Optimal Operation of Finite-time Tricycles with Heat Conduction Losses

Raj K. Pathria
Department of Physics,University of Waterloo
Waterloo, Ontario N2L 3G1, Canada

Peter Salamon
Department of Mathematical Sciences
San Diego State University
San Diego, CA 92182, USA

James D. Nulton
Department of Mathematics
San Diego City College
San Diego, CA 92101, USA

Abstract

We consider the issue of in-principle limits to the finite-time operation of a cycling working fluid acting as an agent in the transfer of heat among three heat reservoirs. The set of feasible operations of such a heat engine is explicitly described and its boundary is chara cterized as the set of operations which allocate time optimally among the heat conduction br anches for a wide variety of cost functions. One point on this boundary represents the oper ation that maximizes the heat output at the high-temperature heat reservoir. There is a natura l notion of efficiency for such an operation and the result in this case generalizes the well-kn own result of Curzon and Ahlborn for the efficiency of a Carnot-like heat engine at maximu m power.

PACS: 05.70.Ln, 05.70.–a, 44.90.+c

I. Introduction

The present paper continues the remarkable story of heat engines operating in finite time. The problem has been treated in many ways by many authors over the past twenty years and yet it appears to be an inexhaustible vein of interesting structure [1, 2, 3, 4, 5]. These efforts can be classified into two categories:

(1) Simple models designed to understand the basic physics of the limitations imposed on the set of operations of a heat engine by the constraint of finite time [3, 4]. The aim in these studies is to find in-principle bounds on the net effects of a process.

(2) More realistic models approaching the operation of real heat engines as engineering systems [5]. The aim of these studies is to identify and model loss mechanisms in currently operating heat engines.

In the present paper we pursue the former goal by studying a special model heat engine in order to gain an understanding of its possible operations. The model that we treat here is an extension of our previous work [1, 2] to heat engines working between three heat reservoirs. For ready reference, engines working between three heat reservoirs have been dubbed "tricycles" [6, 7, 8, 9, 10, 11] and provide interesting generalizations of the two heat-reservoir and one work-reservoir case.

The framework we use has several features which have become standard for exploring in-principle limitations on finite-time operations. One such feature is to assume that our processes all are *endoreversible*. This term, coined by Rubin [12], literally means "internally reversible" and allows us to treat the subsystems participating in a process as being at all t imes in quasi-equilibrium states. For the processes considered here, such subsystems are the working fluid and the three heat reservoirs. For the purpose of finding bounds, the assumption of endoreversibility can be rigorously and generally justified through a theorem of Orlov [13], which is proved by considering the optimal control of thermodynamic processes. Such control is pursued subject to the constraints imposed by the dynamical equations describing the time evolution of our subsystems. A general result in optimization theory guarantees that if constraints are eliminated, the resulting optimum can only improve. This is sufficient to imply that calculations using endoreversible processes provide rigorous bounds which must be obeyed by any process.

A second feature of our framework is to assume that only the transport of heat across a boundary is irreversible. This is a strong assumption since it allows us to let all other processes occur reversibly and at arbitrary rates. The justification lies again in the fact that the results bound what can happen in real processes. The advantage here is that this feature allows us to focus our attention on the heat exchange branches and take any adiabatic branches to be instantaneous and reversible [14].

Our discussion is thus reduced to an idealization of real systems that is useful as a physical limit, i.e., in the same spirit as a reversible process. Rather than having to deal with distributed systems with infinite degrees of freedom, we deal at each instant with quasi-equilibrated subsystems with a finite number of degrees of freedom. The analysis can, however, be even further simplified. For instance, it was shown in [15] that, under quite general conditions, the optimization of heat exchange with a constant-temperature heat reservoir is obtained when the working fluid temperature is kept constant [16]. This is, in fact, a special case of a general theorem obtained by Rosonoer and Tsirlin [4]. Their general result says that for a process in which both the objective function and the constraints can be expressed as time i ntegrals of the thermodynamic functions of state, the optimal control is always piecewise constant — taking on at most m+1 values, where m is the number of constraints.

Our results for a tricycle reduce to those for more conventional heat engines in the limit as the temperature of one of the reservoirs tends to infinity. In this limit, one heat reservoir becomes a work reservoir which is able to supply entropiless energy at any temperature [6]. Our results thus generalize the well known results of Novikov-Curzon-Ahlborn [5, 17] to tricycle processes. Explicit expressions are derived for many interesting quantities which bound the values of these quantities in real heat engines.

A surprising feature of our analysis is the fact that we can derive a host of results without needing to specify the objective function. The analysis can be carried out assuming only that the objective function can be written in terms of the net effects and that the total time is constrained. The finite-time constraint specifies the boundary of the set of possible operations and any reasonable objective function pushes us to this boundary.

The theory of heat engines working between three heat reservoirs has useful realizations. A common example is a propane powered refrigerator often found in recreational vehicles. In this instance, heat flows from the highest temperature (the flame) to the atmosphere to power as much heat removal from the lowest temperature as possible. The reverse process also has commercial applicability. For example, domestic heat can be provided by the conduction of heat at an intermediate temperature (a subterranean reservoir) to the lowest temperature (the atmosphere), rejecting some of it to the highest temperature. In this case the objective is to produce as much heat as possible at the highest temperature. In view of the fact that conventional heat engines producing work are a special case of this example as the hottest temperature tends to infinity, we will refer to this instance as a *heat engine*.

The layout of the present paper is as follows. In section II we carry out the optimization of the time allocation among the three heat-exchange branches. Surprisingly, we find that we should always use the same total time for heat absorption as for heat discharge. In section III we consider the case of different objective functions and represent the set of optimal operations in a planar diagram similar to the one employed in references [1] and [2]. In section IV we relate these operations back to the temperatures of the working fluid during the cycle. In section V, we complete our analysis by considering operations which are relatively less interesting in the sense of economically desirable control but give a broader perspective on the complete set of optimal operations. The paper ends with section VI where we make some concluding remarks.

II. Optimal time allocation

We consider a system undergoing a tricycle of operations conducted over a total time τ during which it exchanges heat with three reservoirs at fixed temperatures T_1^o, T_2^o and T_3^o; see Fig. 1. To fix ideas, we choose T_j^o's such that $T_3^o > T_1^o > T_2^o$, so that in the limit $T_3^o \to \infty$ the third reservoir assumes the role of a "work reservoir" [with which energy is exchanged *without* entropy production, see Eq. (2)] and our process reverts to a conventional cycle involving only two reservoirs at temperatures T_1^o and T_2^o. Denoting the times spent in contact with the individual reservoirs as τ_1, τ_2, and τ_3, respectively, the amounts of heat transferred from the reservoirs to the system, assuming Newton's law of heat transfer, would be

$$Q_j = \kappa_j \tau_j (T_j^o - T_j), \qquad j = 1,\ 2,\ 3, \tag{1}$$

where T_j's are the temperatures of the system held fixed during the three periods of contact, while κ_j's are the thermal conductances associated with the relevant surfaces of contact; for s implicity, we take all κ_j's to be the same and denote them by a common symbol κ. The resu lting entropy changes of the reservoirs are then given by

$$\sigma_j^o = -\frac{Q_j}{T_j^o} = -\frac{\kappa\tau_j(T_j^o - T_j)}{T_j^o}, \qquad (2)$$

while the corresponding changes for the system, assuming endoreversibility, would be

$$\sigma_j = \frac{Q_j}{T_j} = \frac{\kappa\tau_j(T_j^o - T_j)}{T_j}. \qquad (3)$$

From (2) and (3) it follows that

$$\frac{1}{\sigma_j} + \frac{1}{\sigma_j^o} = -\frac{1}{\kappa\tau_j}, \qquad (4)$$

so that

$$\sigma_j = -\frac{\sigma_j^o\kappa\tau_j}{\sigma_j^o + \kappa\tau_j}. \qquad (5)$$

We note that in the limit $\tau_j \to \infty$, $\sigma_j \to -\sigma_j^o$ and the process involved becomes reversible; o therwise, it is necessarily irreversible. We also note that the cycle under study must confor m to the obvious constraints

$$\sum_j Q_j = 0, \qquad \sum_j \sigma_j = 0, \qquad \sum_j \tau_j = \tau. \qquad (6a,b,c)$$

If the performance of our cycle is characteristic of a heat engine, then it would be nat ural for us to optimize the quantity Q_3; if, on the other hand, it is characteristic of a refrigerat or (or a heat pump), then we would like to optimize Q_2 (or Q_1). To accommodate all these c ases, we may optimize an arbitrary function $f(Q_j)$ — subject, of course, to the constraints (6)

. Expressing all our quantities in terms of the external variables σ_j^o and τ_j, the Lagrangian o f the problem takes the form

$$L(\sigma_j^o,\ \tau_j) = f(\sigma_j^o) - \lambda\sum_j T_j^o \sigma_j^o - \mu\sum_j \frac{\sigma_j^o \kappa\tau_j}{\sigma_j^o + \kappa\tau_j} - \nu\left(\sum_j \tau_j - \tau\right), \tag{7}$$

where λ, μ and ν are the Lagrange undetermined multipliers. Optimization with respect to τ_j 's leads to the conditions

$$0 = -\,\mu\kappa \frac{(\sigma_j^o)^2}{(\sigma_j^o + \kappa\tau_j)^2} - \nu\,, \qquad j = 1,\ 2,\ 3\ . \tag{8}$$

This shows that the constants μ and ν must be of opposite signs. Introducing a set of dime nsionless parameters u_j, defined by the relations

$$u_j = \frac{\kappa\tau_j}{\sigma_j^o}\,, \tag{9}$$

conditions (8) may be written as

$$1 + u_j = \varepsilon_j\sqrt{-\mu\kappa/\nu}\,, \qquad j = 1,\ 2,\ 3, \tag{10}$$

where ε_j = +1 or –1, depending on the actual direction of heat transfer between the system a nd the *j*th reservoir. To see this explicitly, we first observe that the quantity

$$\frac{1 + u_j}{u_j} = \frac{\sigma_j^o + \kappa\tau_j}{\kappa\tau_j} = -\frac{\sigma_j^o}{\sigma_j} = \frac{T_j}{T_j^o} > 0\ . \tag{11}$$

Next, we infer, from Eqs. (9) and (11), that both u_j and $(1 + u_j)$ have the same sign as σ_j^o. Equation (10) then tells us that $\varepsilon_j = +1$ or -1 according as σ_j^o is positive or negative (which indeed is related to the direction of heat transfer). This leads one naturally to consider eight different process types, as discussed in the Appendix, based on the signs of the ε_j's. The types that evoke most interest are:

(i)

A process of type VI, which pertains to a <u>heat engine</u> with $Q_1 > 0$, $Q_{2,3} < 0$ and hence $\sigma_1^o < 0$, $\sigma_{2,3}^o > 0$; it follows that for this case $\varepsilon_1 = -1$ while $\varepsilon_2 = \varepsilon_3 = +1$. Equation (10) then leads to the important results

$$u_2 = u_3 = -2 - u_1 \, , \tag{12}$$

so that, while $u_{2,3} > 0$, $u_1 < -2$.

(ii)

A process of type III, which pertains to a <u>refrigerator</u> (or a <u>heat pump</u>), with $Q_1 < 0$, $Q_{2,3} > 0$ and hence $\sigma_1^o > 0$, $\sigma_{2,3}^o < 0$; it follows that now $\varepsilon_1 = +1$ whereas $\varepsilon_2 = \varepsilon_3 = -1$. Once again, Eq. (12) holds (though now $u_1 > 0$ while $u_2 = u_3 < -2$).

Two other possibilities, that evoke lesser interest, are considered later in Sec. V.

Returning to the question of optimal time allocation, as determined by conditions (10) in conjunction with constraints (6), we observe that, in view of Eqs. (5) and (9), the constraint (6b) can be written as

$$\sum_j \frac{\kappa\tau_j}{1+u_j} = 0 \,. \tag{13}$$

Combining (13) with (10), we obtain the remarkable result

$$\sum_j \varepsilon_j \tau_j = 0 \tag{14}$$

which shows that, regardless of the nature of the function $f(\sigma_j^o)$ and regardless of the cycle chosen, the optimal time allocation is such that the system spends as much time absorbing h eat from the reservoir(s) as it spends rejecting it to the reservoir(s). Thus, for heat engines a s well as refrigerators (and heat pumps)

$$\tau_1 = (\tau_2 + \tau_3) = \frac{1}{2}\,\tau \,. \tag{15}$$

We now recall constraint (6a), which states that

$$\sum_j T_j^o \sigma_j^o = \kappa \sum_j \left(\frac{T_j^o \tau_j}{u_j} \right) = 0 \,. \tag{16}$$

Equations (15) and (16), along with the fact that $u_2 = u_3$, yield the optimal τ_j's:

$$\tau_1 = \frac{1}{2}\tau, \qquad \tau_2 = \frac{1}{2}\tau \frac{u_1 T_3^o + u_2 T_1^o}{u_1(T_3^o - T_2^o)}, \qquad \tau_3 = \frac{1}{2}\tau \frac{-u_1 T_2^o - u_2 T_1^o}{u_1(T_3^o - T_2^o)} \,. \tag{17a,b,c}$$

The quantities Q_j $[\, = -T_j^o(\kappa\tau_j / u_j)\,]$ are then given by

$$Q_1 = -\frac{1}{2}\,\kappa\tau \frac{T_1^o}{u_1}\,, \tag{18a}$$

$$Q_2 = -\frac{1}{2}\,\kappa\tau\,\frac{T_2^o}{T_3^o - T_2^o}\left(\frac{T_3^o}{u_2} + \frac{T_1^o}{u_1}\right), \tag{18b}$$

$$Q_3 = \frac{1}{2}\,\kappa\tau\,\frac{T_3^o}{T_3^o - T_2^o}\left(\frac{T_1^o}{u_1} + \frac{T_2^o}{u_2}\right), \tag{18c}$$

while the net rate of entropy production is given by

$$D = \frac{1}{\tau}\sum_j \frac{\kappa\tau_j}{u_j} = \frac{1}{2}\,\kappa\left(\frac{1}{u_1} + \frac{1}{u_2}\right) \tag{19a}$$

$$= -\frac{\kappa}{u_1 u_2} > 0 \tag{19b}$$

because u_1 and u_2 are of opposite signs.

The foregoing expressions are optimal insofar as time allocation on the three branch es of the cycle is concerned. They are still functions of the parameters u_1 and u_2 which allo w room for further optimization — depending on what purpose our cycle is supposed to ac hieve. To appreciate the end results, it seems advisable to first put the results of this section in a geometrical framework that would enable us to see the various aspects of this problem i n a somewhat broader perspective.

III. Representation in the (*P*, *D*)-plane and further optimizations

In the spirit of paper II, we examine the results of the previous section in the (*P*, *D*)-plane where *P* is the analog of the "power generated by a heat engine" while *D* is the net rate of entropy production in the cycle (also called the "degradation"). By Eqs. (18c) and (19a), we have

$$P = -\frac{Q_3}{\tau} = -\frac{1}{2}\,\kappa\,\frac{T_3^o}{T_3^o - T_2^o}\left(\frac{T_1^o}{u_1} + \frac{T_2^o}{u_2}\right) \tag{20}$$

and

$$D = \frac{1}{2}\,\kappa\left(\frac{1}{u_1} + \frac{1}{u_2}\right). \tag{21}$$

Solving (20) and (21), we get

$$\frac{1}{u_1} = -\frac{2(\widetilde{P} + T_2^o D)}{\kappa(T_1^o - T_2^o)}, \qquad \frac{1}{u_2} = \frac{2(\widetilde{P} + T_1^o D)}{\kappa(T_1^o - T_2^o)}, \tag{22a,b}$$

where $\widetilde{P}$ denotes the "effective power" of the cycle, viz.

$$\widetilde{P} = P\,\frac{(T_3^o - T_2^o)}{T_3^o}; \tag{23}$$

note that, as $T_3^o \to \infty, \widetilde{P}$ becomes synonymous with P. Now, substituting (22) into (19b), o r else using the fact that $u_1 + u_2 = -2$, we obtain the desired relationship

$$(\widetilde{P} + T_1^o D)(\widetilde{P} + T_2^o D) - \frac{1}{4}\,\kappa\,(T_1^o - T_2^o)^2 D = 0 \tag{24}$$

which represents a hyperbola in the $(\widetilde{P}, D)$-plane, with asymptotes

$$\widetilde{P} + T_1^o D = -\frac{1}{4}\,\kappa\,(T_1^o - T_2^o) \tag{25a}$$

$$\widetilde{P} + T_2^o D = \frac{1}{4}\,\kappa\,(T_1^o - T_2^o). \tag{25b}$$

Eq. (24) is plotted in Fig. 2, where it appears as the hyperbolic arc DCOAB. Points on this hyperbola represent all possible processes for which the time allocation is optimized; in fact, these processes can be parameterized by the single quantity u_2/u_1 [see Eqs. (17)].

The reversible cycle, which requires all τ_j's to be infinite, is represented by the origin O of this plane; by Eqs. (12), (20) and (21), this corresponds to both u_1 and u_2 being infinite in magnitude while the ratio u_2/u_1 approaches –1. The analog of the Curzon-Ahlborn cycle is depicted by point A where $\overset{\frown}{P}$ is maximum; by Eqs. (12) and (20), this corresponds to the ratio u_2/u_1 being equal to $-\sqrt{T_2^o/T_1^o}$, with

$$u_1 = -\frac{2\sqrt{T_1^o}}{\sqrt{T_1^o}-\sqrt{T_2^o}}\,, \qquad u_2 = \frac{2\sqrt{T_2^o}}{\sqrt{T_1^o}-\sqrt{T_2^o}}\,. \tag{26}$$

The corresponding values of P and D turn out to be

$$P_A = \frac{1}{4}\,\kappa\,\frac{T_3^o}{T_3^o - T_2^o}\,(\sqrt{T_1^o}-\sqrt{T_2^o})^2\,, \qquad D_A = \frac{1}{4}\,\kappa\,\frac{(\sqrt{T_1^o}-\sqrt{T_2^o})^2}{\sqrt{T_1^oT_2^o}}\,, \tag{27a,b}$$

while

$$(Q_1)_A = \frac{1}{4}\,\kappa\tau\sqrt{T_1^o}\,(\sqrt{T_1^o}-\sqrt{T_2^o})\,. \tag{28}$$

These results are essentially the same as the ones obtained previously, except that the "power attained" now seems enhanced by the factor $T_3^o/(T_3^o - T_2^o)$. The reason for this apparent enhancement of power lies in the fact that what this engine delivers to reservoir 3 is not "true mechanical work" — it is only "high-grade heat" which, if converted into true mechanical work (by utilizing the best means available and the coldest reservoir available) would produce no more than the amount $Q_3(1 - T_2^o/T_3^o)$, thus bringing the true efficiency of this cycle down to the value attained in the Curzon-Ahlborn cycle.

The general expression for the power efficiency η can be written down with the help of Eqs. (18a) and (18c), with the result

$$\eta = -\frac{Q_3}{Q_1} = \frac{T_3^o}{T_3^o - T_2^o}\left(1 + \frac{u_1 T_2^o}{u_2 T_1^o}\right). \qquad (29)$$

Now, apart from the two special cases noted above, we may also mention the "complete wast e" process, depicted by point B in Fig. 2, which corresponds to $\eta = 0$ and hence to the ratio u_2/u_1 being equal to $-T_2^o/T_1^o$; this means that now

$$u_1 = -\frac{2T_1^o}{T_1^o - T_2^o}, \qquad u_2 = \frac{2T_2^o}{T_1^o - T_2^o}, \qquad (30)$$

while

$$P_B = 0, \qquad D_B = \kappa \frac{(T_1^o - T_2^o)^2}{4T_1^o T_2^o}. \qquad (31a,b)$$

With $Q_3 = 0$, this process implies a direct transfer of heat from reservoir 1 to reservoir 2; in fact, the same is true of all processes lying on the straight line OB.

We now turn our attention to the case where our tricycle acts as a refrigerator ($Q_1 < 0$, $Q_{2,3} > 0$); Eqs. (12) and (18) now tell us that (i) $u_1 > 0$, (ii) $u_2 < -2$, while (iii) the ratio u_2/u_1 lies between -1 and $-T_3^o/T_1^o$. The limit $u_2/u_1 \to -1$ corresponds to the reversible case ($u_1 \to \infty$, $u_2 \to -\infty$), with the coefficient of performance

$$\omega = \frac{Q_2}{Q_3} = \frac{T_2^o(u_1 T_3^o + u_2 T_1^o)}{T_3^o |u_1 T_2^o + u_2 T_1^o|} \qquad (32)$$

$$\longrightarrow \frac{T_3^o - T_1^o}{T_3^o}\,\omega_R, \qquad (33)$$

where ω_R is the coefficient of performance of the corresponding Carnot refrigerator:

$$\omega_R = \frac{T_2^o}{T_1^o - T_2^o}\ . \tag{34}$$

Once again, the reduction factor, $(T_3^o - T_1^o)/T_3^o$, appearing in (33) can be understood in terms of the fact that what we are utilizing here is not "true mechanical work" but only "high-grade heat" provided by a source at temperature T_3^o; for transfer to reservoir 1, heat Q_3 is no more effective than an amount of work $W = Q_3(1 - T_1^o/T_3^o)$. The limit $u_2/u_1 \to -T_3^o/T_1^o$ corresponds to $Q_2 \to 0$ and hence to a direct transfer of heat from reservoir 3 to reservoir 1; this limit is depicted by point D in Fig. 2; it can be shown that $Q_2 = 0$ for all processes lying on the straight line OD.

Between the two extremes, O and D, lies a special case, depicted by point C, where Q_2 is optimal; this happens when the ratio $u_2/u_1 = -\sqrt{T_3^o/T_1^o}$, so that

$$u_1 = \frac{2\sqrt{T_1^o}}{\sqrt{T_3^o} - \sqrt{T_1^o}}\ , \qquad u_2 = -\frac{2\sqrt{T_3^o}}{\sqrt{T_3^o} - \sqrt{T_1^o}}\ , \tag{35}$$

while

$$\omega_C = \frac{T_2^o(\sqrt{T_3^o} - \sqrt{T_1^o})}{\sqrt{T_3^o}(\sqrt{T_1^o T_3^o} - T_2^o)}\ . \tag{36}$$

In the limit $T_3^o \to \infty$, the points C and D both run off to infinity and we recover the situation studied in II.

IV. An alternative representation of the tricycle

For a broader understanding of the situation under study, we may relate the external quantities $\widetilde{P}$ and D with the temperatures T_j of the system itself as it goes through the tricyc le process described above. First of all, in view of relations (11) and (12), we observe that

$$\frac{T_2}{T_2^o} = \frac{T_3}{T_3^o} \; ; \qquad (37)$$

it is, therefore, sufficient to consider only one of these quantities, say T_2/T_2^o. Secondly, we i ntroduce parameters β and β^o defined through the standard relations

$$\beta = \frac{T_2}{T_1} , \qquad \beta^o = \frac{T_2^o}{T_1^o} , \qquad (38)$$

and note that the ratio

$$\frac{\beta}{\beta^o} = \frac{T_2/T_2^o}{T_1/T_1^o} = \frac{(1+u_2)/u_2}{(1+u_1)/u_1} = -\frac{u_1}{u_2} , \qquad (39)$$

so that

$$u_1 = -\frac{2\beta}{\beta - \beta^o} , \qquad u_2 = \frac{2\beta^o}{\beta - \beta^o} . \qquad (40)$$

Equations (20), (21) and (23) then become

$$\widetilde{P} = \frac{1}{4}\, \kappa\, T_1^o \, \frac{(\beta - \beta^o)(1-\beta)}{\beta} \qquad (41)$$

and

$$D = \frac{1}{4}\,\kappa\,\frac{(\beta - \beta^{o})^2}{\beta\beta^{o}} \tag{42}$$

which, apart from the replacement of P by $\widetilde{P}$, are the same as expressions (28b) and (29b) o f [2], with the parameter $\alpha = 1$ (that corresponds to optimal time allocation). The power effi ciency η, as given by Eq. (29), now takes the form

$$\eta = \frac{T_3^o}{T_3^o - T_2^o}\,(1-\beta)\,, \qquad \beta^{o} < \beta < 1\,, \tag{43}$$

which agrees with expression (26) of [1], except for the extra factor arising from the temper ature, T_3^o, of the hottest reservoir. It is quite straightforward to see that the points O, A and B of Fig. 2 correspond to

$$\beta = \beta^{o}\,, \ \ \sqrt{\beta^{o}} \ \ \text{and} \ \ 1\,, \tag{44}$$

respectively.

It will be noted that expressions (41) and (42) apply to a refrigerator cycle as well, b ut with $\beta < \beta^{o}$. The coefficient of performance of this cycle, as given by Eq. (32), now take s the form

$$\omega = \frac{\beta - \gamma^{o}}{1-\beta}\,, \qquad \gamma^{o} = \frac{T_2^o}{T_3^o} \tag{45}$$

which may be compared with Eq. (43) of [2]. The points O, C and D now correspond to

$$\beta = \beta^{o}\,, \ \ \sqrt{\beta^{o}\gamma^{o}} \ \ \text{and} \ \ \gamma^{o}\,, \tag{46}$$

respectively.

V. Other feasible processes

For completeness we develop the main features of the tricycle processes of type II, see Appendix, in which the <u>net</u> transfer of heat takes place from the hotter reservoirs 3 and 1 to the coldest reservoir 2 and of type IV in which the <u>net</u> transfer takes place from the hottest reservoir 3 to the colder reservoirs 1 and 2. A special feature of these processes is that they allow optimization of the heat, Q_1, exchanged by the reservoir at the intermediate temperature T_1^o.

(i)

For a process of type II, $Q_{1,3} > 0$, $Q_2 < 0$ and hence $\sigma_{1,3}^o < 0$, $\sigma_2^o > 0$; it follows that for this case $\varepsilon_1 = \varepsilon_3 = -1$ while $\varepsilon_2 = +1$. Accordingly, $u_1 = u_3 < -2$ while $u_2 > 0$ such that $u_2 + u_3 = -2$. We now get

$$\tau_1 = \frac{1}{2}\tau\frac{u_2 T_3^o + u_3 T_2^o}{u_2(T_3^o - T_1^o)}, \qquad \tau_2 = \frac{1}{2}\tau, \qquad \tau_3 = \frac{1}{2}\tau\frac{-u_2 T_1^o - u_3 T_2^o}{u_2(T_3^o - T_1^o)} \tag{47}$$

whence

$$Q_1 = -\frac{1}{2}\,\kappa\tau\,\frac{T_1^o}{T_3^o - T_1^o}\left(\frac{T_2^o}{u_2} + \frac{T_3^o}{u_3}\right), \tag{48a}$$

$$Q_2 = -\frac{1}{2}\,\kappa\tau\frac{T_2^o}{u_2}, \tag{48b}$$

$$Q_3 = \frac{1}{2}\,\kappa\tau\,\frac{T_3^o}{T_3^o - T_1^o}\left(\frac{T_2^o}{u_2} + \frac{T_1^o}{u_3}\right). \tag{48c}$$

The quantities $\widetilde{P}$ and D are now given by

$$\widetilde{P} = -\frac{1}{2}\kappa\frac{T_3^o - T_2^o}{T_3^o - T_1^o}\left(\frac{T_2^o}{u_2} + \frac{T_1^o}{u_3}\right) \tag{49}$$

and

$$D = \frac{1}{2}\,\kappa\left(\frac{1}{u_2} + \frac{1}{u_3}\right) = -\frac{\kappa}{u_2u_3} > 0\,, \tag{50}$$

respectively. Eliminating u_2 and u_3 from the equations, we obtain the relation

$$\left(\widetilde{P}\frac{T_3^o - T_1^o}{T_3^o - T_2^o} + T_1^oD\right)\left(\widetilde{P}\frac{T_3^o - T_1^o}{T_3^o - T_2^o} + T_2^oD\right) - \frac{1}{4}\,\kappa\,(T_1^o - T_2^o)^2D = 0\,, \tag{51}$$

which may be compared with Eq. (24) that holds for processes of type III and VI.

Equation (51) is plotted in Fig. 2 where it appears as the hyperbolic arc BF (that is found to be continuous with the previous curve DCOAB, though with a discontinuous slope at the point B). The extremities B and F of this arc are characterized by the respective values $-T_1^o/T_2^o$ and $-T_3^o/T_2^o$ of the parameter u_3/u_2 so that while $Q_3 = 0$ at B, $Q_1 = 0$ at F; in fact, it can be shown that $Q_1 = 0$ for all processes on the straight line OF. Between the extremes B and F, there may exist a point E where Q_1 is optimal (actually, a maximum). By Eqs. (48), this happens where $u_3/u_2 = -\sqrt{T_3^o/T_2^o}$ and is possible only if $T_1^o < \sqrt{T_2^oT_3^o}$. If $T_1^o \to \sqrt{T_2^oT_3^o}$, the point E approaches B.

(ii)

For a process of type IV, $Q_{1,2} < 0$, $Q_3 > 0$ and hence $\sigma_{1,2}^o > 0$, $\sigma_3^o < 0$; it follows that for this case $\varepsilon_1 = \varepsilon_2 = +1$ while $\varepsilon_3 = -1$. Accordingly, $u_1 = u_2 > 0$ while $u_3 < -2$ such that $u_1 + u_3 = -2$. We now get

$$\tau_1 = \frac{1}{2}\tau\frac{-u_1T_3^o - u_3T_2^o}{u_3(T_1^o - T_2^o)}\,, \qquad \tau_2 = \frac{1}{2}\tau\frac{u_1T_3^o + u_3T_1^o}{u_3(T_1^o - T_2^o)}, \qquad \tau_3 = \frac{1}{2}\tau \tag{52}$$

whence

$$Q_1 = \frac{1}{2}\,\kappa\tau\,\frac{T_1^o}{T_1^o - T_2^o}\left(\frac{T_2^o}{u_1} + \frac{T_3^o}{u_3}\right), \tag{53a}$$

$$Q_2 = -\frac{1}{2}\,\kappa\tau\,\frac{T_2^o}{T_1^o - T_2^o}\left(\frac{T_1^o}{u_1} + \frac{T_3^o}{u_3}\right), \tag{53b}$$

$$Q_3 = -\frac{1}{2}\,\kappa\tau\,\frac{T_3^o}{u_3}\,. \tag{53c}$$

The quantities $\widetilde{P}$ and D are now given by

$$\widetilde{P} = \frac{1}{2}\kappa\,\frac{T_3^o - T_2^o}{u_3} \tag{54}$$

and

$$D = \frac{1}{2}\,\kappa\left(\frac{1}{u_1} + \frac{1}{u_3}\right) = -\frac{\kappa}{u_1 u_3} > 0\,, \tag{55}$$

respectively. Eliminating u_1 and u_3 from these equations, we obtain the relation

$$\widetilde{P}^2 - (T_3^o - T_2^o)\widetilde{P}D - \frac{1}{4}\,\kappa\,(T_3^o - T_2^o)^2 D = 0\,, \tag{56}$$

which differs significantly from the corresponding equations, (24) and (51), obtained for ot her processes.

Equation (56) is also plotted in Fig. 2 where it appears as the hyperbolic arc DF (that is fo und to be continuous with the previous curves OCD and BF, though with discontinuous slopes at the points D and F). The extremities D and F of this arc are characterized by the respective values $-T_3^o/T_1^o$ and $-T_3^o/T_2^o$ of the parameter u_3/u_1 so that while $Q_2 = 0$ at D, $Q_1 = 0$ at F. Between the e xtremes D and F, there may exist a point E' where Q_1 is optimal (actually, a minimum). By Eqs. (

53), this happens where $u_3/u_1 = -\sqrt{T_3^o/T_2^o}$ and is possible only if $T_1^o > \sqrt{T_2^o T_3^o}$. If $T_1^o \to \sqrt{T_2^o T_3^o}$, the point E' approaches D.

VI. Conclusions

The present paper generalizes the analysis of references [1] and [2] to heat engines working between three heat reservoirs (tricycles). The results obtained here reduce to the previous ones in the limit as the hottest reservoir's temperature tends to infinity. For three finite temperatures, our results provide generalizations of the Novikov-Curzon-Ahlborn result to the case of tricycles.

Our geometric analysis of the time-optimal tricycle operations shows that the hyperbola in the (P,D) plane obtained in references [1] and [2] becomes a convex region bounded by three different hyperbolas. As in papers [1] and [2], the points on the boundary of this region represent exactly those processes which allocate time optimally among the different branches of the process.

One surprising feature of our analysis is that the optimal time allocation among the branches in a tricycle can be obtained without specifying the cost function and by assuming that such cost depends only on the net effects of the process. Previous results based in Riemannian geometric structures on the set of equilibrium states [18, 19] have also hinted that the optimal time allocation during a process can be obtained in considerable generality. These previous analyses have obtained optimal time allocation only for the objective of minimizing total entropy production. In this sense, the present work extends those results, though the optimal time allocation for tricycles does not seem to come from a Riemannian metric.

One feature of our processes (as well as of those in references [1] and [2]) is that they possess a minimum time. Roughly stated, if a certain amount of heat exchange must occur through a set of given conductances, then it must take at least a certain amount of time. Reference [2] dwelt on this point and introduced the idea of time efficiency for such processes. In that sense, all processes discussed in the present paper have a time efficiency of one, and this is exactly what places them on the boundary of the feasible region in our *P-D* diagram. As discussed in [2], having a time efficiency of one corresponds to taking place in minimum time. The well known duality between an objective function and a constraint for an optimization problem indeed allows a general reformulation of finite-time thermodynamics problems in terms of minimum time. This reformulation is surprisingly powerful and we have used it to explore a much larger class of problems in this area. The results of that study will be reported in a subsequent paper.

Acknowledgments

R.K.P. would like to acknowledge financial support from the Natural Sciences and Engineering Research Council of Canada. R.K.P. and J.D.N. acknowledge the hospitality of the Department of Mathematical Sciences, San Diego State University.

Appendix

In this appendix we summarize all sets of operations accessible to a tricycle. Considering the directions of arrows in Fig. 1, we clearly have eight possibilities for which signs of the Q_j's, the amounts of heat transferred from the various reservoirs to the system, are listed below; note that these signs are the same as those of the σ_j's and hence are opposite to the ones pertaining to the σ_j^o's.

	I	II	III	IV	V	VI	VII	VIII
Reservoir 3	+	+	+	+	–	–	–	–
Reservoir 1	+	+	–	–	+	+	–	–
Reservoir 2	+	–	+	–	+	–	+	–

It is straightforward to see that cases I and VIII violate the principle of energy conservation while cases V and VII violate the principle of increase of entropy (remember that, by assumption, $T_3^o > T_1^o > T_2^o$). Cases II and IV (being the opposite of the cases VII and V, respectively) are downright dissipative and hence are of lesser interest in the present study. The cases that evoke most interest are III and VI which, on the whole, exchange heat between the reservoir at the intermediate temperature, T_1^o, on one hand and the reservoirs at the highest and the lowest temperatures, T_3^o and T_2^o, on the other; of these, the former pertains to a refrigerator (or a heat pump), the latter to a heat engine.

References

1.

Nulton J D, Salamon P and Pathria R K 1993 Carnot-like processes in finite time. Theoretical limits *Am. J. Phys.* **61** 911-6

2.

Pathria R K, Nulton J D and Salamon P 1993 Carnot-like processes in finite time. Applications to model cycles *Am. J. Phys.* **61** 916-24

3.

Sieniutycz S and Salamon P (ed) 1991 *Finite-Time Thermodynamics and Thermo economics, Advances in Thermodynamics* vol 4 (New York: Taylor and Francis)

4.

Rozonoer L I and Tsirlin A M 1983 Optimal control of thermodynamic processes. I, II, and III *Avtomat. Telemekh.* **1** 70-9; **2** 88-101; **3** 50-64

5.

Bejan A 1996 Entropy generation minimization: the new thermodynamics o f finite- size devices and finite-time processes *J. Appl. Phys.* **79** 1191-1218

6.

Andresen B, Salamon P and Berry R S 1977 Thermodynamics in finite tim e: extremals for imperfect heat engines *J. Chem. Phys.* **66** 1571-7

7.

Rubin M H and Andresen B 1982 Optimal staging of endoreversible heat e ngines *J. Appl. Phys.* **53** 1-7

8.

Yan Z and Chen J 1988 An Optimal Endoreversible Three-Heat-Source Re frigerator *J. Appl. Phys.* **65** 1-4

9.

Chen J 1994 Performance of Absorption Refrigeration Cycle at Maximum Cooling Rate *Cryogenics* **34** 997-999

10.

Chen J 1994 Optimal Heat-Transfer Areas for Endoreversible Heat Pumps *Energy* **19** 1031-1036

11.

Chen J C and Wu C 1996 The R-ε Characteristics of a Three-Heat-Source Refrigeration Cycle *Applied Thermal Engineering* **16** 901-905

12.

Rubin M H 1979 Optimal configuration of a class of irreversible heat engi nes. I *Phys. Rev. A* **19** 1272-6,

Rubin M H 1979 Optimal configuration of a class of irreversible heat engines. II *Phys. Rev. A* **19** 1277-81

13.

Orlov V N and Berry R S 1990 Power output from an irreversible heat engi ne with a non-uniform working fluid *Phys. Rev. A* **42** 7230-5

14.

It is quite easy to take such analysis a step further and take into account the f
act that any needed adiabats may require a fixed amount of time, see for instance,
Agrawal D C, Gordon J M and Huleihil M 1994 Endoreversible engines with finite
-time adiabats *Indian J. Eng. Mater. Sci.* **1** 195-8

15.

Salamon P and Nitzan A 1981 Finite-time optimizations of a Newton's law Carnot
cycle *J. Chem. Phys.* **74** 3546-60

16.

The constant temperatures of the working fluid and the reservoir are, of cour
se, not equal to each other if the process is to take place in finite time.

17.

Curzon F L and Ahlborn B 1975 The efficiency of a Carnot engine at maximum p
ower output *Am. J. Phys.* **43** 22-24

18.

Salamon P and Berry R S 1983 Thermodynamic length and dissipated avai
lability *Phys. Rev. Lett.* **51** 1127-30,
Nulton J, Salamon P, Andresen B and Qi A 1985 Quasistatic processes as step equili
brations *J. Chem. Phys.* **83** 334-338

19.

Spirkl W and Ries H 1995 Optimal finite-time endoreversible processes *P
hys. Rev. E* **52** 3485-9

Figure captions

Figure 1 The operation of a tricycle.

Figure 2

All possible processes for a tricycle depicted in the $(\widetilde{P}, D)$-plane. Th e arc OAB pertains to processes of type VI (heat engines), arc OCD to type III (refrigerators and heat pumps), arc BF to type II and arc DF to type IV; f or the classification of these processes, see Appendix.

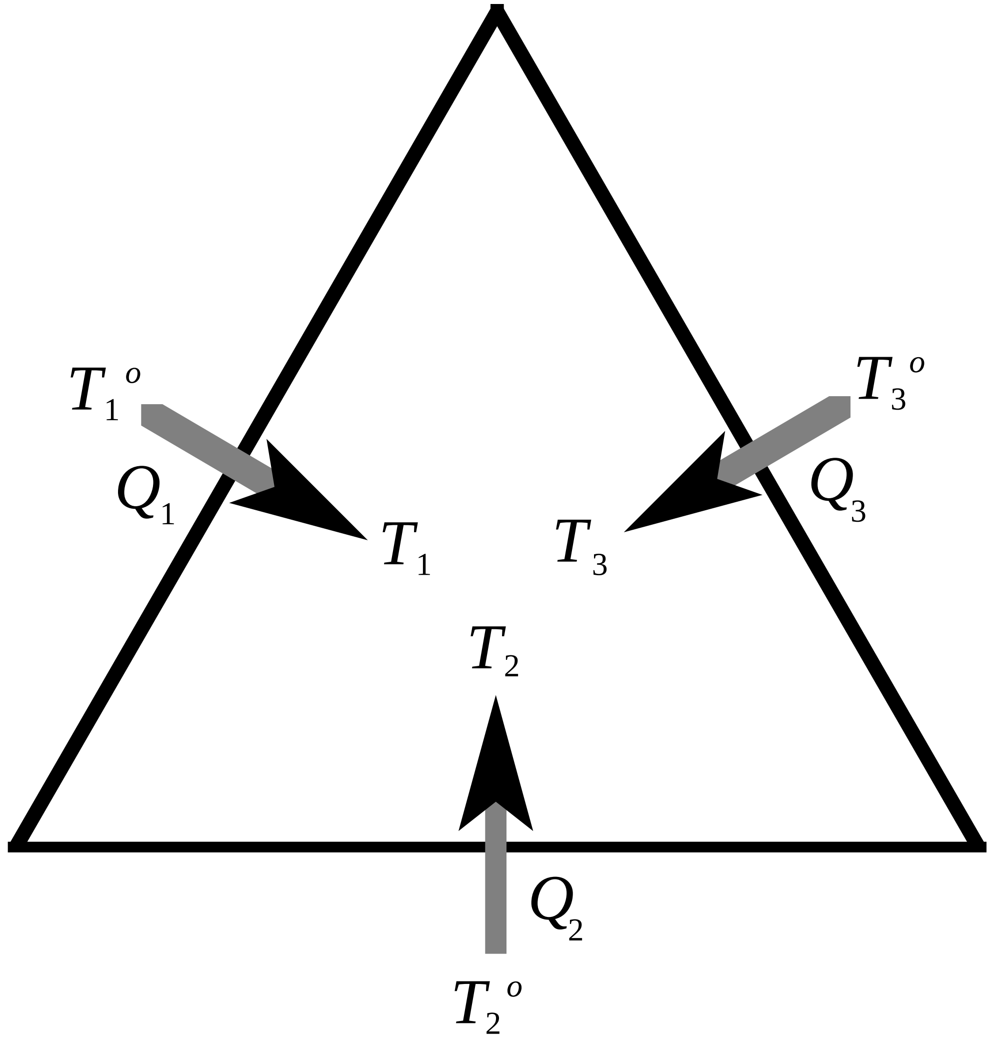

Figure 1

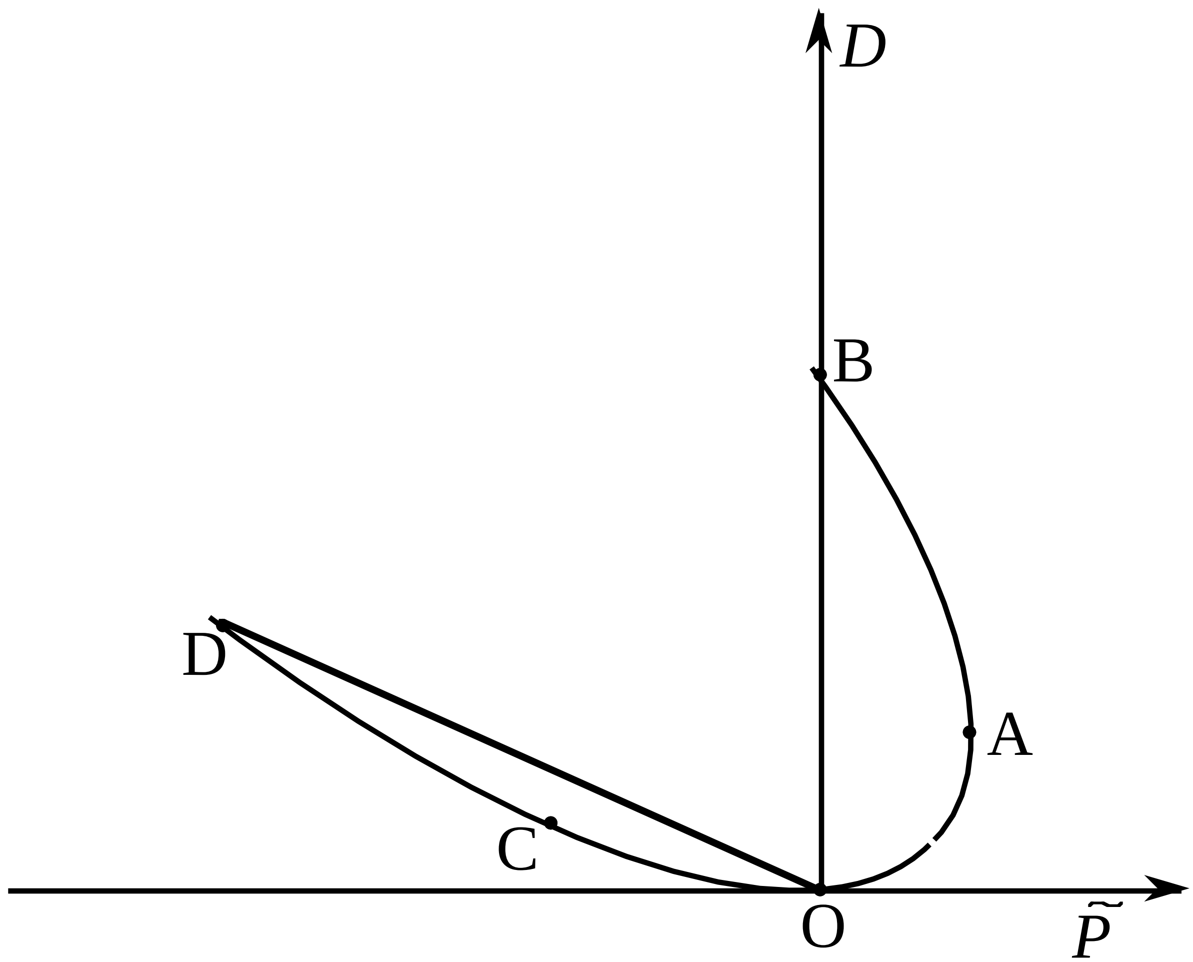

Figure 2

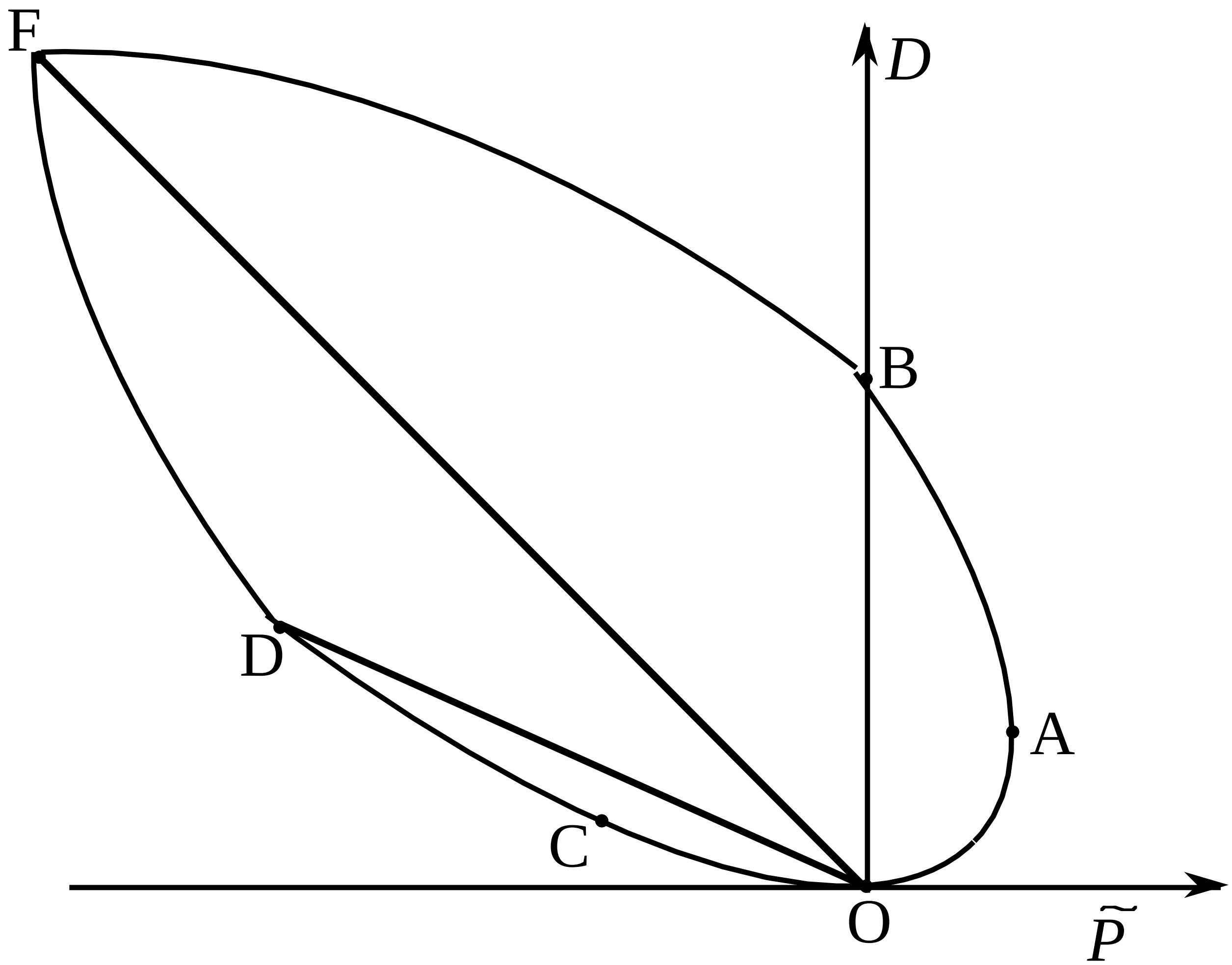

Figure 3