difficulties do students face when extending their numeric domains from whole numbers to integers, and are there ways of reasoning that can help students overcome these challenges?
3. What are cognitive obstacles and cognitive affordances within the domain of integers?

Theoretical Perspectives Guiding Our Research

An historical perspective. For the purposes of this study, we are interested primarily in uses of history that can help us reconceptualize what rich integer understanding should entail and better understand the construction of the field of integer knowledge, though we acknowledge that there are many other reasons for integrating history and mathematics education—see Fauvel and van Maanen (2000) and Furinghetti (2007) for a broader rationale. One way researchers have approached the use of history is through the idea of *parallelism*, a term referring to parallels between the historical evolution of a mathematics topic and students' current thinking about that same topic (e.g., Brousseau, 1997; Gallardo, 2002; Sfard, 1995; Thomaidis & Tzanakis, 2007). Parallelism, or as some refer to it, *recapitulation,* is the idea that one knows how individuals learn (and how to best teach students) on the basis of the historical development of a particular mathematical topic. For example, Thomaidis and Tzanakis (2007) offered parallelism as an explanation for similarities they observed in historical writings and modern secondary students' reasoning about the ordering of numbers, and Sfard (1995) linked parallels between the historical development of algebra with students' current understanding of symbolic representations, particularly in equations with parameters.

Other scholars, however, have strongly critiqued parallelism, arguing that the circumstances and environments surrounding historical development are significantly different from the circumstances and environments surrounding learners today; thus, modern cultures cannot reproduce the historical formation of ideas (e.g., Furinghetti & Radford, 2008; Radford & Puig, 2007; Schubring, 2011). "Given that the environment changes, it becomes difficult to maintain that the children's intellectual development will undergo the same process as the one children experienced in the past" (Furinghetti & Radford, 2002, p. 635). For us, these differing perspectives raised the question of whether similarities in history and students' modern-day reasoning might be observed because these conceptualizations are inherent in a given mathematical topic or for other reasons. We believe that modern mathematics and current social practices cannot be separated from the historical advances and developments that have spurred us onto different ways of thinking—ways of thinking that are now viewed as common and natural. What we now accept as mathematics is influenced by what was accepted and done historically. Thus, contemporary learning is not independent of historical development. Yet, it may not necessarily follow the path of history.

Our position is that an understanding of the history of integers is a source by which one might develop better and deeper knowledge of integers and the key conceptual ideas necessary for robust integer understanding. For example, one way we might understand negative numbers is as indicative of a process to be performed—of a subtraction yet to be completed—as opposed to a mathematical object in its own right. Historically, we find that the idea of *subtractive numbers* was one of the earlier conceptions mathematicians had for negative numbers (Henley, 1999). The sign - preceding -3 indicated the intention of subtracting 3, but the specific minuend was not given; thus subtractive numbers were understood as “quantities to be subtracted” (Henley, 1999, p. 647; see also Gallardo, 2002).

We look to the historical record as one way to identify cognitive obstacles and cognitive affordances to which teachers and researchers may need to attend within the realm of negative numbers. In this capacity, the historical perspective serves to help us identify possible sources of insight and confusion. Moreover, we agree with Herscovics (1989) that modern students’ difficulties may or may not be related to the historical development of a concept, but there is evidence that historical conceptions might shed light on persistent student challenges (see also Radford & Puig, 2007; Thomaidis & Tzanakis, 2007).

A children’s mathematical thinking perspective. In the past 30 years, many scholars have taken a children’s thinking perspective. Research in this vein is diverse and includes, for example, the Cognitively Guided Instruction (CGI) literature (e.g., Carpenter et al., 1999; Carpenter, Fennema, Peterson, Chiang & Loef, 1989; Fennema et al., 1996), Steffe’s work on rational number (Steffe, 1994, 2002, 2004; Steffe & Olive, 2010), and Fuson and her colleagues’ work on supporting *ten-structured thinking* (Fuson & Briars, 1990; Fuson, Smith, & Lo Cicero, 1997; Lo Cicero, Fuson, & Allexsaht-Snider, 1999). One approach to research on children’s mathematical thinking involves designing studies and developing models of cognition with an eye toward using those research findings to support teachers. For example, Carpenter and Moser’s early research on children’s thinking about addition and subtraction provided the foundation for CGI (Carpenter, Hiebert, & Moser, 1981; Carpenter & Moser, 1984; Carpenter, Moser, & Romberg, 1982). CGI proponents then used these research-based models of children’s thinking to “help teachers develop an understanding of their own students’ mathematical thinking and its development” (Fennema et al., 1996, p. 406). Initially, the content focus of CGI was on whole number operations (addition, subtraction, multiplication, and division), but it has been expanded to include base-ten concepts, algebraic reasoning (Carpenter et al., 2003), and fractions and decimals (Empson, 1999, 2001; Empson & Levi, 2011). Within each of these mathematical topics, research-based models of students’ mathematical thinking are used to help teachers make instructional decisions that support the development of more sophisticated conceptual understanding on the basis of a student’s current understanding and strategy.

Broadly speaking, the research we present in this article is designed to develop models of children's integer reasoning that, in the future, may be useful to teachers. In the realm of integers, research-based frameworks detailing and describing students' ways of reasoning do not exist. In our project, we addressed this need by carefully studying children's thinking about integers and integer operations to add to the knowledge base within mathematics education.

Cognitive obstacles and affordances. We use the constructs of cognitive obstacles and cognitive affordances to develop an initial framework of integer reasoning. The notion of cognitive obstacles can be traced to the French philosopher Gaston Bachelard (1938/2002) and his theory of epistemological obstacles in the development of scientific thinking. Bachelard viewed obstacles as the heart of cognition, both for individual learning and the historical development of scientific thought. Essentially, an *obstacle* is knowledge that is useful in solving a certain type of problem but, when applied to a new problem or context, is inadequate or leads to contradictions (see also Brousseau, 1997; Cornu, 1991; Herscovics, 1989). For example, children sometimes overgeneralize their experiences with natural numbers and assume that "multiplication makes bigger" for all numbers (Bell, Swan, & Taylor, 1981, p. 405; see also Graeber & Campbell, 1993). The idea that a product should be larger than its factors can become an obstacle to learning when, for example, a student is faced with a problem of the form $4 \bullet \square = 3$ or is asked to multiply by a rational number between 0 and 1 and maintains that multiplication makes larger. Obstacles play a critical role in learning because they compel the learner to modify and adapt some aspect of his or her thinking to resolve the contradiction.

Bachelard's (1938/2002) conception of epistemological obstacles set forth in *The Formation of the Scientific Mind* was primarily philosophical and theoretical. He used the construct of obstacles to address questions related to the epistemology of scientific thinking and how scientific knowledge advances. Others, like Brousseau (1997), have adapted Bachelard's work to the teaching of mathematics and have defined *epistemological obstacles* differently. In Brousseau's conceptualization, epistemological obstacles are viewed as rooted in the very nature of the knowledge of a topic and independent of one's culture, society, and learning environment. Although scholars like Brousseau classified different types of obstacles (e.g., genetic, epistemological, didactic—see Brousseau, 1997; Cornu, 1991; Herscovics, 1989), we use the term *cognitive obstacles* more generally to mean understanding or knowledge that once supported a learner's thinking but may impede the learning of a new concept. This understanding may be present in an individual learner's construction of knowledge or in the historical growth of knowledge in a particular field. We view both sources as sites for cognitive obstacles, and we do not distinguish the obstacle's source with specific terminology as some researchers do.

Our goals, however, are not restricted to obstacle identification within the realm of integers; we also want to identify ways of reasoning and knowledge that lead to successful problem solving with integers, which we call *cognitive affordances.*

Gibson (1986) used the term *affordance* to describe the unique relationship between an individual and his or her environment. He explained that a chair or a ledge or even a rock may afford sitting, so long as one perceives that the object can be sat upon. Similarly, a hammer may afford driving and removing nails, or even digging, but only if those affordances are perceived. Thus, the affordances of a context, situation, tool, location, object, or problem vary on the basis of what an individual perceives, notices, and experiences. In the context of integers, we seek particular kinds of affordances, namely, ways of reasoning that offer learners the ability to solve problems they are otherwise unable to solve.

Methods

In this study, we used two complementary methodological approaches and two sets of data to address our research questions. We used historical data from writings and publications in the realm of mathematics and empirical data from clinical interviews with children ages 6–10 to answer our research questions. Coupling our historical analysis with individual, clinical interviews of children enabled us to identify (a) cognitive obstacles for mathematicians and children; (b) ways of reasoning that helped mathematicians and children to overcome these obstacles, which we call cognitive affordances; and (c) similarities and differences between mathematicians' and children's integer understanding.

Because we were interested in children's reasoning when the children were initially learning about integers and expanding their numeric domains to **Z** (the set of all integers), we decided to study the operations of addition and subtraction prior to formal integer instruction. Though we realize that multiplication and division with signed numbers are critical facets of integer understanding, we limited the scope of our research to explore addition and subtraction in sufficient depth.

In the following sections, we describe our two data sources and our analyses. We begin by discussing the selection of our historical texts and then describe the students with whom we worked, the settings in which we worked, and the nature of the interview and interview tasks we used to collect data on children's thinking. Because of the integrated and interrelated nature of our analysis, we begin the analysis section with an overview of our general process and methods. This section is followed by a description of the development of the coding scheme we used to analyze both the historical mathematical texts and the children's thinking data and the analysis methods we used for both sets of data. We conclude the methods section by explaining our process for identification of cognitive obstacles and affordances.

Historical Data

To answer our first research question about obstacles and affordances present in integers' historical development, we identified and analyzed primary and secondary historical sources that addressed negative numbers. Our analysis of the secondary sources (a) helped us to situate the development of integer reasoning within the broader historical context of mathematics; (b) sensitized us to the importance of the increasing generality of algebraic approaches and their corresponding symbolisms, the role of enumeration and magnitude in the concept of number, and the existence of operational rules for integer arithmetic long before justification of those rules; and (c) helped us to identify potential primary sources from which we could identify integer-specific obstacles and affordances. However, these secondary sources did not provide sufficient information or detail for us to identify obstacles and affordances directly from their writings.

On the basis of our own knowledge of the history of integers and other scholars' work in this area, we generated a list of potentially important original historical mathematical texts that addressed integers, integer arithmetic, or both. For each historical document on the list, we located and read the original text or English translation. Our goal was neither to conduct a comprehensive historical analysis nor to add to the knowledge base on the historical development of integers, but instead to consider historical development from a new perspective for the purpose of identifying persistent, significant, historical challenges as well as reasoning that afforded periods of progress. Thus, our selection of historical texts was not exhaustive but purposeful and informed by other researchers' work in this area. The 13 texts we analyzed are listed in Table 3 in the Findings section. The historical texts span from the 1st century through the Middle Ages and to the close of the 19th century. The integer-related writings represent the views of both Eastern and Western mathematicians from a variety of countries and perspectives; some texts were influential and well-known writings within mathematics, whereas others were more obscure and lesser known writings.

Though we present a narrative of the development of integer understanding, we claim neither that thought across mathematicians was homogeneous nor that the development was continuous, smooth, sequential, and cumulative (Schubring, 2011). In fact, the mathematical texts (e.g., Frend, 1796; Wallis, 1685) clearly showed that authors and subgroups/other individuals within their contemporaneous mathematics communities expressed disagreement. Although our historical data are a collection of mathematical texts that represent the understanding and views of the respective authors, we also take a social and interactive stance on the historical development of integers (Radford, 1997; Schubring, 2011). The creation of mathematics involves negotiation within one's mathematical community to determine the correctness and acceptance or rejection of mathematical ideas. Our claim, then, is that the ideas reflected in the texts we analyzed were not isolated to a few mathematicians but that their thinking reflected prevailing thought in the larger community while still allowing for disagreements among individuals, some of which are acknowledged in the texts themselves.

Children's Thinking Data

Setting and participants. Forty-seven elementary school children ages 6–10 (Grades 1–4 in the United States) participated in our study. We wanted to pursue children's reasoning in depth, but we also wanted to interview enough children to identify prevalent and persistent ways of reasoning. Other researchers had previously used 20–30 clinical interviews in studies of students' integer conceptions (Chiu, 2001; Peled, 1991). Using these numbers as a baseline, we increased the number of interviews to accommodate the larger age range in our study. Students were drawn from nine classes located in elementary schools in Texas and California. We chose young children, as opposed to middle school or older students, because they had received no formal instruction on integers and might be more likely than older students to reveal intuitive and informal ideas about negative numbers. Instruction on integers typically begins in the United States during Grade 5, with 10- to 11-year-olds. However, some students we interviewed had knowledge of integers or were in the process of developing integer knowledge.

We conducted the interviews at three elementary schools in three school districts, all of which had above-average standardized test scores when compared with their state's respective mean. Two of the elementary schools, Schools B and C in Table 1, were high-performing schools in terms of standardized state exam scores. The other school, School A, had a Science, Technology, Engineering, and Mathematics (STEM) focus and was designated a NASA Explorer School, meaning that scientific experimentation was integrated throughout the curriculum. Each of these schools was located in a large city with a diverse population. School demographic information is summarized in Table 1.

Student participants were purposefully chosen with the help of the classroom teacher to represent different levels of mathematical understanding. Classroom teachers identified 2–4 students they considered to be on grade level with respect to mathematical proficiency, 2–4 students they considered to be below grade level, and 2–4 students above grade level. Using these criteria, teachers selectively sent

Table 1
School Demographic Information and Standardized Test Scores

Demographic characteristics	**School A**	**School B**	**School C**
Academic Progress Indicator (API)/ Accountability Rating	807[a] (API score)	918[a] (API score)	Exemplary Rating[b]
% Hispanic students	37%	26%	26%
% White students	2%	59%	52%
% African American students	54%	4%	14%
% Asian students	3%	4%	6%
% Economically disadvantaged students	100%	22%	39%
Number of students interviewed	8	32	7
Number of participating teachers	3	5	1

[a]State API average is 790 points. [b]Highest of four possible performance ratings as determined by percentage of students on campus meeting academic standards on standardized state exam.

home permission forms, and all students who returned permission forms were interviewed. We did not observe classroom mathematics instruction; consequently, we know little about the kinds of mathematics instruction occurring in most of the classrooms. However, we do know, on the basis of teacher self-report, that two of the classrooms (one from School C and one from School B) regularly implemented a problem-solving instructional approach that incorporated whole-class discussion of children's strategies.

Interview tasks. The 50–70 minute clinical interviews (Ginsburg, 1997) were conducted at the children's school sites during the school day and were videotaped. We chose to use clinical interviews in an attempt to understand the integer reasoning of individual children. Following Ginsburg (1997),

> The interviewer, observing carefully and interpreting what is observed, has the freedom to alter tasks to promote the child's understanding and probe his or her reactions; the interviewer is permitted to devise new problems, on the spot, in order to test hypotheses; the interviewer attempts to uncover the thought and concepts underlying the child's verbalizations. (p. 39)

In these interviews, we sought to follow the child's thinking, posing follow-up questions in the moment that were based on the child's ideas. As a result, the interviews were not standardized; that is, we did not pose the same problems to every child. The strength of these interviews is that they enabled us to engage with a child to promote understanding and make visible his or her ways of reasoning. For us, like Ginsburg (1997), "the clinical interview seems to provide rich data that could not be obtained by other means" (p. 39).

Because we were trying to understand children's reasoning about a topic with little existing research to guide us, part of the process involved identifying problems or tasks that provided children significant opportunities to engage with integers. We had no such collection of tasks when we began conducting these interviews. Thus, the use of clinical interviews helped us to identify tasks with the potential to yield rich information about children's integer reasoning in interview settings.

Across the 47 clinical interviews reported in this study were four consistent categories of tasks: introductory questions (asking children to name large/small numbers and to count backward), open number sentences, contextualized problems that could be solved using negative integers, and comparison problems. Note that students were not asked the same questions within each of these categories. Consistent with the nature of clinical interviews, we followed the child's lead and responded specifically to his or her emerging ideas. See the Appendix for examples of problems from each category.

Although context was not a major focus, we did pose contextualized problems as well as a series of comparison tasks during many clinical interviews. We used multiple integer-related contexts across the 47 interviews (e.g., comparing scores for sporting events, gains and losses, money and lending, elevation differences, and happy and sad thoughts). However, we found that young children usually did not explicitly use negative integers in their solutions but instead reasoned about the magnitudes, or absolute values, of the numbers regardless of their signs. As a result, these tasks in contexts did not provide a meaningful entry point for most students to engage with negative integers.

Open number sentences (i.e., problems of the form $-3 + \square = 6$ and $\square + 6 = 4$, with the location of the unknown varying) were the most prevalent type of question we posed across all interviews. We chose to focus on open number sentences because these types of problems are consistent with the abstract nature of negative numbers and the formal approach that eventually ushered in their historical acceptance. Additionally, these types of tasks had been used in other research designed to explore children's mathematical thinking—for example, see Carpenter, Franke, and Levi's (2003) *Thinking Mathematically* and the research of Peled and colleagues (1991; Peled et al., 1989). For each open number sentence, students were asked to read the problem aloud and solve the problem if they thought it had a solution. Interviewers did not introduce the word *negative* but instead used the students' terminology (minus 3, dash 3, etc.). Open number sentences were posed to all students, but not all students were asked to complete the same open number sentences. Initially, we were unsure whether decontextualized problems such as the open number sentences would prove useful in revealing students' integer understanding. Early interviews revealed that these tasks successfully encouraged students to engage with negative-number concepts; the remainder of the interviews helped us to elaborate on how these tasks were useful, identify what conceptions we might be able to uncover with these tasks, and uncover students' varied ways of reasoning. We report here on data from only the open number sentences in the interview. For analyses of other types of tasks, see Bishop, Lamb, Phillipp, Schappelle, and Whitacre (2011), Lamb et al. (2012), and Whitacre et al. (2012).

Data Analysis

Before describing details related to data analysis, we provide a general overview of the development of the analytic framework used to code and analyze both data sources. First, a single coding scheme was developed using the historical data as well as the children's thinking data; this set of codes was used to analyze both sets of data. A list of final codes and their definitions are found in Table 2. Second, we highlight the dialectic nature of the analysis process: That is, the analysis of the historical texts informed the analysis of children's thinking and the analysis of children's thinking informed the analysis of the historical texts. We did not analyze the historical texts first and then analyze the children's thinking data or vice versa; instead, the analyses were iterative and interrelated inasmuch as insights from each data source informed our understanding and analysis of the other.

On the one hand, the affordance of order-based reasoning was not a code that initially emerged from our analyses of historical texts. But after seeing many children successfully use counting strategies and other types of order-based reasoning, we were sensitized to the importance of this approach and better able to recognize

Table 2
Integers Coding Scheme

Ways of reasoning categories	Definitions
Order	In this way of reasoning, one leverages the sequential and ordered nature of numbers to reason about a problem. Using an order-based way of reasoning, one places integers in a sequence and can include the use of counting strategies or a number line with motion/movement. Counting strategies include counting forward or backward by ones (or another incrementing amount). For example, when solving $3 - 5 = \square$, one might say, "Three, 2 (puts up one finger), 1 (puts up second finger), 0 (puts up third finger), -1 (puts up fourth finger), -2 (puts up fifth finger). So it's -2." When using number lines, one typically treats the start and result as locations on the number line and the change as a distance.
Magnitude	This way of reasoning is characterized by one's relating numbers and, in particular, negative numbers to a countable amount or quantity. Magnitude-based reasoning is tied to ideas about cardinality and the view of a number as having magnitude or substance. At times, negative numbers may be related to contexts (e.g., debt) or evoke the idea of opposite (directed) magnitudes. Opposite magnitudes include, for example, the ideas of (a) directional segments (e.g., vectors), (b) a time-certain event and the periods before and after this event has occurred, and (c) losing and gaining amounts.
Logical Necessity/ Formal	In this way of reasoning, one takes a formal approach to problem solving, leveraging the ideas of structural similarity, well-defined expressions, and fundamental mathematical principles (e.g., commutativity, negation). This way of reasoning includes generalizing beyond a specific case by making a comparison to another, known, problem and appropriately adjusting one's heuristic so that the logic of the approach remains consistent. One may reason about a problem involving negative numbers (or make a generalization about operating with negative numbers) by making a comparison to a similar problem for which an answer is known and extending that reasoning to this new domain of negative numbers. Formal approaches can be reflected in algebraic approaches and the search for generalizable solutions. For examples of this code, see the Logical Necessity and Formalisms section in the Findings.
Computational	In a computational way of reasoning, one uses a procedure, rule, or calculation to arrive at an answer to a problem involving negative numbers, either as part of the problem statement or as appearing in the solution set. Computational ways of reasoning about negative quantities can be present when solving a variety of algebraic and arithmetic problems, including solving systems of equations, finding zeros of functions, and finding sums and products of negative values.

Limited	This category of reasoning reflects incomplete or limited views of negative numbers. At times, the domain of possible solutions is locally restricted to nonnegatives. Additionally, these strategies may not be based upon appropriate mathematical foundations.
Addition Cannot Make Smaller	This *limited* subcode is related to conceptualizations of addition as joining or increasing a set. It is seen when one overgeneralizes that addition always makes larger and claims that a problem for which the sum is less than one of the addends has no answer. One response for $6 + \square = 4$ might be that there is no answer because it is impossible to add 6 to any number and get a number smaller than 6. The domain of possible solutions appears to be restricted to whole numbers and the effect (or possible effect) of adding a negative number is not considered.
Subtraction Cannot Make Larger	This *limited* subcode is related to conceptualizations of subtraction as separating or removing objects. It is seen when one overgeneralizes that subtraction always makes smaller and claims that a problem for which the difference is larger than the minuend has no answer. For example, when solving $5 - \square = 8$, one might respond that there is no answer because it is impossible to subtract a quantity from 5 and get a larger number. Again, the domain of possible solutions seems to be restricted to whole numbers and the effect (or possible effect) of subtracting a negative number is not considered.
Subtrahend < Minuend	This *limited* subcode is most often applicable to responses to problems of the form $3 - 5 = \square$, for which one maintains that solving the problem is not possible or the problem is nonsensical because 5 is larger than 3. How can one take something bigger from a smaller number? The basic argument is again related to an underlying conceptualization of subtraction as separating or removing. We also refer to this code as removing something from nothing or removing more than one has.
Negatives Rejected	This *limited* subcode applies to responses showing evidence that negative quantities exist for the individual but that these quantities are rejected as legitimate numbers. This code also applies when, for example, negative solutions occur but are accepted as legitimate solutions only if they have a magnitude-based interpretation in the problem.

it in historical texts. Consequently, we were able to name Wallis's (1685) use of the number line as more than just the tool itself and as indicative of an underlying way of reasoning that leveraged the idea of order. On the other hand, formal reasoning initially emerged from the historical texts but was often defined in general terms. Only when we saw students using fundamental mathematical properties to extend their number system to negative integers were we able to operationalize this way of reasoning. Without the historical analysis, we would have been unlikely to identify these children's strategies as examples of formal mathematical reasoning.

But without the children's thinking data, we would have been unable to exemplify formal reasoning using specific strategies. Some codes initially emerged from the historical data and others from the children's thinking data, but all codes were further refined with the other data source.

For both data sources, we used open coding and the constant comparative method (Strauss & Corbin, 1998) to identify emergent, distinguishing themes and features of mathematicians' and students' reasoning about integers, focusing on persistent difficulties and the related ways of reasoning as well as ways of reasoning associated with successful engagement with integer tasks. Initially, we developed four codes (*Order*, *Magnitude*, *Logical Necessity/Formal*, and *Limited*) representing critical, yet different, ways of reasoning. As we continued to analyze and reanalyze both sets of data, this initial set of codes was revised and expanded. The final set of codes reported in Table 2 includes the four initial codes listed above and five new codes: a *Computational* way of reasoning code (which describes mathematicians' computational approaches to integer-related tasks) and four subcodes within the *Limited* category that were developed to characterize the ways of reasoning reflected in the data in more detail.

Analysis of historical data. For the historical mathematical analysis strand, our goal was to identify potential obstacles to and affordances for conceptual change. After creating a list of potential primary sources, as described above, we identified passages wherein negative numbers were discussed in each historical text on our list. We located these passages by consulting other historical syntheses for potential locations; using the text's table of contents, when available, to direct us to applicable chapters or sections; and skimming texts to get a sense of the mathematical topics that were covered and where the topic of integers might appear to identify sections for a more focused reading. After identifying historical mathematical texts within which (a) negative numbers were explicitly discussed or (b) problems involving negative integers were posed and solved, we recorded whether and how these problems were solved, the topic within which negative numbers arose (e.g., negative roots when solving equations, numeric computations), challenging problems, and ways in which mathematicians were able to overcome these problems if, indeed, they were. We then used the coding scheme in Table 2 to code 13 historical texts, seeking evidence for each of the five ways of reasoning codes and the relevant subcodes and looking for evidence of cognitive obstacles and affordances. The unit of analysis for the historical data was the mathematician and the related historical text.

Analysis of children's thinking interview data. We began analysis by open coding of the videotaped interviews of students (Strauss & Corbin, 1998). We worked directly from the video recordings and student work, transcribing as needed. The focus of this coding was on children's underlying ways of reasoning about integers and operations with integers. Initially, we had low expectations that young children would even be able to engage with these problems.

As described earlier, the development of our coding scheme was necessarily iterative and interconnected as we continued to read additional historical texts and interview more children. After the coding scheme was finalized, each integer problem posed to every child was coded for the underlying way of reasoning the child used in his or her solution (again, see Table 2 for the list of codes). Note that a child could, in theory, be assigned multiple codes for a single open-number-sentence problem, but, in practice, most students were assigned only one code per problem.

Identification of cognitive obstacles and affordances. We used two related, yet different, approaches to identify obstacles in the historical data and obstacles for children in the interview data. In our interviews with children, we identified ways of reasoning as cognitive obstacles if that way of reasoning was associated with an inability to solve, or difficulty in solving, integer problems. In particular, we found evidence for obstacles in children's responses that were assigned codes in the *Limited* category in Table 2. In relation to the historical data, Cornu (1991) suggested that "periods of slow development and the difficulties which arose . . . may indicate the presence of epistemological obstacles" (p. 159). We looked in historical writings for evidence not only for codes in the *Limited* category in Table 2 but, additionally, that those codes were assigned for multiple mathematicians and over long periods of time (i.e., over centuries). We also interpreted disagreements across mathematicians as well as contradictions within a mathematician's own writings as indications of cognitive obstacles.

To document cognitive affordances, we used the codes in Table 2 to identify (a) the ways of reasoning observed in the interview data that were present when students engaged successfully with integer tasks and (b) the ways of reasoning mathematicians leveraged that afforded progress historically in overcoming these obstacles. Our final list of cognitive obstacles and affordances included those identified in either the interview data with children or historical mathematicians' writings. In the end, we identified three cognitive obstacles, three cognitive affordances, and one way of reasoning that functioned as both an obstacle and an affordance.

Findings

On the bases of our historical analysis of mathematical texts and our clinical interviews with children, we document cognitive obstacles and affordances related to integer reasoning. We begin by sharing results from the historical analysis. This section is followed by an analysis of the children's understanding, including the challenges children faced while they reasoned about integers and powerful ways of reasoning that helped them to move beyond paradoxes that negative integers presented.

Negative Numbers—What Can We Learn From History?

For more than 1,000 years, mathematicians pondered, struggled with, operated on, rejected, and eventually came to accept negative numbers. Not unlike their problematic, unwanted, and barely acknowledged cousins—the irrational numbers—negative numbers were a source of great consternation, confusion, and controversy. "Negative numbers troubled mathematicians far more than irrational numbers did, perhaps because negatives had no readily available geometrical meaning and the rules of operation were stranger" (Kline, 1980, p. 118). Negative numbers were theoretically plausible yet practically impossible.

However, with Fermat's and Descartes's creation of analytic geometry, as well as the need for more formalized and generalizable solutions to problems, mathematicians saw the rise of algebra and a corresponding acceptance of negative numbers (Freudenthal, 1983; Kline, 1980; Thomaidis & Tzanakis, 2007). This acceptance, though, was not without objections from some of the best mathematicians of their day. Negative numbers were referred to as "incongruous" (Bháscara, 12th century, as cited in Colebrook, 1817, p. 217), "fictitious" (Girolamo Cardano, 16th century, as cited in Cardano, 1545/1968, p. 11), and "false" or "defects" (René Descartes, 17th century, as cited in Descartes, 1637/1925, p. 159). Table 3 below displays results from our historical analysis of mathematical writings across approximately 1,800 years. For each of the writings, we summarize the surrounding text within which the topic of integers arose and then use the coding scheme described in Table 2 to identify negative-number challenges as well as productive ways of reasoning that afforded progress.

Barriers to accepting negative numbers. Why were mathematicians slow to accept negative numbers? The first and most fundamental reason for mathematicians' reluctance to accept negative quantities is reflected in the *Negatives rejected* code seen throughout Table 3—the lack of a physical, concrete, or geometrically meaningful representation. If numbers represented a countable number of objects and negative numbers were "less than nothing" (Descartes, 1637/1925, p. 159; Frend, 1796, p. x; Wallis, 1685, p. 264), then one could not have some number of objects that was smaller than the absence of any objects (i.e., zero). For example, how can one have a negative number of monkeys (Bháscara II, 1150; as cited in Colebrooke, 1817), buy a negative amount of cloth from a merchant (Chuquet, 1484; as cited in Flegg et al., 1985), or have a negative number of denari (Fibonacci, 1202/2002)? These impossibilities were accepted only if a reasonable explanation for the negative solution could be generated in the given problem context.[1] In other words, to be accepted, a negative solution required a magnitude-based interpretation.

Second, mathematicians struggled with negative numbers because they were the result of nonsensical operations. For example, how can one remove something from nothing? Though related to the necessity of having a physical, concrete

[1] Fibonacci (1202/2002) interpreted negative solutions as debits, and Chuquet (as cited in Flegg et al., 1985) did similarly, explaining a negative amount of cloth as a credit from the merchant, though Chuquet was not bothered by isolated negative quantities. Bháscara II (as cited in Colebrooke, 1817) recognized that negative solutions existed, but he did not accept them as valid, calling them *incongruous* and stating, "People do not approve a negative absolute number" (p. 217).

Table 3
Coding of Primary Data Sources for Historical Analysis

Author	Date	Title	Description and applicable codes
Diophantus of Alexandria	3rd century	*Arithmetica,* Book V	*Addition cannot make smaller, Negatives rejected* Negative quantities arose in solving algebraic equations with unknowns; author described them as absurd and rejected them (Heath, 1964).
Brahmagupta	628	*Brâhmasphuta Siddhânta*	*Computational* Rules of adding, subtracting, multiplying, and dividing with negative quantities are stated (Colebrook, 1817).
Bháscara II	1150	Víja-Ganita volume of *Siddhanta Siromani*	*Computational, Magnitude (owing/having), Negatives rejected* Negative quantities occurred as solutions to context problems yielding nonsensical results—e.g., a negative quantity of monkeys (Colebrook, 1817). This solution was rejected because it was a "negative absolute number" (p. 217). However, Bháscara II gave rules for performing computations with negative quantities. Additionally, the words for *negative* and *affirmative* are translated literally as *debt* and *wealth* (p. 131) and are referred to as *directed,* which commentators interpreted in terms of contrary directions (East/West) and time (before and after).
Leonardo Fibonacci (Leonardo Pisano)	1202	*Liber Abaci*	*Magnitude* (*owing/having*)*, Computational, Negatives rejected* Negative quantities arose as solutions to context problems solved nonalgebraically, in one's working from known quantities and given relationships to unknown values (Fibonacci, 1202/2002). Fibonacci accepted negative solutions *only* if they could be interpreted as debits (debt). For example, an intermediate solution for one problem called for a man to have a debit of 15 bezants (a form of currency). However, the interpretation is confusing because it called for the man to give part of his money (in this case, debt) to another man to buy a horse.

Table 3 (*continued*)
Coding of Primary Data Sources for Historical Analysis

Nicolas Chuquet	1484	*Triparty en la science des nombres*	*Magnitude (owing/having), Computational* Negative quantities arose as solutions to context problems involving systems of equations (Flegg, Hay, & Moss, 1985). Chuquet used the ideas of credit, debt, and giving to interpret negative quantities. In one problem, for example, the solution called for purchasing -2.5 pieces of cloth from a merchant. This quantity did not make sense, practically speaking (which Chuquet acknowledged), but was a correct result algebraically. His interpretation of the negative solution was that the -2.5 pieces were bought from the merchant on the basis of a prior credit.
Girolamo Cardano	1545	*Ars Magna* (*The Great Art*)	*Computational, Formal* In his treatise on the rules of algebra, Cardano (1545/1968) described negative solutions as fictitious or false. Negative quantities arose as solutions to polynomial equations, specifically to cubic and quartic equations solved using algebraic methods. Cardano appears to have been using the language of the day in describing negative roots as false but does not appear to have been troubled by them.
René Descartes	1637	*La Géométrie*	*Computational, Formal* In Book III of *La Géométrie,* Descartes (1637/1925) described his theory of equations and their roots. He accepted and operated with negative roots, though he referred to them as false roots or defects (-5 was the defect of 5). In his discussion of the Rule of Signs, he explained how to transform an equation so that all false roots become true and all true roots become false.
Blaise Pascal	1669	*Pensées*	*Subtrahend < Minuend, Negatives rejected* Pascal's (1669/1941) statement, "I know some who cannot understand that to take four from nothing leaves nothing" (p. 25), was found in a collection of unpublished writings addressing philosophy and religion published posthumously.

Table 3 (*continued*)
Coding of Primary Data Sources for Historical Analysis

Isaac Newton	1684	*Universal Arithmetic: Or, A Treatise of Arithmetical Composition and Resolution*	*Computational, Magnitude* (*owing/having, directional segments*), *Formal* Newton (1684/1966) defined numbers as ratios of like quantities; he used ideas of directional line segments, difference between, and money owed vs. gained as interpretations of negative numbers. He also stated the sign rules for products of negative numbers.
John Wallis	1685	*A Treatise of Algebra, Both Practical and Historical*	*Order, Formal, Magnitude* (*owing/having*) Wallis (1685) accepted negative quantities but acknowledged that they presented difficulties. Wallis is often credited as being one of the first mathematicians to use a version of our modern number line. He used a number line representation and the idea of order as well as the notion of opposite magnitudes (gain/loss) to explain negative quantities. Wallis described a common contemporary view that subtracting a larger number from a smaller number was impossible as was conceiving of a magnitude as less than nothing. He explained that this paradox could be resolved if one took a formal, algebraic view.
Jean le Rond d'Alembert	1751	*Encyclopédie, Négatif*	*Subtrahend < Minuend, Subtraction cannot make larger, Negatives rejected, Computational, Magnitude* Calculations with negative quantities were accepted but questions surrounding their meaning remained (d'Alembert (1751/2011). D'Alembert viewed negative quantities in terms of opposite, or directed, magnitudes and rejected the idea of negatives as less than nothing. However, he allowed for isolated negative quantities but only when understood to be in a "false position" that needed to be rewritten or revised so that the appropriate positive solution could arise.

Table 3 (*continued*)
Coding of Primary Data Sources for Historical Analysis

William Frend	1796	*The Principles of Algebra*	*Subtrahend < Minuend, Negatives rejected* Frend's (1796) mission for his book was to make the study of algebra clear by rejecting "strange ideas of number" (p. xi) and scientific principles that cannot be understood "without reference to metaphor" (p. x). The chief offender was negative numbers. Frend rejected negative quantities in and of themselves and as solutions to equations.
Augustus De Morgan	1902	*On the Studies and Difficulties of Mathematics*	*Subtrahend < Minuend, Negatives rejected, Computational, Magnitude (owing/having and directional segments), Formal* De Morgan (1902) described 3 – 8 as absurd and rejected the definition of negative quantities as less than nothing. De Morgan could, however, compute with negative numbers and viewed signs as operations, not as indicative of positive or negative quantities. He used the idea of opposite magnitudes (gains and losses and before and after) and directional segments or measurements and saw the value in negative quantities in the realm of algebra.

representation of a number, this challenge was distinct in that it hinged on the seeming impossibility of removing more than one possessed, whether working within real-world contexts or in the formal realm of mathematics (this obstacle is reflected in the *Subtrahend < Minuend* code listed in Table 3). Additionally, we view this struggle as tied to an interpretation of zero as an *absolute zero*, in which zero represents none or nothing (Glaeser, 1981; Hefendehl-Hebeker, 1991). William Frend described this struggle in the preface of his 1796 book *The Principles of Algebra:*

> Though the whole world should be destroyed, one will be one, and three will be three; and no art whatever can change their nature. You may put a mark before one, which it will obey: it submits to be taken away from another number greater than itself, but to attempt to take it away from a number less than itself is ridiculous. Yet this is attempted by algebraists, who talk of a number less than nothing, of multiplying a negative number into a negative number and thus producing a positive number. . . . This is all jargon, at which common sense recoils. (pp. x–xi)

Similarly, 17th-century mathematician Blaise Pascal (1669/1941) in his *Pensées* somewhat dismissively said, "I know some who cannot understand that to take four from nothing leaves nothing" (p. 25). He continued, stating that some "first principles are too self-evident for us" (Pascal, 1669/1941, p. 25). For Pascal, if zero was, in fact, understood to be nothing, how could someone remove something (e.g., four) from nothing (zero)? In Pascal's mathematical world, this subtraction was not possible, and so the result was nothing (i.e., zero).

Third, the existence of negative numbers made the routine interpretation of addition as the joining of subsets to create a larger set problematic, as seen in the two related codes *Addition cannot make smaller* and *Subtraction cannot make larger*. Hence, Diophantus, in the 3rd century, characterized the equation $4x + 20 = 4$ as "absurd" because the four units as the result of the summation "ought to be some number greater than 20" (Heath, 1964, p. 200). We find this idea confirmed as late as the 18th century. Jean le Rond d'Alembert, who wrote the definitive text regarding negative numbers, *Négatif*, for the *Encyclopédie*, argued that although an answer existed to the problem $x + 100 = 50$, the problem itself was incorrectly formulated:

> According to the rules of algebra we have $x + 100 = 50$, so that $x = -50$. This shows that the quantity x is 50 and that instead of being added to 100 it *must* [our emphasis] be subtracted. . . . In computations, negative quantities actually stand for positive quantities that were supposed in an incorrect position. The "-" sign before a quantity is a reminder to eliminate and correct an error made in the assumption, as the example just given demonstrates very clearly. (d'Alembert, 1751/2011, pp. 72–73)

Though d'Alembert could find the solution to $x + 100 = 50$ algebraically, he argued that the original equation should have involved subtraction instead of addition and been written as $100 – x = 50$. In summary, mathematicians historically grappled with what we describe as three cognitive obstacles related to negative numbers: the lack of a physical, tangible, concrete representation for quantities less than nothing; the problem of removing more than one has; and situations counterintuitive to interpretations of addition and subtraction as joining and separating.

The eventual acceptance of negative numbers. What led to the eventual acceptance of negative numbers in the 18th and 19th centuries? To answer this question, one must first understand how the conception of number itself developed and changed over time. Dating back to Aristotle and Euclid, mathematicians historically distinguished quantities as *numerable* or *measurable*[2] (Ross, 1924; see also Freudenthal, 1983; Klein, 1968; Neal, 2002). This distinction was driven by the practical and perceptual difference between continuous and discrete objects (five miles as compared to five sheep) as well as theoretical and philosophical concerns. The Greek word for number, *arithmos*, carried with it a narrower meaning than our modern-day understanding of number (Klein, 1968; Neal, 2002; Schubring, 2005). *Arithmos* was defined as a numerable (countable), finite quantity; in the

[2] According to Aristotle, a *numerable quantity* was "divisible into non-continuous parts" (i.e., discrete parts defined by unity, or 1), whereas a *measurable quantity* was "divisible into continuous parts" (Ross, 1924, p. 323). A *number*, then, was a finite, numerable quantity.

Greek conception, numbers consisted only of what we now describe as the natural numbers (i.e., discrete, positive units). This distinction between numerable and measureable quantities (otherwise conceived as numbers and magnitudes) held on through the centuries, influencing mathematical thought well into the 1800s. Clearly, both Eastern and Western mathematicians made use of other quantities, including what we now call rational and irrational numbers, but these quantities were not given status as arithmos. There was a distinction between the specific domain of quantities mathematicians labeled as legitimate numbers and the quantities with which they might work or perform calculations—as seen in Table 3 in the high number of *Computational* codes and their co-occurrence with the *Negatives rejected* code. In fact, computational reasoning was one of four ways of reasoning about negative numbers that, historically, enabled some mathematicians to successfully engage with negative numbers and eventually led to their acceptance (see Table 3). The other three are order-based ways of reasoning (e.g., the number line; see Wallis, 1685), logical necessity/formalisms (seen most often in algebraic approaches to problems and the development of generalized solutions), and magnitude (e.g., using ideas of opposite magnitudes, often in context).

During the 16th and 17th centuries, the concept of number, in general, was in flux. Tension between the practical usefulness of new kinds of numbers and classical theoretical concerns remained (Neal, 2002). For example, in the late 1600s, the English mathematician John Wallis (1685) used the equation $x + a = 0$ with $a > 0$ to define negative quantities as all x such that $x = -a$. He went on to explain integers in terms of a number line consistent with our modern number line, using the ideas of motion and movement as well as opposite magnitudes. Wallis accepted negative quantities even though they did not conform to the classical conception of number, but realized that for many, they presented difficulties. In discussing the idea of imaginary quantities, Wallis said, "But it is also impossible, that any quantity (though not a supposed square) can be *Negative*. Since that it is not possible that any *Magnitude* can be *Less than Nothing*, or any *Number Fewer than None*" (p. 264). For some, negatives were not accepted as quantities because they were less than nothing and thus had no magnitude.

With the rise of algebraic notation, the widespread use of symbols to represent parameters, and the generalized solutions and systemized procedures these new representations occasioned (due largely to Vieta and later Descartes), mathematicians more readily treated numbers as mathematical symbols separate from the countable, discrete objects they referenced (Ifrah, 2000; Klein, 1968; Neal, 2002; Schubring, 2005). Abstraction and generalization further blurred the distinction between number and variable, as well as between arithmetic and algebra, and broadened mathematicians' ideas about number itself. Meanwhile, practical applications of the day—including problems in optics, commerce, the motion of free-falling objects, and navigation—drove mathematicians to devise new calculational techniques (e.g., Napier's logarithms), notational systems (e.g., Stevins's infinite decimal expansions), and abstract symbolisms, all of which required the use of new and different kinds of numbers (Neal, 2002). Thus, mathematical formalisms, together with the demands of applied and practical problems, ushered in a paradigm shift wherein the field's conception of *number* was broadened, allowing not only for negative numbers but rational, irrational, and even imaginary numbers.

One might conclude that by the time of Descartes, in the first half of the 17th century, and certainly by the time of Newton and Leibniz, in the second half of the 17th century, that the mathematics community would have finally converged upon a view of number that we have taken as understood today. But such was not the case. Mathematicians continued to argue about the meaningfulness of negative numbers, and only in the 19th century, when the formal view of mathematics won out over the magnitude view, were the negative numbers, once and for all, accepted as legitimate mathematical objects by the entire mathematics community.

Children's Ways of Reasoning About Negative Numbers

We now turn to children's integer reasoning and examine challenges and successes students have in understanding negative numbers. In this section, we describe our findings with respect to children's conceptions of and thinking about integers. We begin by highlighting conceptual challenges students faced and end the section by describing the ways of reasoning some children used to overcome conceptual difficulties and reason in relatively sophisticated ways.

Conceptual challenges children faced when operating with negative integers. After analyzing interviews of the students (none of whom had prior, school-based, formal instruction on integers), we found that the introduction of negative numbers presented significant conceptual contradictions to most students, contradictions that might require students to reconceptualize the idea of number itself. In fact, many of the students had difficulty initially engaging with open number sentences involving negative numbers.

To illustrate, we share students' responses for the three most frequently posed open number sentences across all 47 interviews (the most commonly posed problems were $3 - 5 = \square$, $4 + \square = 3$, and $5 - \square = 8$). For the problem $3 - 5 = \square$, 26 of the 46 children (56.5%) who were asked to solve this task gave a correct answer, but eight of the 20 students who missed the problem $3 - 5 = \square$ gave answers of *not possible* or 0, which received the *Subtrahend < Minuend* code. Other students, after expressing that the problem was confusing, then tried to apply a recalled fact and answered 2. Fifteen of the 41 students (36.6%) who were asked to solve $4 + \square = 3$ answered correctly. Students who missed the problem most often claimed this problem was "not a real problem" or impossible, answers that received the *Addition cannot make smaller* code (24 of 26 students). Alternatively, some children found the sum or difference of the two given numbers and gave an answer of 1 or 7. Finally, none of the 25 students asked this question got $5 - \square = 8$ correct; 12 of the 25 students who missed the problem explained that subtraction should not make the difference larger (reflected in the *Subtraction cannot make larger* code) or they gave answers of 3 or 13 by finding the sum or difference of the given numbers.

On the basis of students' responses to these problems and other number sentences posed during clinical interviews, we identified three consistent obstacles young children faced when they solved problems involving negative numbers: (a) a view of *number* as representing something concrete and countable in conflict with being asked to represent numbers that were "less than nothing" (or "under zero"); (b) trying to remove more than one has (e.g., 3 – 5, the result of which is a negative number) or removing something from nothing (e.g., 0 – 5); and (c) contradictions to generalizations children formed about the ways addition and subtraction function in whole numbers, namely that addition does not make smaller and subtraction does not make larger. In Table 4 we list each of these obstacles, their related codes from Table 2, and illustrative examples of children's ways of reasoning. Additionally, we have identified related responses from mathematicians.

We have found that young children can, at times, resemble early mathematicians in their struggles to understand and accept negative numbers. For example, children's definitions of *number* often sounded like mathematicians' ideas of *arithmos*. Although they may entertain the existence of other kinds of quantities, children are reluctant to give them full number status. Violet's and Elena's statements in

Table 4
Children's Negative-Number Obstacles and Similarities to Mathematicians

Obstacle	Children's responses	Mathematicians' related responses
The existence of quantities less than nothing and the lack of a tangible, concrete, or realistic interpretation for negative numbers (*Negatives rejected*)	"Negative numbers aren't really numbers because we don't really count with them in school. And there's no negative 1 cube (holds up a Unifix® cube)." A number is "how you know how much something is." (Violet, second grade) "A number shows how much of something you have; … [-8] is not actually a number because it's less than a number. . . . It just doesn't really have any volume for it, like what it has." (Elena, third grade) "Zero is nothing and negative is more nothing." (Rebecca, second grade)	The Indian mathematician Bháscara II explained, "People do not approve a negative absolute number"; thus, negative solutions were considered "incongruous." (Colebrook, 1817, p. 217) Fibonacci and Descartes did not accept negative solutions unless the result could be interpreted as something positive. "Above all, he [the student] must reject the definition still sometimes given of the quantity -*a*, that it is less than nothing. It is astonishing that the human intellect should ever have tolerated such an absurdity as the idea of a quantity less than nothing; above all, that the notion should have outlived the belief in judicial astrology and the existence of witches, either of which is ten thousand times more possible." (De Morgan, 1902, p. 72)

Table 4 (*continued*)
Children's Negative-Number Obstacles and Similarities to Mathematicians

Removing something from nothing or removing more than one has (*Subtrahend* < *Minuend*)	"Three minus five is zero because you have 3 and you can't take away 5, so take away the 3 and it leaves you with zero." (Sam, first grade) When asked to solve 3 – 4 and 3 – 3, Sam answered 0 to both. "Three minus five doesn't make sense because three is less than five." (Nola, first grade) When solving 3 – 5 = □, Andrew (second grade) replied, "How come there's three and take away five? I don't have enough. 'Cuz look, there's three (holds up 3 fingers) and I cannot take away five 'cuz there's not enough."	"I know some who cannot understand that to take four from nothing leaves nothing." (Pascal, 1669/1941, p. 25) "3 – 8 is an impossibility; it requires you to take from 3 more than there is in 3, which is absurd. If such an expression as 3 – 8 should be the answer to a problem, it would denote either that there was some absurdity inherent in the problem itself, or in the manner of putting it into an equation." (De Morgan, 1902, pp. 103–104)
Counter-intuitive situations involving routine interpretations of addition and subtraction (*Addition cannot make smaller; Subtraction cannot make larger*)	"4 + □ = 3 is not a real problem. It's not true" (he crossed out the problem). "Four minus 1 would equal 3." (Brad, first grade) In response to the problem 6 + □ = 4, Brian (first grade) said, "What's that plus for? Isn't it supposed to be a minus?" In response to the problem 5 – □ = 8, Ryan (first grade) said, "I wouldn't be able to do it because it would always be behind 8 if it was minus something. Because if it was minus 0 it would be 5. It [the difference] would always be behind 8."	Diophantus claimed that the equation $4x + 20 = 4$ was "absurd" because the 4 was less than the 20 units that were added. (Heath, 1964, p. 200) D'Alembert (1751/2011) argued that the equation $x + 100 = 50$ should have involved subtraction instead of addition and been written as $100 - x = 50$.

Table 4 that negative numbers are not really numbers appear to be rooted in the idea that numbers are "used for counting" or "to know how many." We define an understanding of number that appears to be equated with numeration, counting, and magnitude and, consequently, restricted to whole numbers as a *magnitude-based* or *cardinal view* of number—by this definition, we mean an understanding of number as a tangible amount of something, tied to numeration and counting sets of objects. In fact, each of the children's cognitive obstacles described in Table 4 seems to stem from a magnitude-only view of number. For example, the cognitive obstacle of removing something from nothing rests on the assumption that integers represent something—a countable something. Similarly, the overgeneralizations that addition cannot make smaller and subtraction cannot make larger draw upon interpretations of adding and subtracting as joining and separating countable sets of objects. For this reason, we identify a magnitude-based way of reasoning as a fourth cognitive obstacle for children, in addition to and related to the three previously mentioned obstacles.

Mathematicians, too, were reluctant to give negative quantities status as numbers, but not because their only understanding of number was as representing cardinal sets of objects. Their objection was rooted in ancient Greek conceptions of number and distinctions among arithmetic, algebra, and geometry. Interestingly, some mathematicians, almost paradoxically, used magnitude-based interpretations to reason about integers productively. For example, De Morgan (1902) referred to the idea of opposite magnitudes in explaining that

> a negative solution indicates that the nature of the answer is the very reverse of that which it was supposed to be in the solution; for example, . . . if we supposed that *A* was to receive a certain number of pounds, it [-*c*] denotes that he is to pay *c* pounds. (p. 121)

Yet, De Morgan called the expression 3 – 8 "absurd" on the basis of the same magnitude-based view of numbers (p. 104). For De Morgan, magnitude was both an obstacle and an affordance!

We have shared cognitive obstacles for both children and mathematicians, and these obstacles may be expected given each group's early experiences with number. However, we all know that mathematicians no longer struggle with these conceptual obstacles related to negative numbers. They understand that addition does not necessarily result in a larger sum and accept the possibility of subtracting a larger number from a smaller number. How can one explain this acceptance? Modern mathematics can be characterized by a formal algebraic approach based on foundations laid in the 1800s when mathematicians came to fully embrace formalisms and unifying structures. In a Kuhnian sense, the old paradigm with its strict division between geometry and arithmetic, number and magnitude, and the theoretical and concrete was swept away with successive algebraic extensions of the concept of number (Kuhn, 1962). Likewise, not all the students in our study reasoned about number in the ways described in Table 4. In the next section, we describe ways in which children have displayed powerful and successful ways of reasoning about integers.

Children's successful ways of reasoning about integers. Some students not only had negative integers in their domains but also could correctly solve problems like $3 - 5 = \square$ and $-2 + \square = 4$. We have found, perhaps surprisingly, that children as young as age 6 can overcome cognitive obstacles and mathematical contradictions that many of their peers could not. Below, we present three ways of reasoning that children used appropriately to correctly solve integer problems: leveraging ordering relations for sets of numbers, employing the concept of logical necessity, and, perhaps most surprisingly, seeing negative numbers as having magnitude. We share these ways of reasoning not because they are common but because we believe that they are mathematically powerful approaches to integer reasoning that all children should eventually be able to understand and apply when solving integer tasks. Across all 47 children, 45% (or 21 children, including students from each of the three schools represented) used ordering relations, logical necessity, or magnitude to successfully solve at least one problem during the course of the interview. That is not to say that these ways of reasoning characterized these young students' thinking, but they occurred with enough regularity that we believe that teachers and researchers could productively attend to and build on them. Further, we propose that these types of reasoning are deeply mathematical; they are foundational for reasoning about numbers and mathematics in more sophisticated, formal, and structured ways, yet they are developmentally appropriate for children this age. In the following sections, we explain each of these ways of reasoning; relate them to the modern, formal, mathematical approaches in which they are grounded; and describe how children used these mathematical principles in their solutions.

Ordering relations. The idea of order is a basic principle of our number system. Lakoff and Núñez (2000) identified order as a foundational component of mathematical cognition; arithmetic as "motion along a path" is one of their four "grounding metaphors" (p. 21). They suggested that negative numbers are constructed as point locations within this motion metaphor, using the idea of symmetry on the number line. Some young students leveraged the idea of order, though not in its formal form, when solving problems involving integers. In particular, these students imposed some kind of ordering on **Z** and then used the ordinal, or positional, nature of numbers in their strategies. In the remainder of this section we briefly sketch the mathematical background of ordering relations and consider how some students' strategies leveraged the mathematical ideas of order in developmentally appropriate ways.

Children's initial experiences with ordering occur when they learn to count and reason about *before* and *after* and *smaller* and *greater*. Mathematicians have formalized this idea and developed a variety of ways to order sets using ordering relations. For different ordering relations, one uses different criteria to compare elements. In some orders, like the familiar less-than ordering of **N** (the natural numbers), one can compare every pair of elements. But in other orders, like the subset ordering of **S** (the power set), one cannot.

Of course, none of the young children we interviewed made these distinctions, nor did they treat **Z** in a set-theoretic way. However, we do see underlying mathematical ideas of order and ordering relations reflected in some children's ways of reasoning. In particular, 43% of students in our study (20 of 47) used the ordinal and sequential nature of integers at least once in solutions that involved counting or the use of the number line, the context of motion and movement, or both. For example, many students who correctly solved the problem $3 - 5 = \square$ used a counting-back strategy. Typically students counted backward starting at 3, saying, "Three, 2 (put up one finger), 1 (put up second finger), zero (put up third finger), -1 (put up fourth finger), -2 (put up fifth finger). The answer is -2." In this strategy, -2 is a position; it is the place one lands when starting at 3 and counting back 5; it is the number before -1 and after -3.

Similarly, we saw Lynn use a counting strategy while she debated whether to count up or down to solve $-8 - 3 = \square$. Note that in her explanation she used the language of "counting down," and in her counting she uses 8, 7, 6, 5 rather than -8, -7, -6, -5, attaching the "negative" after completing the counting sequence.

> I'll just start counting by, start counting down at 8 because it's negative. Eight, 7 (holds up one finger), 6 (holds up two fingers), 5 (holds up three fingers). Negative 5. (Pause.) Wait, I think it's switched. I don't know. At first I was thinking since it was a minus [the subtraction symbol in -8 – 3] so it would have to be a minus. But now I'm thinking since this one was a plus (she points to the previous problem $-3 + 6 = \square$) and I had to do minusing, that this one [$-8 - 3 = \square$] is plus on the negative numbers. (Lynn correctly counted up 6 units from –3 for the previous problem but because the absolute values of the numbers decreased, she said it was "like minusing" to her.) I want to change my answer and count up now. Eight, 9 (holds up one finger), 10 (holds up two fingers), 11 (holds up three fingers). Negative 11. (Lynn, Grade 2)

Lynn used a counting-down strategy, but because her starting point was in the negative numbers, she struggled to decide which way was "down." She eventually decided that -8 – 3 did not operate like 8 – 3 after comparing the former to an earlier problem involving addition. Because she had counted from -3 to -2 to -1 and so forth when solving -3 plus 6, then she needed to "count up" (what we traditionally would describe as counting down) to solve -8 minus 3 because, as she explained, "Minusing is plus on the negative numbers." Lynn's explanation is a reminder that when one operates with negative numbers, counting does not feel quite the same as with positives (see also Ball, 1993; Bishop et al., 2013). In particular, when the magnitude of a negative number increases, the number itself gets smaller; hence, she described counting from -8 to -11 as "counting up."

Other students leveraged the ordered nature of integers by using a number line. For example, when solving $3 - 5 = \square$, Lucy initially made stacks of three and five unifix cubes but quickly turned to a number line. She put a pointer at 3 and moved it 5 spaces to the left while counting the number of movements aloud (1, 2, 3, 4, 5). She answered 02 (her invented notation for -2) and said, "That number line actually helped me a lot" (Lucy, Grade 1). Her response after being asked whether the cubes helped her was telling: "When I used cubes, I mean what could they help me with this? How am I gonna do it?" For Lucy, -2 appeared to represent not a countable number of objects but instead a location on the number line that existed as part of an ordered sequence of numbers.

Though not as formalized as the modern mathematical definition of ordering relations, young children's conceptions of the mathematical idea of order were used successfully to solve new and novel problems (to them) involving integers. Thus, we identified ordering relations, or an order-based understanding of negative numbers, as a cognitive affordance for children's integer reasoning.

Logical necessity and formalisms. We use the term *logical necessity* to describe a way of reasoning wherein students use a more formal approach to problem solving that leverages the ideas of structural similarity, well-defined expressions, and fundamental mathematical principles (e.g., commutativity, negation). This way of reasoning includes generalizing beyond a specific case by making a comparison to another known problem and appropriately adjusting one's heuristic so that the underlying logic of the system and approach remains consistent. Hefendehl-Hebeker (1991) described this idea using the phrase "the principle of the permanence of formal laws" (p. 30; see also Freudenthal's [1983] algebraic-permanence principle; both permanence principles are based on Hermann Hankel's and George Peacock's formulations of this principle). She explained the principle's meaning, saying, "Formulas valid in the system of natural numbers are to remain valid in the extended number systems" (Hefendehl-Hebeker, 1991, p. 31). The key characteristic of this principle is maintaining consistency with what we know to be true for whole numbers. In other words, new types of numbers should not operate in ways that violate properties for whole numbers.

Logical necessity is a fundamental idea in the discipline of mathematics. For example, when employing proof by contradiction, one proves a proposition to be true *indirectly*, by showing that its negation leads to a contradiction of what is already known. Consider, for a moment, the product of two negative numbers. First, take the Peano axioms as the basic, underlying assumptions for behavior of natural numbers and also require these relationships to hold for extensions of the natural numbers (in this case $\mathbf{Z}$). One can then argue that the product of two negative numbers is positive because it must be! It is a logical necessity or outgrowth of the structure, underlying assumptions, and rules of operation for our number system. Were it not so (i.e., evoking a proof by contradiction), inconsistencies such as $1 = 0$ would appear, and our number system as we know it would break.

Approximately 15% of students in our study (7 of 47) used logical necessity at least once during the interview. We see a hint of logical necessity in Lynn's counting strategy above when she compared addition to subtraction and reasoned that the two operations cannot behave in the same way. In other words, one cannot count down for both addition and subtraction of a positive number; they must be well-defined operations yielding unique values when given the same input. A clearer example is seen in Terrell's way of reasoning about the problem $-5 - -3 = \square$. Before solving this problem, he solved $-5 + -1 = \square$, answering -6. He then reasoned that,

> -5 – -3 . . . it would probably be something, it would probably be, um, minus two [his term for -2] because if you add, if you use addition with this (pointing to the numbers -5 and -3 in the original number sentence), it would be farther from positive numbers. So if you do the opposite it should be closer. (Terrell, Grade 4)

Here, Terrell compared the operations of subtraction and addition, holding the numbers -5 and -3 constant. Terrell reasoned that if addition moves one further from the positive numbers, then its opposite, subtraction, should move one closer to the positive numbers. Terrell even extended this reasoning to the problem -7 – -9 = $\square$, correctly reasoning that the answer was "regular 2."

Ryan, a first grader, claimed that problems of the form 5 + $\square$ = 3 were not possible to solve early in his interview. Ryan had negative numbers in his numerical domain, could correctly order them, and solved problems like 3 – 5 = $\square$, -4 + 7 = $\square$, and $\square$ – 5 = -8 successfully using order-based ways of reasoning. Ryan, however, was troubled by 5 + $\square$ = 3, asking, "If you add something, how does it get to 5? If it's 5 plus, then it's [the sum] always past 3."

Toward the close of the interview, we posed the problem -2 + 5 = $\square$ to Ryan. His answer was 3, which he obtained by using a counting strategy, counting up five from -2. We then asked him to consider the problem 5 + -2 = $\square$. He answered, "Three," explaining,

> Because it's pretty much the same thing (points to -2 + 5). Five plus negative 2 and negative 2 plus 5. If you add the same things, and you just say 5 first and [negative] 2 second, it's still the same thing. . . . You always add the same things together.

Ryan used a fundamental principle of mathematics, the commutative property, to reason about a possible meaning for adding a negative number. He assumed that all numbers, even negative numbers, obey the commutative property for addition; consequently, his answer had to be 3. For his newly expanded numeric system to be consistent and logical, 3 was the necessary answer. Immediately, we asked Ryan to solve 6 + $\square$ = 4, which Ryan compared to the problem he had previously solved, 5 + -2 = 3, saying, "It's kinda like that one [he points to 5 + -2 = 3] because there's plus a negative. Six plus negative 2 goes two back." When reminded that he had earlier said that this kind of problem was "impossible," he smiled and said, "Now it's plus a negative. So if it's a negative number, you have to fill it back in. . . . It's like minus."

By using the principle of logical necessity, some children were able to approach and use numbers in a more formal, algebraic way, leveraging key mathematical ideas about inverses and fundamental algebraic properties to solve problems previously unsolvable for them. This way of reasoning, which is integral to modern mathematics, helped some students to conjecture about meanings for adding and subtracting negative numbers and then to accept or reject those conjectures on the basis of whether the structure and logic of the system were preserved. We see this as a powerful way of reasoning for young children and, therefore, identify logical necessity and formalisms as a cognitive affordance for children's integer reasoning.

Magnitude. For most children, a magnitude-based understanding of numbers was challenging to extend in ways to help them reason about negative integers, hence its inclusion earlier as an obstacle. However, some students were able to view negative integers as representing countable amounts; in fact, approximately one fourth of the students in our study (11 of 47) productively used magnitude-based strategies at least once during the interview. Consider one of our first graders, who treated -8 as eight "negative 1s" or as equivalent to -1 + -1 + -1 + -1 + -1 + -1 + -1 + -1. When solving $-8 - -1 = \square$, he answered -7 and explained, "You take away a negative because there is eight negative 1s altogether so that would make negative 8. And you take away just 1 of it and now it's negative 7." This student reasoned about a meaning for subtracting a negative number by treating -8 as comprised of a countable number of negative 1s. He could then remove one of those objects, reducing the number of negative 1s in his set. Just as he could count oranges, he could count negative 1s.

Another student, Jacob, when presented with the problem $-7 - \square = -5$, answered negative two. He explained his answer using unifix cubes.

> Well, this one I need little cubes. . . . [It] would be like real numbers, but just add the minus sign. You just do 7 plus, well actually, 7 minus 2 equals 5. That's the answer for real numbers. So I just added a negative to all of them, and there is my answer.

Here we see Jacob treat negative numbers like "real" numbers to productively reason about and solve problems involving negative integers. In particular, he treated -7 and -5 as having magnitude and representing a countable number of objects that could be represented with cubes. Jacob had earlier stated, "The negative numbers are like that [the real numbers] because you just add a minus sign." Similarly, Carlee, a fourth grader, treated negative numbers like "normal" numbers when solving $-5 - -3 = \square$. She answered, "Negative 2," and explained, "Five minus 3 is an easy fact for me. So, um, using negatives it will probably be the same thing like using normal numbers. It will probably be the same thing, but with negatives it'll probably be negative 2."

We highlight these strategies because of the seeming paradox they present. Magnitude-based, or cardinal, views of number may initially seem to limit the possibility of reasoning about negative numbers. Yet, some students could view negative numbers as having magnitude, but not necessarily as less than nothing, and representing tangible amounts (e.g., amount owed, depth of a hole, the number of negative 1s). Moreover, they used these ideas to reason about operations with negative numbers in robust, mathematically correct ways. As a result, we identify a magnitude-based way of reasoning as both a cognitive affordance and an obstacle.

Discussion and Implications

Using data from historical writings of mathematicians and student interviews, we have identified a total of three cognitive obstacles, three cognitive affordances, and one type of integer understanding that can function as either an obstacle or affordance. We see these obstacles and affordances, which are listed in Table 5, as foundational components in an initial framework of integer reasoning.

Our findings indicate that the primary difficulties surrounding negative numbers are related to four cognitive obstacles. First, numbers are used primarily for numeration and counting objects in our early mathematical experiences, hence the first obstacle, viewing numbers solely from a magnitude-based perspective. Second, the cardinal nature of numbers corresponds to a tangible and concrete representation of a number. For example, 7 is the number of toy bears I have in a pile; -7 does not typically carry the same kind of meaning, which brings us to the second cognitive obstacle. Negative numbers, when conceived of as less than nothing, often defy concrete, tangible representation. Third, how can one remove more than one has (as in the problem 3 – 5) or remove something from nothing? And fourth, we struggle with negative numbers because their very existence seemingly contradicts deeply held notions about the meanings of the operations of addition and subtraction. When expanding their numeric domain to include negative numbers, learners are confronted with the fact that addition can, in fact, make smaller and subtraction can make larger. Our data show that these are challenges learners are likely to face while they struggle to make sense of and operate with negative numbers. Further, we claim that students should be given opportunities to grapple with these contradictions because only in their resolution will students develop a robust understanding of this new kind of number.

Despite the existence of these cognitive obstacles, we found that particular ways of reasoning may help learners to think more productively and successfully about

Table 5
Cognitive Obstacles and Affordances Identified in Children's Thinking and Historical Texts

	Children's thinking	**Historical texts**
Cognitive Obstacles		
Magnitude	X	X
Existence of quantities less than nothing (*Negatives rejected*)	X	X
Removing more than one has (*Subtrahend < Minuend*)	X	X
Overgeneralizations that addition cannot make smaller and subtraction cannot make larger	X	X
Cognitive Affordances		
Magnitude	X	X
Order	X	X
Logical Necessity/Formal	X	X
Computational		X

negative numbers. First, we found that understanding numbers as ordered—that is, as locations on a number line or elements in an ordered sequence—supported both mathematicians and children to engage with integer tasks using number lines and counting sequences. Second, we found that some students and mathematicians used magnitude-based reasoning as an affordance; they appropriately treated numbers as countable amounts or as representing objects. Third, we identified logical necessity/formal ways of reasoning as an affordance for integers wherein one looks for and makes use of underlying structures and principles to reason about integers. Formal ways of reasoning extend well beyond the domain of integers as this way of reasoning supports learners to engage in fundamental mathematical practices. And fourth, although computational reasoning functioned as a cognitive affordance for mathematicians historically, it did not for students in our study, most likely because they had not yet learned integer arithmetic. Because we did not interview older children who had already learned integer arithmetic, we do not know whether or how computational reasoning would function in these students' thinking about integers. We identified four cognitive affordances in the domain of integers present in the historical data and the interview data: order, magnitude, logical necessity/formal, and computational ways of reasoning. These four affordances along with the previously mentioned obstacles focus on ways of reasoning, a topic that is largely absent from existing integer research. Additionally, our identification of order and magnitude as affordances builds on earlier work by Peled and colleagues (1991; Peled et al., 1989) and Chiu (2001) by documenting specific ways in which children engaged in these ways of reasoning in more and less productive ways. In summary, we have developed an initial, research-based framework for integers that highlights two key aspects of integer reasoning, obstacles and affordances, and that adds to the knowledge base within mathematics education.

Relating Children's and Mathematicians' Integer Reasoning

In this study, we highlight the value of using two seemingly disparate data sources to better understand the conceptual field of integers and develop an initial framework of integer reasoning. The children's thinking data enabled us to exemplify and clearly describe specific strategies and details related to cognitive obstacles and affordances. But we found that by understanding the historical affordances and constraints faced by mathematicians who were thinking about integers, we were able to more clearly see and make sense of children's mathematical thinking. We are not suggesting that children's reasoning develops in parallel with the historical development of the mathematical ideas but that the analyses were interrelated, mutually dependent, and shared interesting similarities. Although we are sharing what we deem to be useful similarities between children's difficulties with negative numbers and mathematicians' struggle to accept negative quantities as numbers, we also recognize three fundamental differences between children and early mathematicians. First, generally speaking, mathematicians could compute with negative numbers and understood operations with them long before they gave negative quantities the status of number. In the case of the young children we studied, not only was accepting negative quantities as numbers a difficulty, but they also were unable to compute with negative numbers. So, unlike some mathematicians who could find negative solutions to problems but then rejected those solutions, many of the children in our study could not obtain negative solutions. Second, some mathematicians could conceive of negative quantities as formal objects disconnected from numeric systems and leverage algebraic representations. And third, for many children a magnitude-based understanding of number was their only understanding of number. For mathematicians, it was not that they could not conceive of negative numbers but that they held differing, and sometimes competing, conceptions simultaneously.

We also know that the children in our study will come to accept negative quantities as numbers long before adulthood, but unlike mathematicians of old, our students today are encultured into a world in which negative numbers appear not only in school mathematics but also in temperatures, measures below sea level, and the like. Students today also have access to models, tools, and representations (e.g., the number line and chip models) that were products of successful historical approaches to integers. Thus, there are important differences between the historical path to accepting negative quantities as negative numbers and children's paths to accepting negative numbers today.

Despite the differences surrounding historical integer reasoning and children's current thinking, learners today appear to encounter similar obstacles and affordances, perhaps because residual aspects of historical development are embedded into our current representations, tools, and ways of reasoning (Radford & Puig, 2007; Schubring, 2011). Ideas that were, historically, catalysts for productive integer reasoning are now accepted as routine mathematical practice. For example, consider the affordance of order-based reasoning. We see an order-based understanding of number reflected in Wallis's (1685) use of the number line in his text *A Treatise of Algebra Both Historical and Practical*. Wallis is often credited as being one of the first mathematicians to use a version of our modern number line, coordinating both positive and negative number lines. Henley (1999) described the development of this type of understanding of negative numbers as a paradigm shift for mathematicians. Negative numbers could thereafter be viewed as quantities to the left of zero on the number line (as opposed to less than nothing), and their ordering meant that the statement -2 < -1 could be interpreted to mean that -2 comes before -1 (not restricted to the interpretation that -2 is smaller in magnitude than -1). What was groundbreaking then is seen as commonplace now. Students today can access a representation, centuries in the making, that presents them with an alternative view of numbers as ordinal.

The number line and counting strategies that students encounter and productively use today are the product of historical cognition reflected in what is commonly accepted as standard mathematical practice and embedded in current tools and representations. These tools provide opportunities for students to treat negative numbers as positional and as sequentially related to other numbers and not necessarily as having to represent a tangible amount. Thus, we are not surprised to see this similarity between modern students' thinking and the historical development of integers. Radford and Puig (2007) explained this type of relationship with their *Embedment Principle*, which posits that similarities exist because the historical dimension is embedded into our current social mathematical practices and the resultant semiotic representations and tools we use today (see also Furinghetti & Radford, 2002, 2008; Schubring, 2011). One way to explain the relationship between historical thinking about integers and children's mathematical thinking is that students today encounter the products of historical cognition deposited in tools, representations, and commonly accepted ways of reasoning and leverage them to transform their own integer reasoning in powerful ways. Although our study and the data we collected were not designed to determine the nature of the relationship between children's current thinking and the historical development of integers, we see this as an important area for future research.

Limitations and Future Directions for Research

We believe that many of the limitations of our study point toward new directions for research. For example, our clinical interview was not standardized, and because not all students were asked to solve the same open number sentences, we could not link the correctness of student responses and ways of reasoning to specific problem types. A promising direction for future research would be to determine if particular obstacles and affordances are more and less prevalent for specific problem types; which problems, in general, were easier and harder for students; and if a child's grade level is related to success with different types of problems. Additionally, our sample did not include low-performing schools (as measured by standardized state tests). Although we included students who were identified by their teachers as below grade level with respect to mathematical proficiency, we do not know what differences might exist if interviews had been conducted at low-performing schools. On a related note, we know little about the instruction in classrooms and thus cannot explain the emergence of certain affordances documented in our study. We do know, however, that in the academic year in which we conducted the clinical interviews, the teachers had not provided any formal, school-based instruction on integers. Consequently, future research should be situated in a wider variety of schools, and researchers might seek to identify links between instruction and the emergence of cognitive affordances and obstacles. Finally, the historical analysis of mathematics texts includes context problems, whereas our analysis of the clinical interviews did not. By including context problems in our analyses, we might possibly have identified more and different magnitude-based approaches from children or perhaps even obstacles and affordances different from those we report here.

In closing, although we did not set out in this study to determine how best to teach integers, we understand that readers may look for instructional implications from our work. A conclusion that one might draw from our study is that teachers ought to teach integers to children using order-based or formal views of number, but we do not embrace this conclusion! On the basis of our work with children, together with our study of the historical development of integers, we conclude that many views of numbers will arise for learners. We believe that students should be encouraged to make sense of and negotiate among these different ways of reasoning and to develop multiple ways of understanding numbers. Thus, one instructional goal should be for children to approach negative numbers using ordering relations, magnitude, and logical necessity/formal views of number and to flexibly move among these ways of reasoning.

We encourage teachers, researchers, and curriculum writers to remember that children are extraordinary problem solvers and are capable of powerful mathematical reasoning when given the opportunity. Our findings point to the potential benefits of encouraging students to grapple with big mathematical ideas and the historical contradictions that puzzled mathematicians of old. Children proved more than capable of engaging with these problems and the formal mathematical reasoning they fostered. Ryan, Terrell, and Lynn, for example, encountered novel problems, and their approaches resembled those of current-day mathematicians when they used inherent mathematical structures and the tools of the discipline itself for insight into possible solutions. These children have taught us not to avoid complex mathematical ideas but to use students' questions and points of confusion as opportunities.

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APPENDIX

Sample Interview Questions

1. Introductory Questions

   a) Name a big number. Name a bigger number. Name a small number. Name a smaller number. (*If child names 0, ask her if she can name a number smaller than zero*.)

   b) Please count backward starting at 5. (*If child stops at 0 or 1, ask*, "Can you keep counting back?")

2. Open Number Sentences

a) 6 – 11 = ☐	b) 6 + ☐ = 4	c) -11 + 6 = ☐
d) 5 – ☐ = 8	e) ☐ – 6 = -3	f) -5 – -2 = ☐
g) 6 + -3 = ☐	h) ☐ + -4 = –9	i) -9 + ☐ = -4

3. Contextualized Problem

   a) Yesterday you borrowed $8 from your friend to buy a school t-shirt. Today you borrowed another $5 from the same friend to buy lunch. What's the situation now? (*If needed*, "Does your friend owe you money, do you owe your friend money, or is it some other situation?")

   b) Can you write an equation or number sentence that describes this story? (*After child writes an equation* ask, "Can you explain how this number sentence [or equation] relates to the story?")

   c) I am going to show some equations that other students wrote for this problem. They may or may not match yours. Please write yes next to the equation if you think it matches the story about you and your friend or write no if it does not match. Then explain your response.

$$-8 + -5 = -13$$
$$8 + 5 = 13$$
$$-8 - 5 = -13$$

4. Comparison Tasks

   For each pair, circle the larger, write "=" if the two quantities are equal, or write "?" if there is not enough information to determine which is larger.

a) 3	-7	b) 0	-9
c) -5	-6	d) -100	-5
e) – -3	-3	f) ☐	- ☐