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Change of State Variables in the Problems of Parametric Control of Oscillators

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Abstract—The problems of optimal parametric control of a single oscillator and an assembly of quantum oscillators were solved and used by way of example to demonstrate the potentialities of the method of transition to the new variables of the state space of the controlled system.

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1. INTRODUCTION

Solution of the optimal control problem

$$I = \int_0^\tau f_0(x, u, t)dt \to \max \tag{1}$$

on conditions that

$$\dot{x}_i = f_i(x, u, t), \quad x \in R^n, \quad u \in V_n \subset R^m, \quad i = 1, \dots, n, \tag{2}$$

where f_i is continuous in u and continuously differentiable in x, t and V_u is a compact, may be substantially facilitated by passing to new state variables $y(x)$. We call (1), (2) the original problem and that with the state variables y, the transformed problem.

We list the aims that may be reached as the result of such transition:

(1) Part of the variables y may be made independent of all or some of the control actions.

(2) Some of the variables y may be used as controls with reduction of the problem dimensionality.

(3) Detection of the function of phase variables which is constant for any permissible controls or in the class of controls satisfying the optimality conditions.

(4) For the problems where the right-hand sides of the differential equations are not explicitly time-dependent, the problem dimensionality may be reduced if the rate of variation of one of the variables y_ν retains its sign for all or the optimal values of its arguments. In this case, dt may be replaced by dy_ν. In the original problem such a variable needs not to exist.

(5) If the right-hand sides of the differential equations for $y_j,\ j = 0, \dots, n$, are independent of the state variable y_ν, then the νth equation may be rejected or replaced by an integral constraint if on the boundary of the interval the values of this variable are fixed.

(6) Problem (1), (2) is substantially simplified if the boundary conditions and control constraints enable on to construct the domain W of the reachable states of the transformed problem and at that prove that the boundary of this domain is the optimal solution of the transformed and, consequently, original problem.

Some of the aforementioned aims may be reached in the estimation problem with a wider set of the permissible solutions, rather than in the original problem (1), (2). V.I. Gurman and his students solved some important applied problems using namely this change of the state variables [1].

On the strength of Eqs. (2), the rate of variations of the phase variable $y_j(x)$ is as follows:

$$\dot{y}_j = \sum_{i=1}^{n} \frac{\partial y_j}{\partial x_i} f_i(x, u, t). \tag{3}$$

Conditions like sign constancy or independence of the right-hand sides of one or another variable must be imposed to reach the aforesaid aims. The conditions lead to partial derivative equations whose solution defines the desired change.

No set of rules to determine the desired change of variables exists, but there are, however, certain methods and practices facilitating selection of the dependencies $y(x)$. These methods are demonstrated below by solving two problems of oscillator control:

—pendulum slowing-down (swinging) by varying the distance of its center of gravity to the point of suspension (the problem of swing);

—energy extraction from a system of quantum oscillators by varying the Hamiltonian parameter.

Both problems may be regarded as those of extraction of the given energy in a minimal time or, in terms of the "optimization thermodynamics," the maximal power problems [2]. The first problem was considered in [3], but the solution obtained there was incorrect. The second problem was solved in [4] with the use of the *Maple* package for a special form of the boundary conditions. As is shown in what follows, the passage to new state variables substantially simplifies solution of these problems.

2. PARAMETRIC OSCILLATOR SLOWING-DOWN

Let us consider the problem of optimal speed for a controlled system like

$$\dot{q} = p, \quad \dot{p} = -uq, \quad 0 < u_1 \leqslant u \leqslant u_2, \tag{4}$$

where q is the pendulum deviation, p is its speed, and the coefficient u which corresponds to the squared frequency of the natural oscillations depends on the mass and distance from the point of suspension to the center of gravity. The latter can be varied in order to influence the system motion.

Given are the initial state of the system p_0, q_0, the value of total energy at the terminal instant τ and the control constraints

$$p^2(\tau) + u_0 q^2(\tau) = \overline{E}, \quad 0 < u_1 < u_0 < u_2. \tag{5}$$

The value of u_0 is fixed. Needed is to determine a control law $u^*(t)$ such that

$$\tau \to \min_u . \tag{6}$$

Since the initial state is given, the initial value of energy $E_0 = p_0^2 + u_0 q_0^2$ is fixed as well, and the problem corresponds to the extraction of the maximal power $N^* = \dfrac{E_0 - \overline{E}}{\tau^*}$. For a constant value of u, on the phase plane the trajectories make up a portrait of the center type (system of ellipses); time profile of the phase state corresponds to the clockwise motion for which $e(t) = p^2(t) + uq^2(t) =$ const.

Although time is not involved explicitly in the right-hand sides of Eqs. (4), their sign varies, and the problem dimensionality cannot be reduced be replacing time by one of the state variables.

To estimate laboriousness of the solution, we rearrange constraint (5) in the integral form

$$\int_0^\tau \frac{d}{dt}(p^2 + u_0 q^2)dt = \int_0^\tau pq(u_0 - u)dt = \frac{E_0 - \overline{E}}{2}, \tag{7}$$

assume without loss of generality that $u_0 = 1$, and set down the optimality conditions of problem (4), (6), (7) in the form of the principle of maximum [5] under the assumption that the solution is nondegenerate ($\psi_0 = -1$).

The Hamiltonian function

$$H = -1 + \psi_1 p - \psi_2 uq + \lambda pq(1 - u),$$

the equations for conjugate variables

$$\begin{cases} \dot{\psi}_1 = -\dfrac{\partial H}{\partial q} = \psi_2 u - \lambda p(1 - u) \\ \dot{\psi}_2 = -\dfrac{\partial H}{\partial p} = -\psi_1 - \lambda q(1 - u), \quad \psi_1(\tau) = \psi_2(\tau) = 0, \end{cases} \tag{8}$$

and the condition for maximum of the nonsingular solution

$$u^*(t) = \arg\max_u H = \begin{cases} u_1 & \text{for} \quad q(\psi_2 + \lambda p) > 0 \\ u_2 & \text{for} \quad q(\psi_2 + \lambda p) < 0, \end{cases} \tag{9}$$

λ being a scalar selected using condition (7).

A singular solution corresponds to the equality

$$q(\psi_2 + \lambda p) \equiv 0$$

over the time interval of zero measure.

In order to solve the original problem with the use of the principle of maximum, one needs, therefore, to solve the system of Eqs. (4), (5), (7), (8) closed by the requirement (9) and determine a singular solution or prove that there is no such.

We demonstrate how this problem may be simplified by passing to new state variables and introduce new variables $z(p, q)$ and $e(p, q)$ so that in virtue of Eqs. (4) the rate of change of one of them does not change sign. This makes it possible to reduce the problem dimensionality. It is also desirable that the right-hand sides of Eqs. (4) do no involve one of the variables, which enables one to replace the condition in the form of a differential equation by the integral equality. We select the new variables as

$$z = -\frac{p}{q}, \quad e = \ln(p^2 + q^2).$$

In virtue of Eqs. (4), their rate of change is as follows:

$$\dot{z} = \frac{\dot{q}p - \dot{p}q}{q^2} = \frac{p^2 + uq^2}{q^2} = z^2 + u, \quad z_0 = -\frac{p_0}{q_0}, \tag{10}$$

$$\dot{e} = \frac{2(p\dot{p} + q\dot{q})}{p^2 + q^2} = 2\frac{-pqu + qp}{q^2(1 + z^2)} = 2\frac{z(u - 1)}{1 + z^2}, \tag{11}$$

$$e_0 = \ln(p_0^2 + q_0^2), \quad \overline{e} = \ln \overline{E}.$$

Since the right-hand side of Eq. (10) is positive for all permissible z and u, the variable z may be used as an argument instead of t, thus reducing by one the problem dimensionality. Since the variable e is not involved in the right-hand sides of Eqs. (10), (11), the nature of its variation does not affect the solution. The values of this variable at the initial and terminal time instants are given, which allows one to replace Eq. (11) by an integral constraint.

It follows from conditions (10), (11) that

$$dt = \frac{dz}{z^2 + u}, \quad de = \frac{2z(u-1)dz}{(z^2+u)(1+z^2)}$$

and the original problem is given by

$$\tau = \int_{z_0}^{\overline{z}} \frac{dz}{z^2+u} \to \min_{u,\overline{z}}$$

under the condition

$$J = \int_{z_0}^{\overline{z}} \frac{z(u-1)dz}{(z^2+u)(1+z^2)} = \frac{1}{2}(\overline{e} - e_0) = \frac{1}{2}\ln\frac{\overline{E}}{E_0} < 0 \qquad (12)$$

and the above control constraints.

Therefore, the problem with two differential equations was rearranged in a much simpler problem with one integral constraint instead of two differential equations. Its optimality conditions [6] are expressed in terms of the Lagrange functional

$$S = \tau + \Lambda J \qquad (13)$$

and its integrand

$$L = \frac{1}{z^2+u} + \Lambda\frac{z(u-1)}{(z^2+u)(1+z^2)}. \qquad (14)$$

We set out to prove that the optimal solution of the problem features

(a) *the switching nature of the optimal control, and the ordinate where* z *jumps and the line* $z_r = \dfrac{1}{\Lambda}$ *passing through the second and fourth quadrants of the plane* p, q *are the switching lines on this plane*;

(b) *no single solution.*

The Lagrange multiplier Λ in (13), (14) is nonnegative. Indeed, if we denote by $\tau^*(J)$ the dependence of the minimal process duration vs. the given value of functional (12), then the smaller J, the greater τ^*, and therefore

$$\Lambda = -\frac{d\tau^*}{dJ} \geqslant 0.$$

The optimal control is defined by a condition in the form of the requirements for minimum of L

$$u^*(z,\Lambda) = \arg\min_u L(z,u,\Lambda).$$

We determine the derivative

$$\frac{\partial L}{\partial u} = \frac{1}{(z^2+u)^2}(\Lambda z - 1).$$

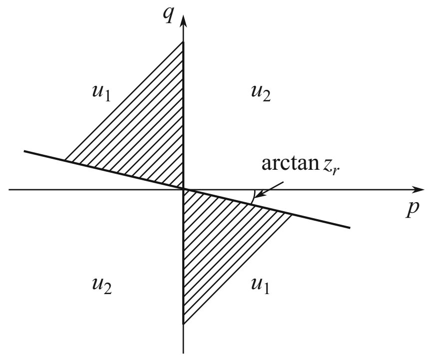

Fig. 1. Control switching lines on the plane p, q.

The sign of the derivative coincides with the sign of the second terms; namely,

$$\frac{\partial L}{\partial u} > 0 \quad \text{for} \quad z > \frac{1}{\Lambda}, \tag{15}$$

$$\frac{\partial L}{\partial u} < 0 \quad \text{for} \quad z < \frac{1}{\Lambda}. \tag{16}$$

If condition (15) is satisfied, then $u^* = u_1$; and if inequality (16) is valid, then $u^* = u_2$. At that, the value of z cannot be equal to $1/\Lambda$ over a time interval of zero measure because $\dot{z} \neq 0$.

The zones of constant optimal control are shown in Fig. 1. The authors of [3] made a groundless assumption that $z_r = 0$ and the switching lines coincide with the coordinate axes. As follows from the assertion that was proved above, this assumption was erroneous.

To determine $\overline{z}$ and Λ or, which is the same, z_r, we have the condition for stationarity of the functional S and condition (12). Let us see what is provided by the first of them:

$$\frac{\partial S}{\partial \overline{z}} = 0 \to \frac{1}{\overline{z}^2 + \overline{u}} = -\Lambda \frac{\overline{z}(\overline{u} - 1)}{(z^2 + \overline{u})(1 + \overline{z}^2)}.$$

Whence it follows that

$$\frac{\overline{z}(\overline{u} - 1)}{1 + \overline{z}^2} = -\frac{1}{\Lambda} = -z_r. \tag{17}$$

The numerator in the left-hand side of this equality must be less than zero, so that for the final time instant

$$\overline{u}^* = \begin{cases} u_2 & \text{for} \quad \overline{z} < 0 \quad \text{(odd quadrants of plane } p, q) \\ u_1 & \text{for} \quad \overline{z} > 0 \quad \text{(even quadrants of plane } p, q). \end{cases} \tag{18}$$

At the intersection of the phase trajectory with the ordinate axis ($q = 0$) the variable z jumps from $z = \infty$ to $z = -\infty$. At that, the optimal control switches from u_1 to u_2 and has an opposite jump on the line with inclination z_r. We assume for definiteness that $z_0 < 0$ and by the stage mean each of the segments of solution consisting of two intervals of constant control. At that, it is always possible to assume that the initial point lies on the ordinate axis of the plane pq at the point $p^0 = \sqrt{E_0} = \sqrt{p_0^2 + u_2 q_0^2}$ because the durations τ^* for the initial state $(p^0, 0)$ and for the initial state p_0, q_0 differ by $\Delta\tau$ which can be readily determined (see (24) below), and the optimal values z_r and $\overline{z}$ remain the same.

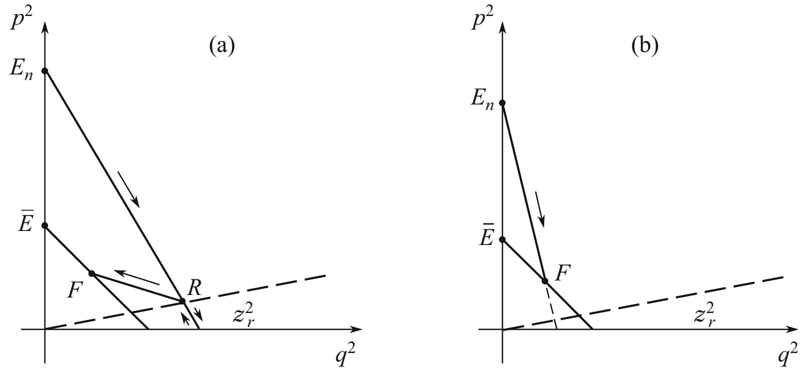

Fig. 2. The nature of the optimal trajectory at the final stage for the cases of (a) one or (b) none switching of control.

The optimal solution consists of n preliminary stages and one final stage. At each of the preliminary stages, z varies from $-\infty$ to ∞ with one switching. At the final stage, the optimal control may be equal to u_2 or u_1 (see (18)) depending on E_0 and $\overline{E}$.

Figure 2 depicts the optimal process in the p^2, q^2 coordinate system. At passing to these coordinates, the phase trajectories corresponding to u_1 and u_2 become straight lines having the tangent of the angle of inclination to the abscissa axis equal to $-u_1$ and $-u_2$, respectively. The clockwise motion corresponds to the motion along the trajectories lying in the first and third quadrants of the plane pq, and the counterclockwise motion, to the motion along the trajectories of the second and fourth quadrants. The bold lines in Fig. 2 show the trajectories corresponding to the final stage, the inclination of the switching line being at that z_r^2.

Let us see the relation between z_r and $\overline{z}$, the duration and variation of energy at the final and preliminary stages.

At the final stage, the conditions

$$\frac{1+z_r^2}{u_2+z_r^2} > \frac{\overline{E}}{E_n} \geqslant \frac{z_r^2+u_1}{z_r^2+u_2}, \tag{19}$$

$$\overline{z}^2 = \frac{\dfrac{E_n}{\overline{E}}(z_r^2+u_1) - u_1(z_r^2+u_2)}{z_r^2+u_2-\dfrac{E_n}{\overline{E}}(z_r^2+u_1)} \tag{20}$$

correspond to the case shown in Fig. 2a.

The conditions

$$1 \geqslant \frac{\overline{E}}{E_n} \geqslant \frac{1+z_r^2}{u_2+z_r^2}, \tag{21}$$

$$\overline{z}^2 = \frac{u_2 - E_n/\overline{E}}{E_n/\overline{E}-1} \tag{22}$$

correspond to the case shown in Fig. 2b.

Let us calculate for each preliminary stage the duration $\tau_i(z_r)$ and the degree of energy reduction E_i/E_{i-1}:

$$\tau_i = \int_{-\infty}^{z_r} \frac{dz}{z^2+u^2} + \int_{z_r}^{\infty} \frac{dz}{z^2+u_1}$$
$$= \frac{1}{\sqrt{u_2}}\left(\arctan\frac{z_r}{\sqrt{u_2}} + \frac{\pi}{2}\right) + \frac{1}{\sqrt{u_1}}\left(\frac{\pi}{2} - \arctan\frac{z_r}{\sqrt{u_1}}\right).$$

According to Fig. 2b, for the energy change during one stage, we immediately establish from the geometrical considerations that

$$\frac{E_i}{E_{i-1}} = \frac{z_r^2+u_1}{z_r^2+u_2} = M(z_r) < 1$$

and

$$\frac{E_n}{E_0} = \left(\frac{z_r^2+u_1}{z_r^2+u_2}\right)^n = M^n(z_r).$$

Therefore, the total duration of the process is equal to

$$\tau = n\left\{\frac{\pi}{2}\left(\frac{1}{\sqrt{u_2}} + \frac{1}{\sqrt{u_1}}\right) + \frac{1}{\sqrt{u_2}}\arctan\frac{z_r}{\sqrt{u_2}} - \frac{1}{\sqrt{u_1}}\arctan\frac{z_r}{\sqrt{u_1}}\right\} + \tau_{f_i}(z_r) \to \min_{n, z_r>0}.$$

Let us consider two variants of the final stage.

(1) The process terminates in the odd (first and third) quadrants of the plane p, q. In this case, inequalities (21) are satisfied, and the duration $\tau_f = \tau_{f1}$ obeys the formula

$$\tau_{f1} = \frac{1}{\sqrt{u_2}}\left(\frac{\pi}{2} + \arctan \overline{z}/\sqrt{u_2}\right). \tag{23}$$

The problem comes to determining the values of the variables $\overline{z} > 0$, $z_r > 0$, n satisfying the conditions

$$z_r = \frac{\overline{z}(1-u_2)}{1+\overline{z}^2},$$
$$\overline{z}^2 = \frac{u_2\overline{E} - E_0 M^n(z_r)}{E_0 M^n(z_r) - \overline{E}},$$
$$M^n(z_r) \geqslant \frac{\overline{E}}{E_0} \geqslant M^n(z_r)\frac{1+z_r^2}{u_2+z_r^2},$$

of which the first condition follows from (17), the second condition, from (22), and the third condition, from (21).

(2) The process terminates in one of the even quadrants of the plane p, q. In this case, the system of relations for $\overline{z} < 0$, $z_r > 0$ and n is given by

$$z_r = \frac{\overline{z}(1-u_1)}{1+\overline{z}^2},$$
$$\overline{z}^2 = \frac{E_0 M^n(z_r)(z_r^2+u_1) - \overline{E}u_1(z_r^2+u_2)}{\overline{E}(z_r^2+u_2) - E_0 M^n(z_r)(z_r^2+u_1)},$$
$$M^n(z_r)\frac{z_r^2+1}{z_r^2+u_2} \geqslant \frac{\overline{E}}{E_0} \geqslant M^{n+1}(z_r).$$

These conditions arise, respectively, from requirements (17), (19), and (20). The duration of the final stage is calculated in this case as follows:

$$\tau_{f2} = \frac{1}{\sqrt{u_2}}\text{arctan}\frac{z_r}{\sqrt{u_2}} + \frac{1}{\sqrt{u_1}}\left(\text{arctan}\frac{\overline{z}}{\sqrt{u_1}} - \text{arctan}\frac{z_r}{\sqrt{u_1}}\right).$$

Of the two variants of the optimal process, that is taken which has smaller total duration.

If the initial state lies off the ordinate axis on a line with the inclination $z_0 < 0$, the optimal process duration as compared with that calculated above is reduced by the value of

$$\Delta\tau = \frac{1}{\sqrt{u_2}}\left(\text{arctan}\frac{z_0}{\sqrt{u_2}} + \pi/2\right). \qquad (24)$$

3. ENERGY EXTRACTION FROM THE SYSTEM OF QUANTUM OSCILLATORS

This problem of the optimization quantum thermodynamics was formulated and solved in [4] with the use of the *Maple* analytical package for a special case of defining the initial conditions. The same reference described the physical sense of the main variables and the conditions imposed on them. Transformation of the state space simplifies the problem and enables one to derive an analytical solution.

Initial formulation: In a minimal time drive the system characterized by the equations

$$\begin{cases} \dot{E} = u(E - L) \\ \dot{L} = -u(E - L) - 2\omega C \\ \dot{C} = uC + 2\omega L \\ \dot{\omega} = u\omega, \quad \omega_1 \leqslant \omega \leqslant \omega_2 \\ E(\tau) = \overline{E} \end{cases} \qquad (25)$$

from the given initial state to the desired terminal state. Here, E, L, C, ω are the state variables having the sense of Hamiltonian, Lagrangian, moment-deviation correlation, and oscillator frequency as averaged over the assembly; at that, $\overline{E} = E(\tau) < E_0$.

The dependence of the variables E, L, C on the oscillation frequency ω, deviations from equilibrium q_i, and oscillator speeds p_i is as follows:

$$\begin{cases} P^2 = \sum_i p_i^2; \quad Q^2 = \sum_i q_i^2 \\ E = P^2 + 0.5\omega^2 Q^2 \\ L = P^2 - 0.5\omega^2 Q^2 \\ C = 0.5\omega(QP + PQ). \end{cases}$$

Since L and C are related with the oscillator oscillation energy and E, with the energy of chaotic motion of molecules, a decrease in E corresponds to extraction of energy at the expense of body cooling.

System (25) has the first integral X which is time-independent along its trajectories:

$$X = \frac{E^2 - (L^2 + C^2)}{\omega^2} = \text{Const} = \frac{E_0^2 - (L_0^2 + C_0^2)}{\omega_0^2}. \qquad (26)$$

One can readily verify by direct differentiation that the rate of variation of this function is zero in virtue of Eqs. (25). In physical terms, X defines the von Neumann entropy S_N which depends on it monotonically:

$$S_N = \ln\left(\sqrt{X - \frac{1}{4}}\right) + \sqrt{X}\,\mathrm{arg\,sinh}\left(\frac{\sqrt{X}}{X - \frac{1}{4}}\right).$$

The constancy of X is indicative that energy extraction from the system at the expense of variations of the oscillation frequency is an adiabatic reversible process. The existence of the minimal time corresponding to the reversible transition from one energy level to another is equivalent to stating that at a time smaller than the minimum one this energy may be extracted only in an irreversible process accompanied by the so-called "quantum friction."

We notice that in the reversible thermodynamics a variation of the gas volume and pressure either in the conditions of complete thermal isolation or immediately, which also results in no heat exchange with the environment, corresponds to the adiabatic process of gas temperature variation.

Let given be the initial values of all variables and, consequently, X. The original problem has an unlimited control u that is involved linearly, and its variable states are interrelated by condition (26). Therefore, the principle of maximum cannot be used directly for solution.

By passing to new variables, we transform the state space

$$z_1 = E + L, \quad z_2 = \frac{E - L}{\omega^2}, \quad z_3 = \frac{C}{\omega} \tag{27}$$

with the aim of eliminating the dependence of the rates of phase variables on the unlimited control u. These variables are selected so as to make the sum of terms depending on u in expressions like (3) to vanish in virtue of Eqs. (25).

In the transformed problem, the initial variables and the value of X are related with the state variables as follows:

$$\begin{cases} C = \omega z_3, \quad E = 0.5(z_1 + \omega^2 z_2) \\ L = 0.5(z_1 - \omega^2 z_2), \quad X = z_1 z_2 - z_3^2. \end{cases} \tag{28}$$

Since the value of X is given, only two variables remain independent.

The initial system state on the plane z_1, z_2 is denoted by S. It is known to lie over the hyperbola $z_1, z_2 = X$. We eliminate the variable z_3 by expressing it in terms of X, z_1, z_2, set down the equations for z_1 and z_2 along the trajectories of system (25) with regard for (27), (28), and obtain

$$\begin{cases} \dot{z}_1 = -2\omega^2 z_3 = \mp 2\omega^2\sqrt{z_1 z_2 - X} \\ \dot{z}_2 = 2z_3 = \pm 2\sqrt{z_1 z_2 - X}. \end{cases} \tag{29}$$

Since u is not involved in these equations, we regard $\omega^2 > 0$ as a control, thus reducing the number of state variables to two and the number of controls to one.

Time is not involved explicitly in the right-hand sides of Eqs. (29), which allows one to simplify the system by using the variable z_2 as an argument. At that, along the system trajectories

$$\frac{dz_1}{dz_2} = -\omega^2. \tag{30}$$

For constant ω, the trajectories on the plane z_1, z_2 are straight lines with negative inclination.

The variations of the frequency ω may be instantaneous. Therefore, the minimal process duration is independent of the value of the initial frequency. We assume that the initial frequency is ω_0, but with an energy reduction it can be immediately changed to ω_1. The same refers to the final value of frequency where a frequency equal to ω_1 corresponds to the energy minimum.

The initial conditions for the variables z_1 and z_2 are given as follows:

$$z_{10} = E_0 + L_0, \quad z_{20} = \frac{E_0 - L_0}{\omega_0^2}.$$

The final values satisfy the equality

$$\overline{z_1} + \omega_1^2 \overline{z_2} = 2\overline{E}.$$

In virtue of the inequality $z_1 z_2 \geqslant X$, $\overline{E}$ is bounded from below. The lower limit is reached at the point of tangency of the hyperbola $z_1 z_2 = X$ and the straight line obeying equality (3). For any fixed value of z_3 and, consequently, the product $z_1 z_2$, the points corresponding to the minimum of energy lie on the straight line $z_1 = \omega_1^2 z_2$. The lower bound of the attainable energy that is reached for $z_3 = 0$ is equal to

$$\overline{E}_{\min} = \omega_1 \sqrt{X}.$$

Energy values smaller than $\overline{E}_{\min}$ *are not reachable from any initial state, that is, there exists a minimal temperature that is attainable in the reversible process.*

The duration of the passage from the initial to the final state is as follows:

$$\tau = \int\limits_{z_{20}}^{\overline{z}_2} \left| \frac{dz_2}{2\sqrt{z_1 z_2 - X}} \right| \to \min. \tag{31}$$

It can be calculated by defining the trajectory $z_1(z_2)$.

We set down the conditions for the principle of maximum for the transformed problem (31), (30), (3) assuming that the differential is nondegenerate ($\psi_0 = -1$):

the Hamilton function

$$H = -\frac{1}{2\sqrt{z_1 z_2 - X}} - \psi(z_2)\omega^2,$$

the optimality conditions

$$\frac{d\psi}{dz_2} = -\frac{dH}{dz_1} = -\frac{z_2}{4(z_1 z_2 - X)^{3/2}} < 0,$$

$$\omega^{2*}(z_2) = \arg\max H = \begin{cases} \omega_2^2 & \text{for } \psi < 0 \\ \omega_1^2 & \text{for } \psi > 0. \end{cases}$$

Since the function $\psi(z_2)$ decreases monotonically and, by the virtue of the condition for the final value z_1, does not vanish at the end of the process, in the course of the optimal process the frequency can change only once from ω_1 to ω_2 for any finite energy smaller than E_0. We denote the switching point by R.

Consequently, we have proved that on the optimal solution the dependence of z_1 vs. z_2 from the switching point R is linear with the minimal inclination, and after this point, with the maximal inclination. The integral (31) can be calculated for each interval as arcsin of some expression.

The initial and final states are denoted in Fig. 3, respectively, by S and F. The sets reachable from the initial state and providing the hit to the final state are bounded by the straight lines

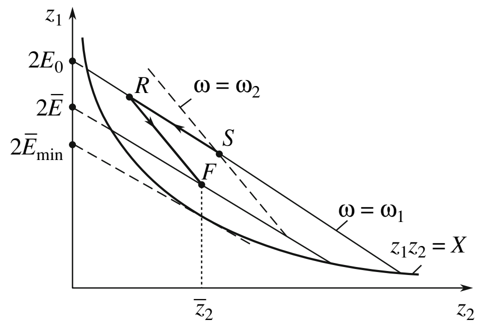

Fig. 3. Domain of attainable values of the state variables and the optimal solution.

outgoing from the initial and final states with inclinations corresponding to the extreme values of the frequency. The trajectory going along the boundaries of the reachability sets either through the switching point R shown in the figure or the switching point lying at the opposite point of the parallelogram corresponds to the optimal process. At the point F, the frequency changes abruptly for ω_1. The tangent to the hyperbola $z_1 z_2 = X$ with inclination corresponding to ω_1 cuts a segment equal to the double minimum of the attainable energy from the ordinate axis.

The coordinates of the switching point R obey the equations

$$z_{1R} = z_{10} - \omega_1^2(z_{2R} - z_{20}),$$
$$z_{1R} = \overline{z_1} + \omega_2^2(\overline{z_2} - z_{2R}).$$

For the coordinate z_{2R}:

$$z_{2R}(\overline{z}_2) = \frac{\omega_2^2\overline{z}_2 - \omega_1^2 z_{20} - z_{10} + \overline{z}_1}{\omega_2^2 - \omega_1^2} = \overline{z}_2 - \frac{2(E_0 - \overline{E})}{\omega_2^2 - \omega_1^2}.$$

The minimal duration τ of body cooling to $E = \overline{E}$ is determined after substitution of the optimal solution in (31):

$$\tau^* = 0.5\left[\int\limits_{z_{20}}^{z_{2R}(\overline{z}_2)} \frac{dz_2}{\sqrt{(2E_0 - \omega_1^2 z_2)z_2 - X}} + \int\limits_{z_{2R}}^{\overline{z}_2} \frac{dz_2}{\sqrt{(2\overline{E} - \omega_2^2 z_2)z_2 - X}}\right].$$

The value of τ^* depends on $\overline{z}_2$ and the given values of energy E at the beginning and end of the process.

The results obtained in the example of [4] where both the initial, S, and the final, F, states lie on the hyperbola $z_1 z_2 = z_{10} z_{20} = X$, follow as a special case of the above relations.

4. CONCLUSIONS

The actual problems of optimal control rarely may be solved with disregard for their specificity. The possibility of such consideration is provided by the change of the state variables enabling sometimes a substantial simplification of the solution. The problems of energy extraction from a single oscillator and an oscillator system by means of the parametric control demonstrate the possibilities of passing to the transformed problem which is much simpler than the original one. The problems of optimal swing slowing-down (swinging) by variation of the length of suspension and energy extraction from the system of quantum oscillators were solved using such change. In the latter case, it was proved that the reversible process of energy extraction can be realized in the quantum thermodynamics by controlling the oscillation frequency if its duration is not smaller than the determined τ^*.

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