Optimal control of the parametric oscillator

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Abstract

We present a solution to the minimum time control problem for a classical harmonic oscillator to reach a target energy E_T from a given initial state (q_i, p_i) by controlling its frequency ω, $\omega_{min} \leqslant \omega \leqslant \omega_{max}$. A brief synopsis of optimal control theory is included and the solution for the harmonic oscillator problem is used to illustrate the theory.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The harmonic oscillator is *the* solvable paradigm system in physics. When optimal control theory came on the scene in the early 1950s, pumping and stopping a harmonic oscillator by applying a controlled external force was one of the first solved examples. As a result, the example has made it into most textbooks on optimal control. Surprisingly, however, the control of an oscillator by varying its frequency is a problem that does not appear to be part of physics or the optimal control literature until very recently. Salamon *et al* [1] treated the problem of controlling a quantum harmonic oscillator in connection with cooling to ultra-low temperatures, e.g. particles in an optical lattice [2]. The classical counterpart of this problem—extracting energy from a *classical* harmonic oscillator by controlling its frequency ω in a given range $\omega_{min} \leqslant \omega \leqslant \omega_{max}$—has not been considered, although a recent paper [3] has treated the problem of such control when one allows the frequencies to become imaginary[5]. Subsequently Chen *et al* [4] also treated the problem, again allowing imaginary frequencies. Such faster than adiabatic cooling of a trapped gas in a gravitational field was verified experimentally by Schaff *et al* [20].

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[5] Strictly speaking they treat the control of a particle in a potential well with the potential $V = \lambda q^2$ by controlling λ, $-\infty \leqslant \lambda \leqslant \infty$.

Controlling the energy in an oscillator by varying its frequency is empirically familiar to anyone who has played on swings as a child. Though closely related, the swing problem is not the same as the harmonic oscillator problem even if one approximates it as a mathematical pendulum. For a swing with rope length ℓ, angular momentum L, mass μ and angle to the vertical ϕ, the Hamiltonian in the small angle approximation is

$$\mathscr{H} = \frac{1}{2\theta}L^2 + \frac{\theta}{2}\frac{g}{\ell}\phi^2, \tag{1}$$

where $\theta = \mu\ell^2$ is the moment of inertia and ℓ is the control. For the harmonic oscillator with displacement q, mass μ and momentum p, the Hamiltonian is

$$\mathscr{H} = \frac{1}{2\mu}p^2 + \frac{\mu\omega^2}{2}q^2, \tag{2}$$

with ω as the control. Although the optimally pumped swing has been treated in the literature [5, 6], the correct solution for minimum time to reach a certain prescribed energy has not been presented[6]. Furthermore, the straightforward identification that maps the swing in the small angle limit to a harmonic oscillator with $\omega = \sqrt{g/\ell}$ gives the wrong answer by a sign; it implies standing when one should squat and squatting when one should stand. This difference is due to the different dependences of the Hamiltonians on the control variables (ℓ for the swing, ω for the oscillator).

Two other recent problems related to the optimal control of harmonic oscillators deserve mention. The problem of minimum time control of a vibrating string has been extensively treated by Il'in *et al* [7] who give explicit solutions for controlling the tension or length of the string and thereby go from any initial displacement $u(x, t_i)$, and velocity $\dot{u}(x, t_i)$ to any desired $u(x, t_f)$ and $\dot{u}(x, t_f)$. The control of the quantum oscillator also plays a role in making squeezed states of light [8].

The goal of this paper is twofold. We treat the problem of optimally controlling a classical harmonic oscillator to reach a certain target energy in minimum time starting from a given initial state. In doing so, we provide a concise introduction to optimal control theory. While not very familiar to the physics community at large, this subject has proved its usefulness for a very wide range of applications including pulse shaping in modern NMR [9, 10], qubit manipulation [24], chemical control [11, 12], robotics [25], disease extinction [26], automobile test drives [27], traffic flow [28] and economics [29, 30], just to name a few. The similarity of optimal control theory to mechanics makes it accessible to physics students, and we are confident that many more untapped uses exist to be discovered. The harmonic oscillator example treated here shows the formalism on an example for which the solutions are easy to understand and which builds physical intuition. As a bonus, the field is rife with related (and open!) problems that are amenable to student formulation and solution.

2. Optimal control theory

Optimal control theory [13] grew out of the mid-twentieth century race to reach outer space. It is in fact the Hamiltonian version of the calculus of variations, particularly suited to problems involving inequality constraints. There is a close analogy between the Lagrangian and Hamiltonian formulations of mechanics and the Lagrangian (calculus of variations) and Hamiltonian (optimal control theory) formulations of optimization. Generally speaking, the

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[6] Notably, as discussed in section 4, the optimal trajectories with a given target energy should not switch at the extreme points of the swing but somewhat after. Private communication with one of the authors of the papers on this optimal control problem (Picolli) resulted in him admitting this inaccuracy in their papers.

Lagrangian formulation allows us to treat a wider class of problems (frictional forces; more complicated differential equation constraints), but when a Hamiltonian formulation is possible, the solution is more easily found.

Optimal control theory considers a dynamical system whose state is described by a vector of state variables $x(t) = (x_1(t), \ldots, x_N(t))$ with the system dynamics given by the system of first-order coupled differential equations:

$$\dot{x} = f(x, u). \tag{3}$$

Here $u(t) = (u_1(t), \ldots, u_R(t))$ is the vector of control variables. These are the variables in the problem that do not have dynamic constraints on them and allow us to choose among feasible solutions of the constraint equations (3). The path $x(t)$ followed by the system in state space is fully determined by an initial state $x_i = x(t_i)$ and the controls $u(t)$. This path can be externally influenced by the choice of the values of the control variables out of the set U of allowed values for $u(t)$.

The problem of optimal control is then to find the controls $u(t)$ for which a given functional V becomes extremal, i.e.

$$V = \int_{t_i}^{t_f} f_0(x, u) \, dt \rightarrow \min_{u(t)}. \tag{4}$$

Note that the final time t_f can be open to optimization as well. In addition, we can specify a target set defined by requiring certain additional equations to hold at the final state $x_f = x(t_f)$

$$h_j(x_f) = 0 \qquad \text{for} \quad j = 1, \ldots, J. \tag{5}$$

For convenience, we introduce an additional state variable x_0 corresponding to the objective V. Its initial condition is $x_0(t_i) = 0$ and its dynamics is taken to be $\dot{x}_0 = f_0$ so the optimality condition (4) becomes $x_0(t_f) \rightarrow \min_{u(t)}$. We next define the optimal control Hamiltonian H (not to be confused with the physical Hamiltonian $\mathcal{H}$) by introducing the time-dependent adjoint (or co-state) variables $\tilde{x}$, one for each state variable:

$$H(x, \tilde{x}, u) = \sum_{n=0}^{N} \tilde{x}_n f_n(x, u), \tag{6}$$

where f_0 is the integrand of the objective function V and $f_n, n \geqslant 1$, are the components of the vector f in the dynamic equations (3). Note that our introduction of the extra state variable x_0 makes the equation $\dot{x}_n = f_n$ hold for $n = 0, 1, \ldots, N$. The adjoint variables are closely related to Lagrange multipliers and we will see their geometric and physical significance in section 7 below.

The necessary conditions for the existence of an extremal path are the canonical equations of motion

$$\dot{x}_n = \frac{\partial}{\partial \tilde{x}_n} H(x, \tilde{x}, u) \qquad n = 0, \ldots, N \tag{7}$$

$$\dot{\tilde{x}}_n = -\frac{\partial}{\partial x_n} H(x, \tilde{x}, u) \qquad n = 0, \ldots, N, \tag{8}$$

and optimality conditions on the values of the controls $u^*(t)$ which are chosen to maximize the Hamiltonian for fixed $x(t)$ and $\tilde{x}(t)$ (Pontryagin's maximum principle):

$$H(x, \tilde{x}, u^*) \geqslant H(x, \tilde{x}, u) \qquad \forall u \in U, \tag{9}$$

where $\tilde{x}_0 \leqslant 0$ is assumed. Note that the resulting controls $u^*(t)$ can be discontinuous.

The optimal controls $u^*(t)$ are then inserted back into the canonical equations of motion. This results in a set of closed coupled differential equations with boundary conditions. These are given by the initial conditions

$$x_n(t_\mathrm{i}) = x_{n,\mathrm{i}} \qquad n = 1, \ldots, N, \tag{10}$$

the conditions on the final state

$$h_j(x_\mathrm{f}) = 0 \qquad j = 1, \ldots, J, \tag{11}$$

and the transversality conditions

$$\tilde{x}_n(t_\mathrm{f}) = \left[\sum_{j=1}^{J} \tilde{h}_j \frac{\partial h_j}{\partial x_n} \right]_{t=t_\mathrm{f}} \qquad n = 1, \ldots, N, \tag{12}$$

where the $\tilde{h}_j$ are the Lagrange multipliers associated with the final state constraints (5). The transversality conditions assure optimality of the choice of point in the target set where our trajectory terminates. This gives $2N + J$ equations for the $2N + J$ variables x_i, $\tilde{x}_\mathrm{f}$, and $\tilde{h}_j$ specifying the boundary conditions for the differential equations (7) and (8).

Note that, exactly as for classical mechanics, the dynamical equations (7), (8) for the state and costate variables imply that

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \sum_{n=0}^{N} \frac{\partial}{\partial x_n} H(x, \tilde{x}, u) \cdot \dot{x}_n + \sum_{n=0}^{N} \frac{\partial}{\partial \tilde{x}_n} H(x, \tilde{x}, u) \cdot \dot{\tilde{x}}_n = 0. \tag{13}$$

Thus the optimal control Hamiltonian is constant. In problems for which the final time t_f is not specified, the constant value of H must be zero, i.e.

$$H = \sum_{n=0}^{N} \tilde{x}_n f_n = 0. \tag{14}$$

We will see a simple geometrical interpretation of this in section 7.

The problem of steering a system to a target set in minimum time is easily realized as a special case of the foregoing formalism by setting $f_0 = 1$ in equation (4). Depending on the problem, the optimal path may consist of several arcs which need to be connected in the so-called switching problem. Across the switchings, the controls can have discontinuities but the state and co-state variables must be continuous.

We close this section by deriving some necessary implications of the foregoing conditions for the adjoint variables $\tilde{x}$. The dynamical equations of the adjoint vector are homogeneous linear equations in $\tilde{x}$ as are all the conditions on the adjoint vector so this vector is only determined up to a constant multiple. In particular it also means that the adjoint vector does not vanish at any instant during the process (a fact needed below) or else it would be zero at all times. The scale for $\tilde{x}$ is usually set by a convention regarding the value of $\tilde{x}_0$. We see from our adjoint equations that

$$\dot{\tilde{x}}_0 = -\frac{\partial H}{\partial x_0} = 0, \tag{15}$$

and so the adjoint variable $\tilde{x}_0$ must be constant since H has no explicit dependence on x_0. Provided $\tilde{x}_0 \neq 0$, we can set its value to -1 and thereby determine the scale for our adjoint vector $\tilde{x}$. Some trajectories can have $\tilde{x}_0 = 0$, and for these trajectories we are free to set the scale of $\tilde{x}$ by some other choice.

3. The minimum time problem

In this paper, we analyse the time-optimal parametric harmonic oscillator. For convenience, we take the mass $\mu = 1$. The dynamics of such an oscillator is described by (cf the physical Hamiltonian (2))

$$\dot{q} = p \tag{16}$$

$$\dot{p} = -\omega^2 q, \tag{17}$$

where q is the position, p is the momentum, and $\omega(t)$ is the frequency of the oscillator, which can be externally changed in the course of time. Here we limit ω to vary freely within the interval $\omega_{min} \leqslant \omega \leqslant \omega_{max}$, with no limitations on $\dot{\omega}$. As the control function $u(t)$ we choose $u = \omega^2$ for this problem. This makes our set of feasible controls $U = \{u; u_{min} \leqslant u \leqslant u_{max}\}$, where $u_{min} = \omega_{min}^2$ and $u_{max} = \omega_{max}^2$.

The goal we want to achieve by the optimal control of $u(t)$ is to reach a given target energy E_T of the oscillator at a given frequency ω_0 in the minimum possible time, starting from a given initial state (q_i, p_i). This can be expressed with the final state condition

$$h_1(q_f, p_f) = p_f^2 + \omega_0^2 q_f^2 - 2E_T = 0. \tag{18}$$

For ease of discussion, we treat the case where we want to reduce the energy in the oscillator, i.e. $E_T < E_i$. The reverse case of pumping energy into the oscillator in minimum time proceeds in exactly the same way.

Before presenting our solution we pause to note that the oscillator sitting still at its equilibrium point, $(q, p) = (0, 0)$, cannot be steered to any other state. This follows from the dynamical equations (16), (17). Nor can $(0, 0)$ be reached in finite time from any other state. Otherwise the reverse of such a trajectory would give a control to reach other states from (0,0). The origin is unique in this respect; any other state of the oscillator can be steered to any other state in finite time.

For this minimum time problem, the integrand of the objective is $f_0 = 1$ and we define $\tau = x_0$. The optimal control Hamiltonian is

$$H = \tilde{\tau} + \tilde{q} p + \tilde{p}(-\omega^2 q) \tag{19}$$

$$= \tilde{\tau} + \tilde{q} p + \sigma u, \tag{20}$$

where we have introduced the switching function $\sigma = -\tilde{p}q$. It follows from the maximality principle (9) that, depending on the sign of σ, the optimal control will be either $u^* = u_{min}$ (for $\sigma < 0$) or $u^* = u_{max}$ (for $\sigma > 0$). If $\sigma = 0$ for an instant, the optimal control is determined as above for all other times and the value of u at the instant has no effect on the state variables, the co-state variables, or the objective and so we can choose its value freely. If the switching function σ vanishes over an interval of time (t_1, t_2), things are more complicated. For this case, the optimal controls are determined from the condition $\sigma(t) = 0$ for $t \in (t_1, t_2)$ in which case all its derivatives $\dot{\sigma}, \ddot{\sigma}, \ldots$ must likewise vanish in (t_1, t_2).

This cannot happen for our oscillator. To see this, suppose $\sigma = -\tilde{p}q$ vanishes over a time interval. It then follows that either $\tilde{p}$ or q vanishes over some subinterval. Suppose it is $\tilde{p}$. In that case $\dot{\tilde{p}} = -\tilde{q}$ would also vanish over this interval. Plugging these values into (14) then gives $\tilde{\tau} = 0$ which would mean that the entire adjoint vector $(\tilde{\tau}, \tilde{q}, \tilde{p}) = 0$, a contradiction. Similarly, if q vanishes over an interval, then $\dot{q} = p$ also vanishes, i.e. the oscillator must be sitting still at its equilibrium position. Since as noted above this state cannot be reached from any other state, this too gives a contradiction. It follows that σ cannot vanish over a finite time interval and all optimal trajectories are piecewise of the form $u^*(t) = u_{max}$ or

$u^*(t) = u_{\min}$. Such solutions are called bang–bang solutions. Bang–bang optimal solutions are very common in physics, in particular when minimum duration or maximum power is sought. Instances of vanishing switching function, interior optima, certainly also exist, often in connection with maximizing an efficiency.

The dynamics for the adjoint variables is given by

$$\dot{\tilde{q}} = u\tilde{p} \tag{21}$$

$$\dot{\tilde{p}} = -\tilde{q}, \tag{22}$$

while the dynamics for the system variables is

$$\dot{q} = p \tag{23}$$

$$\dot{p} = -uq. \tag{24}$$

Along any portion of the optimal trajectory with $u^*(t) = $ constant, these equations are readily integrated in closed form. It remains to determine the switching between the constant values $\omega_{\min}$ and $\omega_{\max}$.

3.1. Some physical reasoning

We know from the switching function σ that we switch between extremes every time $q = 0$. This is exactly what makes physical sense: to reset the frequency of the oscillator when all its energy is kinetic. Changing the frequency at this instant does not change the total energy in the oscillator. Similar physical reasoning leads one to think that the other jumps should occur at $p = 0$ when the kinetic energy is zero since this would achieve the largest change in energy. Whether we are trying to increase or decrease the energy of the oscillator, we are trying to get to the target energy as fast as possible so it makes physical sense that we want the largest change in energy. While in a certain sense this is correct and was the assumed form for the swing problem [5] and the squeezed states problem [8], it is *not* in general the right strategy for a given target energy.

The pair of switches at $q = 0$ and $p = 0$ reduces the energy in the oscillator by a factor of $R = \omega_{\min}^2/\omega_{\max}^2$. If the ratio of the starting energy to the target energy, E_i/E_T, is an integer power of R, switching at $p = 0$ *is* optimal. For these special 'resonant' trajectories, the physical reasoning above holds. It turns out, however, that for any other value of R, we can save some time by staying on the faster branch ($\omega = \omega_{\max}$) a little longer while sacrificing some reduction in the ratio by which we decrease the energy. This subtle feature of the optimal solution was missed by the previous authors [5, 6, 8] but is a very real feature of the physics of the problem. This set of switches occurs at $\tilde{p} = 0$ and does not coincide with $p = 0$ for any but the resonant trajectories.

4. Solution strategies

We begin with some observations. The adjoint variables play a role only in determining the optimal control u^*. Once we know u^*, the dynamical equations for the state variables (24) can be integrated without any further recourse to the adjoint variables. For our problem therefore, we need the values of the adjoint variables only to determine the correct switching points $\tilde{p} = 0$ from arbitrary initial states. To get these, we have to integrate the adjoint equations (21) and (22). These equations are almost identical in form to the dynamical equations for (q, p) and so again integrable in closed form; the difficulty is that we do not have initial conditions to use. For the adjoint variables, we can find final values at any point on the target set but we do
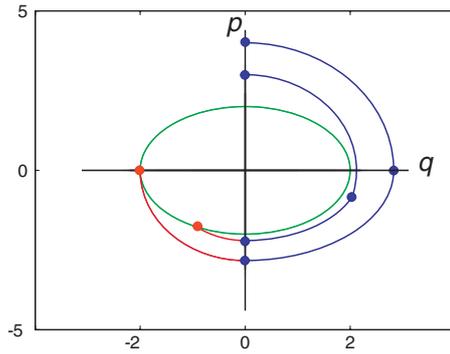
Figure 1. A sample non-resonant optimal trajectory (inner) and a sample resonant optimal trajectory (outer) in phase space. Both trajectories start on the p-axis, i.e. with $q_i = 0$, and end on the ellipse of target energy E_T (green). Switching points are marked with bullets. Both trajectories switch to ω_{max} at $q = 0$, the resonant back to ω_{min} correspondingly at $p = 0$, while the non-resonant trajectory makes that switch a little later at $p < 0$.

not know which point on the target set to start integrating backwards from in order to hit our desired initial state. By any numerical approach, this involves some form of shooting method.

The most direct and intuitively clear solution strategy is one which plays on the strengths of today's students who are much better acquainted with computer software than equation juggling. It proceeds as follows: (1) assume some values of $(\tilde{q}_i, \tilde{p}_i)$ and integrate forward, switching whenever $\tilde{p}q = 0$, (2) evaluate the resulting time τ to reach the target energy E_T as a function of the starting values $(\tilde{q}_i, \tilde{p}_i)$ and (3) directly minimize the resulting time $\tau(\tilde{q}_i, \tilde{p}_i)$. The approach works and gives solutions such as the ones in figure 1. Aided by numerical exploration analytic approaches can also be pushed through (see below).

5. The characteristic slope *m*

Once this solution strategy has been implemented, it is interesting to look at the numerical solutions one gets for a long trajectory, i.e. one with several switches. One such trajectory is shown in figure 2 for the case where the initial energy is higher than the target energy. An immediate conjecture arises: all the $\tilde{p} = 0$ switch points of a trajectory lie on a line $p = mq$. This turns out to be the case as we show below. The $\tilde{p} = 0$ switch points are denoted in the following by $(\cdot)_m^k$, where $(\cdot)$ stands for any of the state or co-state variables $q, p, \tilde{q}, \tilde{p}$, and k indexes these switch points in order along a trajectory. Similarly, the $q = 0$ switch points are denoted by $(\cdot)_0^k$.
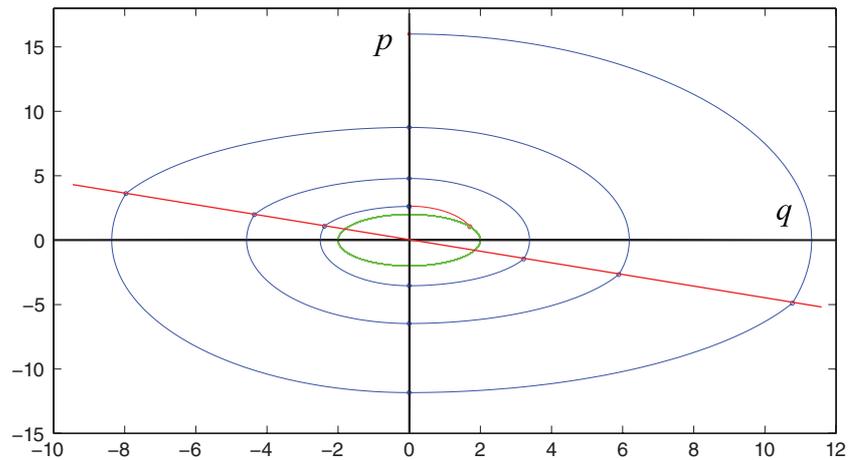
Consider a branch of the trajectory with constant ω starting at (q_i, p_i) and $(\tilde{q}_i, \tilde{p}_i)$ at time $t = 0$. The solution of the dynamical equations (21)–(24) is then

$$q(t) = q_i \cos(\omega t) + \frac{p_i}{\omega} \sin(\omega t) \tag{25}$$

$$p(t) = -\omega q_i \sin(\omega t) + p_i \cos(\omega t) \tag{26}$$

$$\tilde{q}(t) = \tilde{q}_i \cos(\omega t) + \omega \tilde{p}_i \sin(\omega t) \tag{27}$$

$$\tilde{p}(t) = -\frac{\tilde{q}_i}{\omega} \sin(\omega t) + \tilde{p}_i \cos(\omega t). \tag{28}$$

Figure 2. One long trajectory showing the collinearity of all the $\tilde{p} = 0$ switch points, shown with a red line. The $q = 0$ switch points are on the p-axis.

For concreteness as well as convenience, we will in our examples consider a bang–bang trajectory that begins at the point $(0, p_i)$ in the (q, p)-plane. However, for initial points with $q_i \neq 0$, the general conclusions are still the same. The trajectory will after a time t cross the $q = 0$ line. As each part of an optimal trajectory is optimal itself, it remains to determine the appropriate $p(t)$ at which it will do so. This is easily done by the fact that the energy is not changed along that initial branch. A complication which arises is that the initial point might have a $\tilde{p} = 0$ switch point before reaching $q = 0$. In that case the initial branch is extended backwards in time to the previous $q = 0$ switch point. The decision which case applies can only be made by testing both possibilities and then discarding the inconsistent one.

Our aim is to extract energy from the oscillator. To do so, the frequency must be lowered when $q \neq 0$, which in turn means that at the $q = 0$ switch points the frequency is raised to $\omega_{\max}$. We follow this branch for a time t_A to the next switch point with $\tilde{p} = 0$. We will refer to such branches as type A. At the $\tilde{p} = 0$ switch point the frequency is lowered to $\omega_{\min}$. We follow this branch for a time t_B to the next switch point with $q = 0$. We will refer to such branches as type B.

The equations describing a type-A branch are

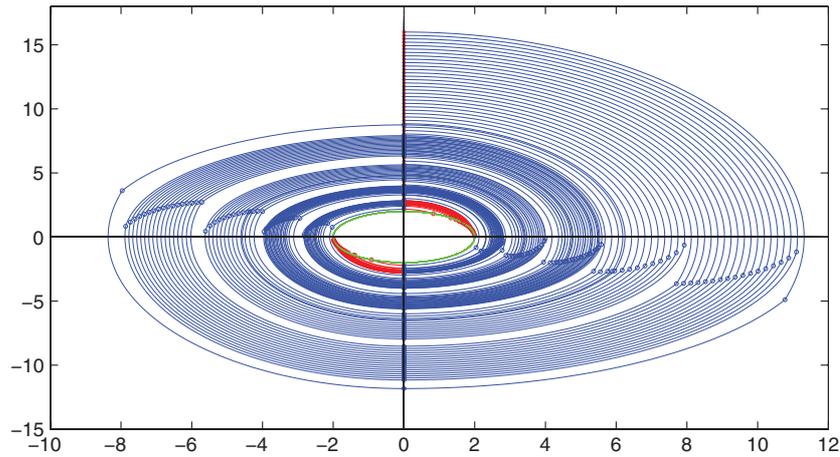$$q_{\mathrm{m}}^k = \frac{p_0^k}{\omega_{\max}} \sin(\omega_{\max} t_A) \tag{29}$$

$$p_{\mathrm{m}}^k = p_0^k \cos(\omega_{\max} t_A) \tag{30}$$

$$\tilde{q}_{\mathrm{m}}^k = \tilde{q}_0^k \cos(\omega_{\max} t_A) + \omega_{\max} \tilde{p}_0^k \sin(\omega_{\max} t_A) \tag{31}$$

$$\tilde{p}_{\mathrm{m}}^k = -\frac{\tilde{q}_0^k}{\omega_{\max}} \sin(\omega_{\max} t_A) + \tilde{p}_0^k \cos(\omega_{\max} t_A) = 0, \tag{32}$$

where we made use of $q_0^k = 0$ and $\tilde{p}_{\mathrm{m}}^k = 0$ for all k. The equations can be rearranged to give

$$\frac{\tilde{q}_0^k}{\tilde{p}_0^k} = \frac{p_{\mathrm{m}}^k}{q_{\mathrm{m}}^k} = \omega_{\max} \cot(\omega_{\max} t_A). \tag{33}$$

Figure 3. Optimal trajectories starting from $q = 0$ at a grid of equi-spaced initial momenta and all ending on the same target energy ellipse (green). Gaps develop between the trajectories when a resonant position is crossed, i.e. when one more A–B branch pair is required.

From here we move along the next branch, a type-B branch. The equations describing this branch are

$$q_0^{k+1} = q_m^k \cos(\omega_{min} t_B) + \frac{p_m^k}{\omega_{min}} \sin(\omega_{min} t_B) = 0 \tag{34}$$

$$p_0^{k+1} = -\omega_{min} q_m^k \sin(\omega_{min} t_B) + p_m^k \cos(\omega_{min} t_B) \tag{35}$$

$$\tilde{q}_0^{k+1} = \tilde{q}_m^k \cos(\omega_{min} t_B) \tag{36}$$

$$\tilde{p}_0^{k+1} = -\frac{\tilde{q}_m^k}{\omega_{min}} \sin(\omega_{min} t_B). \tag{37}$$

These equations can be rearranged to give

$$\frac{\tilde{q}_0^{k+1}}{\tilde{p}_0^{k+1}} = \frac{p_m^k}{q_m^k} = -\omega_{min} \cot(\omega_{min} t_B). \tag{38}$$

Combining (33) and (38) leads to

$$\frac{p_m^{k+1}}{q_m^{k+1}} = \frac{\tilde{q}_0^{k+1}}{\tilde{p}_0^{k+1}} = \frac{p_m^k}{q_m^k} = \frac{\tilde{q}_0^k}{\tilde{p}_0^k} = m, \tag{39}$$

which proves that the quantities p_m^k/q_m^k and $\tilde{p}_0^k/\tilde{q}_0^k$ have the same value m for any two successive $\tilde{p} = 0$ switch points and therefore have the same value for all such switch points. Thus m has the interpretation of the slope of a line through the origin passing through all such switch points (red in figure 2).

5.1. Dependence on initial and target energies

Once this numerical scheme is running, more experimentation is easily performed. For example, an interesting picture emerges if we look at a sequence of trajectories with the same target energy and with initial states spaced evenly along the p-axis as shown in figure 3.
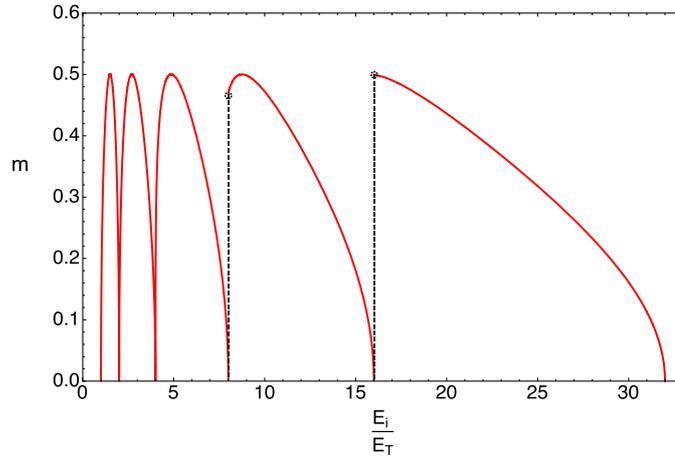
Figure 4. The characteristic slope m for switching points as a function of the total energy reduction ratio E_i/E_T. Note that for values of $E_i/E_T = 2^K$, $m = 0$.

The figure shows some apparent lack of smoothness in the dependence of m on the initial state. It also shows the trajectories spreading out in the vicinity of the resonant trajectories that have $m = 0$. The dependence of the characteristic slope m on the initial state and the final energy is algebraically accessible but its calculation is rather intricate and so we present these calculations in an appendix. It is shown there that m is determined purely by the energy reduction ratio E_i/E_T:

$$E_T^2 m^2 \left(m^2 + \omega_{\max}^2\right)^{2K} = f(m)g(m), \tag{40}$$

where

$$f(m) = E_T \left(m^2 + \omega_{\max}^2\right)^K \omega_{\max}^2 - E_i \left(m^2 + \omega_{\min}^2\right)^K \omega_{\min}^2 \tag{41}$$

and

$$g(m) = E_i \left(m^2 + \omega_{\min}^2\right)^K - E_T \left(m^2 + \omega_{\max}^2\right)^K. \tag{42}$$

Here K is the smallest integer such that

$$\left(\frac{\omega_{\max}^2}{\omega_{\min}^2}\right)^K \geqslant \frac{E_i}{E_T}. \tag{43}$$

K is also the number of A–B branch pairs in the trajectory. Figure 4 shows the resulting dependence of m on E_i/E_T.
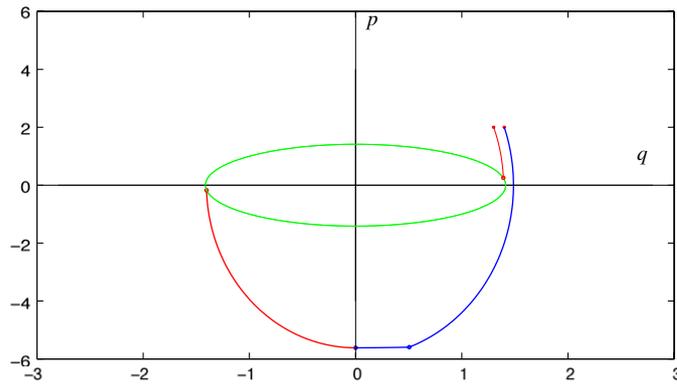
6. Resonant optimal trajectories

The resonant trajectories for the special case

$$\frac{E_i}{E_T} = \left(\frac{\omega_{\max}}{\omega_{\min}}\right)^{2K}, \qquad \text{for some integer } K \tag{44}$$

require $\tilde{\tau} = 0$. For this case the $H = 0$ condition (20) becomes

$$\tilde{q}p - \omega^2 \tilde{p}q = 0, \tag{45}$$

Figure 5. Optimal trajectories for removing energy from two nearby initial points (red stars) to the same final energy (green ellipse). Circles are switching points in ω. The outermost starting point requires an additional A–B branch pair, increasing the duration of the trajectory τ from 0.08 to 0.86, respectively.

and so the $\tilde{p} = 0$ switch points have $\tilde{q} p = 0$ which in turn implies $p = 0$. As we commented above, this is the solution one can arrive at from simple physical reasoning and was the assumed form of the solution in both the previous treatment of the swing and the squeezed light problems [6, 8]. We note that there is a certain sense in which this solution is optimal provided one takes the problem to be not the one to reach a given target energy in minimum time but to reach minimum energy with a given limited number of changes in the frequency ω.

7. The complete surface of solutions

We conclude our treatment of the minimum time control for the classical parametric oscillator by sketching the surface of minimum time $\tau = V_{\min}(q_i, p_i)$. This surface has discontinuities at each resonant trajectory, and getting an accurate graph is complicated by the inherent numerical instabilities in the vicinity of these trajectories. The nature of these discontinuities is illustrated in figure 5 where two nearby initial states give rise to two very different optimal trajectories.

The surface of minimum times appears as a sequence of ramps spiraling down to an elliptical target, with successive ramps filling in the regions between resonant trajectories in a smooth way as shown in figure 6.

It is illustrative to interpret the adjoint variables as well as the condition of maximality (9) in terms of this surface. Recall that the tangent plane to the graph of a function $z = f(x, y)$ has normal vector $\left(-1, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$. The vector of adjoint variables is in fact the normal vector to this surface and the adjoint equations (21) and (22) are just the equations for propagating this normal vector. With our convention $\tilde{\tau} = -1$, this lets us identify $\tilde{q}$ with $\frac{\partial \tau}{\partial q}$ and $\tilde{p}$ with $\frac{\partial \tau}{\partial p}$. We can also identify the vector $(f_0, f_1, f_2) = (1, p, -u^* q)$ with the tangent to this surface along an optimal trajectory. To see this, consider moving the time dt along this trajectory. In that case the vertical coordinate τ changes by

$$d\tau = dt = \frac{\partial \tau}{\partial q}\frac{dq}{dt}\,dt + \frac{\partial \tau}{\partial p}\frac{dp}{dt}\,dt. \tag{46}$$
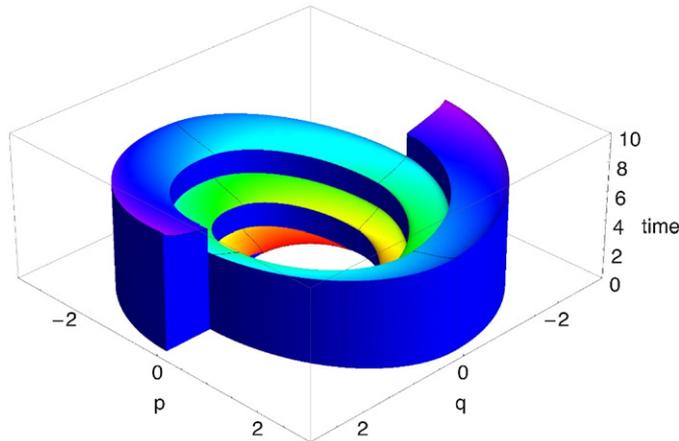
Figure 6. The shortest times necessary to reach the target energy $E_T = 1/2$ (bottom ellipse) of a classical harmonic oscillator starting from initial states (q_i, p_i). The optimal paths use the two frequencies $\omega_{max} = \sqrt{2}$ and $\omega_{min} = 1$, and the target energy is computed with $\omega_T = \omega_{min}$. The vertical cliffs are the resonant trajectories.

Equating the coefficients of dt we find

$$1 = \frac{\partial \tau}{\partial q} \frac{dq}{dt} + \frac{\partial \tau}{\partial p} \frac{dp}{dt} \tag{47}$$

$$1 = \tilde{q}\dot{q} + \tilde{p}\dot{p}, \tag{48}$$

and moving all of this to the right-hand side we find

$$0 = -1 + \tilde{q}\dot{q} + \tilde{p}\dot{p} = (\tilde{\tau}, \tilde{q}, \tilde{p}) \cdot (1, \dot{q}, \dot{p}) = H, \tag{49}$$

giving us the geometric interpretation that goes with the $H = 0$ condition.

Note that the tangency of the vector $(f_0, f_1, f_2)_{u=u^*}$ to the surface holds only for the optimal control u^*; other controls would require longer time to reach the next state along the trajectory and not have the property that the time spent plus the minimum time remaining for the point reached equals the minimum time from the starting point. This means that along the other feasible paths whose tangent is $(f_0, f_1, f_2)_{u \neq u^*}$, all such tangents point above the surface and thus make an angle of more than $90°$ with the downward pointing normal $\tilde{x}$. It follows that for such directions $(f_0, f_1, f_2)_{u \neq u^*}$,

$$H = \tilde{x} \cdot (f_0, f_1, f_2)_{u \neq u^*} = ||\tilde{x}|| \cdot ||(f_0, f_1, f_2)|| \cos(\text{angle} \geqslant 90°) \leqslant 0, \tag{50}$$

illustrating the Pontryagin maximality principle's geometrical underpinnings.

The downward pointing normal discussed in the previous paragraph can only be used when the vertical component of the normal $\tilde{\tau} = -1$. The resonant case $\tilde{\tau} = 0$ happens exactly at the vertical cliffs on the surface and illustrates the need for this case.

8. Conclusions

We hope that we have managed to whet the reader's appetite for the beauty of the oscillator problem and of the power and beauty of optimal control theory. We chose to publish this paper in a journal with focus on pedagogy and teaching of physics at least partly because

there are many interesting and doable questions left unanswered, and we believe that such questions would make suitable projects for the growing number of research experiences for undergraduate (REU) programs.

The problems are particularly suitable for an approach that follows the recipe below, which was amply illustrated in this paper.

(1) Implement a numerical solution.
(2) Explore sample solutions and look for patterns.
(3) Make conjectures and check these conjectures on further examples.
(4) Prove the conjectures analytically.
(5) Iterate 2–4.

Many questions remain unanswered regarding the control of parametric oscillators controlling $\omega_{\min} \leqslant \omega \leqslant \omega_{\max}$. As some examples of the types of questions remaining to be explored, consider

- What is the minimum time between two given states (q_1, p_1) and (q_2, p_2)?
- What is the minimum time from energy E_1 to energy E_2? This means starting and ending at the optimal choice of states with the requisite energies.
- What is the maximum or minimum energy that can be reached from a given state (q_i, p_i) in a given time τ?
- What is the distribution of final states on the target ellipse with energy E_T for the above problems?
- What is the density of optimal trajectories passing through a given area element $\mathrm{d}q\,\mathrm{d}p$ in phase space for the above problems?
- Any of the above problems for the swing/pendulum.

In addition to pulse shaping for NMR and chemical transitions, optimal control theory has been liberally used for examining in-principle limits to the control of thermodynamic processes [16, 18]. The question there is what can be achieved in a constrained time [17, 19]. This is a general issue since proceeding reversibly generally requires infinite time. A question considered there led directly to this work: How generally can one do 'fast adiabatic switching'? The adiabatic theorem [14] assures us that one can do adiabatic switching of external parameters provided we proceed infinitely slowly. The quantum oscillator problem [1] showed that at least for some systems such changes can be performed quickly and showed exactly how long these processes must take. While these features did not show up in the present problem, they do appear in systems of more than one oscillator [15]. Recently, a harmonic oscillator model has been used in combination with dynamical invariants to optimize vibrational cooling in Bose–Einstein condensates. The minimum time solutions can achieve effectively adiabatic transitions in times on the order of or faster than one vibration [21–23]. The calculations were experimentally verified by Schaff *et al* [20].

Appendix

In section 5, we revealed the collinearity of the $\tilde{p} = 0$ switch points and the associated slope m as key structural features of bang–bang trajectories. In equations (40) through (43) we gave an algebraic condition on m that applies in the case of an optimal trajectory for our minimum time problem. We now derive this result.

We are considering a trajectory with $2K + 1$ branches altogether: a sequence of K A–B branch pairs followed by a final branch.

The collinearity proof established

$$m = \omega_{\max} \cot(\omega_{\max} t_A) \tag{A.1}$$

for all type-A branches, and

$$m = -\omega_{\min} \cot(\omega_{\min} t_B) \tag{A.2}$$

for all type-B branches. Thus, for branches of either type, we have

$$\cot^2(\omega t) = \frac{m^2}{\omega^2}, \qquad \cos^2(\omega t) = \frac{m^2}{m^2 + \omega^2}, \qquad \sin^2(\omega t) = \frac{\omega^2}{m^2 + \omega^2}, \quad \text{(A.3)}$$

where t and ω are the waiting time and frequency on the branch.

Equations (A.1) and (A.2) relate m to the waiting time on each of the two branch types. We must establish a similar relation for the waiting time on the final branch. This will require, as an intermediate result, that we express the ratio of the values of p at successive q-switches in terms of m. To that end, consider an A-branch followed by a B-branch. Such a sequence will start at a q-switch k, proceed to a $\tilde{p}$-switch k, and end at a q-switch $k + 1$. Their coordinates in phase space are in the notation of section 5 denoted by $(0, p_0^k)$, (q_m^k, p_m^k), and $(0, p_0^{k+1})$. We know from (39) that $p_m^k / q_m^k = m$; the object is to calculate p_0^{k+1} / p_0^k.

Equation (30) applied to the A-branch (q-switch $k \rightarrow \tilde{p}$-switch k) and (35) applied to the B-branch ($\tilde{p}$-switch $k \rightarrow q$-switch $k + 1$) produce

$$p_m^k = p_0^k \cos(\omega_{\max} t_A), \tag{A.4}$$

$$p_0^{k+1} = -\omega_{\min} q_m^k \sin(\omega_{\min} t_B) + p_m^k \cos(\omega_{\min} t_B). \tag{A.5}$$

Equation (A.5) can be rewritten using (A.2) to replace the sine and p_m^k / m to replace q_m^k:

$$p_0^{k+1} = \frac{m^2 + \omega_{\min}^2}{m^2} p_m^k \cos(\omega_{\min} t_B). \tag{A.6}$$

Now combining (A.4) and (A.6) and using (A.3) to replace the cosines, we have finally

$$\frac{\left(p_0^{k+1}\right)^2}{\left(p_0^k\right)^2} = \frac{m^2 + \omega_{\min}^2}{m^2 + \omega_{\max}^2}. \tag{A.7}$$

There are exactly K A–B branch pairs in the trajectory. If we number the p-coordinates of the consecutive q-shifts $p_0, p_2, \ldots, p_K$, a simple induction yields

$$\frac{p_K^2}{p_0^2} = \left(\frac{m^2 + \omega_{\min}^2}{m^2 + \omega_{\max}^2}\right)^K. \tag{A.8}$$

Now consider the final branch. It is anticipated that the target energy will be achieved as a result of a final down-shift from $\omega_{\max}$ to $\omega_{\min}$. The final branch is therefore part of an A-branch that is terminated as soon as the system encounters a point in phase space at which the shift from $\omega_{\max}$ to $\omega_{\min}$ will produce the target energy E_T. In other words, the A-branch intersects the target ellipse:

$$\tfrac{1}{2}\left(p^2 + \omega_{\min}^2 q^2\right) = E_T. \tag{A.9}$$

See also equation (5).

Since the initial point of this final branch occurs at the end of the K A–B branch pairs, its p-coordinate is p_K. Let t_e be the waiting time before encounter with the target ellipse. Equations (25) and (26) with $q_i = 0$ and $p_i = p_K$ describe motion along this branch, so the final coordinates of the trajectory are

$$q_e = \frac{p_K}{\omega_{\max}} \sin(\omega_{\max} t_e), \tag{A.10}$$

$$p_{\text{e}} = p_K \cos(\omega_{\max} t_{\text{e}}). \tag{A.11}$$

Substituting these into (A.9) and using (A.8) yields our condition on the final waiting time t_{e} in terms of m, the ω's, and the two energies E_{T} and $E = p_0^2/2$,

$$\left(\frac{m^2 + \omega_{\min}^2}{m^2 + \omega_{\max}^2} \right)^K \left(\cos^2(\omega_{\max} t_{\text{e}}) + \frac{\omega_{\min}^2}{\omega_{\max}^2} \sin^2(\omega_{\max} t_{\text{e}}) \right) = \frac{E_{\text{T}}}{E}. \tag{A.12}$$

We are now in a position to impose the minimum time condition. The result to be proved is that m must satisfy the equation

$$E_{\text{T}}^2 m^2 \left(m^2 + \omega_{\max}^2 \right)^{2K} = f(m) g(m), \tag{A.13}$$

where

$$f(m) = E_{\text{T}} \left(m^2 + \omega_{\max}^2 \right)^K \omega_{\max}^2 - E \left(m^2 + \omega_{\min}^2 \right)^K \omega_{\min}^2 \tag{A.14}$$

and

$$g(m) = E \left(m^2 + \omega_{\min}^2 \right)^K - E_{\text{T}} \left(m^2 + \omega_{\max}^2 \right)^K. \tag{A.15}$$

We derive this requirement on m from the necessary condition for the minimum total time

$$\frac{\text{d}}{\text{d}m} (t_{\text{e}} + k(t_{\text{A}} + t_{\text{B}})) = 0. \tag{A.16}$$

The following conditions on the waiting times have been derived in equations (A.1), (A.2), and (A.12):

$$\cot(\omega_{\max} t_{\text{A}}) = \frac{m}{\omega_{\max}}, \qquad \cot(\omega_{\min} t_{\text{B}}) = \frac{-m}{\omega_{\min}}, \tag{A.17}$$

$$\cos^2(\omega_{\max} t_{\text{e}}) + \alpha \sin^2(\omega_{\max} t_{\text{e}}) = \beta(m), \tag{A.18}$$

where

$$\alpha = \frac{\omega_{\min}^2}{\omega_{\max}^2}, \qquad \beta(m) = \frac{E_{\text{T}}}{E} \left(\frac{m^2 + \omega_{\max}^2}{m^2 + \omega_{\min}^2} \right)^K. \tag{A.19}$$

From (A.17) we obtain

$$\frac{\text{d}t_{\text{A}}}{\text{d}m} = \frac{1}{m^2 + \omega_{\max}^2}, \qquad \frac{\text{d}t_{\text{B}}}{\text{d}m} = \frac{-1}{m^2 + \omega_{\min}^2}. \tag{A.20}$$

Equation (A.18) can be rearranged to give

$$\sin(\omega_{\max} t_{\text{e}}) = \sqrt{\frac{1 - \beta(m)}{1 - \alpha}}. \tag{A.21}$$

Differentiating we have

$$\frac{\text{d}t_{\text{e}}}{\text{d}m} = \frac{-1}{2\omega_{\max} \sqrt{(\beta - \alpha)(1 - \beta)}} \frac{\text{d}\beta}{\text{d}m}. \tag{A.22}$$

From (A.19) we have

$$\frac{\text{d}\beta}{\text{d}m} = -2k \frac{E_{\text{T}}}{E} \frac{m \left(\omega_{\max}^2 - \omega_{\min}^2 \right)}{\left(m^2 + \omega_{\min}^2 \right)^2} \left(\frac{m^2 + \omega_{\max}^2}{m^2 + \omega_{\min}^2} \right)^{K-1}, \tag{A.23}$$

and

$$(\beta - \alpha)(1 - \beta) = \frac{f(m) g(m)}{E^2 \omega_{\max}^2 \left(m^2 + \omega_{\min}^2 \right)^{2K}}, \tag{A.24}$$

where $f(m)$ and $g(m)$ are given in (A.14) and (A.15). Now (A.22) can be written as

$$\frac{dt_e}{dm} = \frac{k E_T m \left(\omega_{max}^2 - \omega_{min}^2\right) \left(m^2 + \omega_{max}^2\right)^{K-1}}{\left(m^2 + \omega_{min}^2\right) \sqrt{f(m) g(m)}}. \tag{A.25}$$

From (A.20) we have

$$\frac{dt_A}{dm} + \frac{dt_B}{dm} = \frac{-\left(\omega_{max}^2 - \omega_{min}^2\right)}{\left(m^2 + \omega_{min}^2\right) \left(m^2 + \omega_{max}^2\right)}. \tag{A.26}$$

The desired result (A.13) then follows from (A.16).

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