Optimal control of a collection of parametric oscillators

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(Received 14 December 2012; published 5 June 2013)

The problem of effectively adiabatic control of a collection of classical harmonic oscillators sharing the same time-dependent frequency is analyzed. The phase differences between the oscillators remain fixed during the process. This fact leads us to adopt the coordinates: energy, Lagrangian, and correlation, which have proved useful in a quantum description and which have the advantage of treating both the classical and quantum problem in one unified framework. A representation theorem showing that two classical oscillators can represent an arbitrary collection of classical or quantum oscillators is proved. An invariant, the Casimir companion, consisting of a combination of our coordinates, is the key to determining the minimum reachable energy. We present a condition for two states to be connectable using one-jump controls and enumerate all possible switchings for one-jump effectively adiabatic controls connecting any initial state to any reachable final state. Examples are discussed. One important consequence is that an initially microcanonical ensemble of oscillators will be transformed into another microcanonical ensemble by effectively adiabatic control. Likewise, a canonical ensemble becomes another canonical ensemble.

DOI: 10.1103/PhysRevE.87.062106 PACS number(s): 05.20.−y, 02.30.Yy, 03.67.Ac, 05.20.Gg

I. INTRODUCTION

The parametric harmonic oscillator is an iconic mechanical system with a number of highly interesting features that make it a frequently used classroom example. Recently, the parametric harmonic oscillator was analyzed with respect to the problem of optimally draining the maximum energy from the oscillator in minimum time [1–16]. The original motivation for these efforts derived from the quest to cool physical systems closer and closer to absolute zero. Accordingly, the treatment needed to be at the level of density matrices and quantum control.

The approach in [17] considers a Heisenberg description for which the dynamical algebra of the Harmonic oscillator can be exploited. This dynamical algebra reduces the description of the dynamics of the energy to a three-dimensional system of ordinary differential equations. Specifically, the Hamiltonian augmented by the Lagrangian and a position-momentum correlation form a closed Lie algebra, so other observables become irrelevant in regard to the time evolution of the energy. The present work started with the observation that the constraint equations of the optimal control problem that result by following only the expectation values of these three operators in time hold as well for a collection of classical oscillators as for the quantum oscillator. Thus the optimal control problem is identical for the classical and quantum cases. This leads us to explore this classical regime as a route to further insight.

The classical case does indeed lead to certain interesting surprises. The most notable among these is the fact that one classical oscillator [8] behaves very differently from two or more classical oscillators sharing a frequency. While for a collection of oscillators there is a lowest attainable energy set by the final frequency, for a single oscillator there is no such lowest energy; i.e., all of the energy in the oscillator may be removed, albeit taking longer and longer times as the energy declines. The situation is different for two or more parametric harmonic oscillators. If such a collection of oscillators with different initial conditions is subjected to the same change in their frequency, one has to expect that what takes energy out of one oscillator might put energy into another oscillator due to the phase shift between them. How much energy can be taken out and by what control of the common frequency are two of the open questions addressed in this paper. How this may be achieved in the shortest possible time is another.

Time, in fact, is chiefly what one gains through the optimal control. The adiabatic theorem [18] assures us that we can extract the maximum work possible from the system no matter how we change the frequency provided only that we do it sufficiently slowly. The first and arguably biggest surprise regarding the quantum optimal control [17] was that we can achieve fast, effectively adiabatic processes. By effectively adiabatic, we mean that the process starts and ends in the same states as the adiabatic process would have. The minimum time required depends on the initial and final frequencies as well as the allowed range of intermediate frequencies but is at most a time comparable to one oscillation [17]. This is much faster than methods based on the adiabatic theorem. Subsequent authors have shown that the time required can be made arbitrarily short provided one uses imaginary frequencies ω [2,19]. Here we again [17] restrict the problem we consider to only real frequencies in a given range $0 < \omega_{\min} \leqslant \omega \leqslant \omega_{\max} < \infty$.

Indeed, we believe that one lesson learned by applying optimal control to various physical processes is that more restrictive controls can lead to more interesting answers that reveal more of the physics of the problem. This has been seen, for example, in the control of spin systems where everything is easy for three-dimensional control of the magnetic field but gets more interesting when one is restricted to two-dimensional control [20–22] or even one-dimensional control [23]. With the restrictive control we adopt, we find that trying to proceed faster than the minimum time has a cost in work lost to parasitic oscillations. This feature is lost when one adopts less restrictive controls that allow zero time solutions.

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One classical problem that our analysis sheds light on concerns the various meanings of the term "adiabatic." This word was originally coined for the description of thermodynamic processes for which its use is unambiguous and refers to processes without heat transport. Early quantum theory, however, wreaked havoc on the meaning of this word in connection to changing parameters in a Hamiltonian. Quantum adiabaticity thus means something rather different. It refers to changing a quantum system in a way that keeps the occupation numbers of each state constant. One surprising feature of our optimal control is that it is effectively adiabatic in both senses of the word. How such quantum adiabaticity should be interpreted for the corresponding classical system is one of the findings developed here.

II. COLLECTIONS OF PARAMETRIC HARMONIC OSCILLATORS SHARING THE SAME FREQUENCY

In the present paper we analyze control problems for a collection of N classical parametric harmonic oscillators sharing the same frequency. This, on the one hand, extends the work on one parametric harmonic oscillator [1–5,8,10–12,15], and on the other hand, it builds a bridge to the control of a collection of quantum mechanical parametric harmonic oscillators. Our collection of classical oscillators will be described by the set of $2N$ equations,

$$\dot{q}_n = p_n, \tag{1}$$

$$\dot{p}_n = -\omega^2 q_n, \tag{2}$$

all sharing the same control ω^2. Here q_n and p_n are the dimensionless position and momentum coordinates of oscillator n. For concreteness the units are chosen such that $\hbar = c = m = 1$, where c and m are the speed of light and the mass of the oscillators, respectively. Based on these coordinates one can set up the problem using control theory. Please note that from here on all quantities are thereby dimensionless. Then, in principle, the control problem can be solved following the standard procedures, but it turns out that the natural coordinates (1) and (2) are strongly interdependent and possibly lead to a problem description of unnecessary complexity.

For problems involving the total energy E of the oscillators we avoid this unnecessary complexity (and make the desired connection to the quantum control problem) by again exploiting the dynamic algebra [24] also present in the classical version of the problem. We thus introduce

$$E(t) = \frac{1}{2}\sum_{n=1}^{N} p_n(t)^2 + \omega(t)^2 q_n(t)^2. \tag{3}$$

The differentiation operator $\frac{d}{dt}$ acting on a term $p_n(t)^2$ leads to terms of the type $p_n(t)q_n(t)$ as does its application on $q_n(t)^2$. It is thus not too surprising that the Lagrangian

$$L(t) = \frac{1}{2}\sum_{n=1}^{N} p_n(t)^2 - \omega(t)^2 q_n(t)^2 \tag{4}$$

and the correlation

$$C(t) = \omega(t)\sum_{n=1}^{N} p_n(t)q_n(t) \tag{5}$$

are important coordinates needed in describing the dynamics of the collection. A straightforward calculation shows that the variables $\{E,L,C\}$ form a closed set under time differentiation. We find

$$\dot{E} = \frac{\dot{\omega}}{\omega}(E - L), \tag{6}$$

$$\dot{L} = -\frac{\dot{\omega}}{\omega}(E - L) - 2\,\omega\, C, \tag{7}$$

$$\dot{C} = 2\,\omega\, L + \frac{\dot{\omega}}{\omega}\, C. \tag{8}$$

Later, we will add a differential equation for the frequency ω with a revised definition of the control, $u = u(t) = \dot{\omega}/\omega$,

$$\dot{\omega} = u\,\omega. \tag{9}$$

This is necessary since our differentiations led to the presence of $\dot{\omega}$ in (6)–(8), which is not compatible with the usual formulations of optimal control theory.

We note that Eqs. (6)–(8) are a very remarkable result. They allow us to treat optimal control problems which can be cast in terms of the variables $\{E,L,C\}$ by using a much reduced variable set consisting of only those three variables. This applies for any size N of the collection. In addition it turns out to be exactly the same dynamics as obtained for the expectation values for the quantum equivalents of the energy, the Lagrangian, and the correlation, as is seen by comparing to Eqs. (9)–(12) in [17].

We now seek to complement $\{E,L,C\}$ by further variables such that one obtains a description equivalent to $\{p_1,q_1,\ldots,p_N,q_N\}$. In order to do so we make the observation that, for any pair (i,j) of oscillators of the collection, a direct calculation shows

$$\frac{d}{dt}(p_i q_j - p_j q_i) = -\omega^2 q_i q_j + p_i p_j + \omega^2 q_j q_i - p_j p_i = 0. \tag{10}$$

This means that for any pair (i,j) of oscillators the quantities $r_{ij} = p_i q_j - p_j q_i$ are constants of the motion for arbitrary control. As there are $N(N-1)/2$ such r_{ij}, it is clear that with increasing N there are soon more r_{ij} than there are degrees of freedom for the N oscillator collection which is completely described by the $2N$ variables $\{p_1,q_1,\ldots,p_N,q_N\}$. Thus the r_{ij} cannot be independent of each other in general. A further analysis shows that there are exactly $2N-3$ independent r_{ij}.

Then a natural choice seems to be to choose the r_{ij} on the first subdiagonal $\{r_{i,i+1}\}$ and those on the second subdiagonal $\{r_{i,i+2}\}$ as the independent ones. We thus introduce $R_{1,2} = \{r_{i,i+1}\} \cup \{r_{i,i+2}\}$. It appears that with $R_{1,2}$ being of dimension $2N-3$ and $\{E,L,C\}$ being of dimension 3 one should have a complete description of the collection. However, it turns out that the three quantities $\{E,L,C\}$ are not independent of the $R_{1,2}$; in fact the rank of the Jacobian of the transformation from $\{p_1,q_1,\ldots,p_N,q_N\}$ to $\{E,L,C\} \cup R_{1,2}$ is $2N-1$.

To understand this consider the following. From the set $R_{1,2}$ one can determine all r_{ij} due to the dependence of the remaining ones on those in $R_{1,2}$. Then one can square each

one and sum them up over all possible pairs (i,j),

$$X = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} r_{ij}^2 = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}(p_i q_j - p_j q_i)^2. \tag{11}$$

As a double sum of constants X is also a constant.

Structurally, X is a sum of terms of the kind $p_i p_j q_i q_j$, where i and j are different, the same kind of terms one obtains by squaring C. However, there will be also terms of the form $p_i^2 q_i^2$, which in turn can be obtained by squaring E and L. Thus one can expect that X might be written as a linear combination of E^2, L^2, and C^2. Indeed, one finds

$$X = \frac{E^2 - L^2 - C^2}{\omega^2}. \tag{12}$$

Thus by knowing $\{E,L,C\}$, (any) one of the elements in $R_{1,2}$ can be discarded. That leaves the question of which variable should complement $\{E,L,C\} \cup R_{1,2}$ to get a description fully equivalent to $\{p_1,q_1,\ldots,p_N,q_N\}$. A possible choice is to add as variables the center of mass

$$Q = \sum_{n=1}^{N} q_n \tag{13}$$

and the overall momentum

$$P = \sum_{n=1}^{N} p_n \tag{14}$$

of the collection. For this couple of variables the dynamics is found directly from (1) and (2) by summing over n,

$$\dot{Q} = P, \tag{15}$$

$$\dot{P} = -\omega^2 Q. \tag{16}$$

Note that as $\{E,L,C\}$ is closed under time differentiation so is $\{P,Q\}$. Even though each one of the two would be enough to complement $\{E,L,C\}$ and $R_{1,2}$ reduced by one r_{ij}, it is more convenient to use both and reduce $R_{1,2}$ by two r_{ij}. A choice keeping the symmetry between the oscillators is to use $R_{2,3} = \{r_{i,i+2}\} \cup \{r_{i,i+3}\}$ instead of $R_{1,2}$. Then $\{E,L,C\} \cup \{P,Q\} \cup R_{2,3}$ is one set of variables which allows a complete description of the collection equivalent to $\{p_1,q_1,\ldots,p_N,q_N\}$.

Armed with our new coordinates, we note that, except for the overall movement of the center of mass of our collection of oscillators, the reduced dynamics in terms of E, L, and C captures the full motion of our collection.

III. THE MINIMUM ENERGY PROBLEM

The coordinate X in (12) plays a special role in two respects. Since it can be expressed in terms of r_{ij}, it is constant. Since it can be expressed in terms of E, L, and C, it can be used to reduce the dimension of the space on which the dynamics of interest in this problem and its control take place. This has been exploited in [14,16].

Recently [25], X has been dubbed the *Casimir companion*, and its close relationship to the Casimir operator has been emphasized. The Casimir companion has been shown to be generally useful for optimal control problems of quantum systems with a dynamical symmetry. Its value uniquely determines the von Neumann entropy of the system.

Our present concern, however, is to use the value of X to find the lowest energy $E_{\mathrm{f}}^{\min}$ that can be reached from an initial state $(E_{\mathrm{i}},L_{\mathrm{i}},C_{\mathrm{i}},\omega_{\mathrm{i}})$. By the invariance of X, we must have

$$\frac{E_{\mathrm{i}}^2 - L_{\mathrm{i}}^2 - C_{\mathrm{i}}^2}{\omega_{\mathrm{i}}^2} = \frac{E_{\mathrm{f}}^2 - L_{\mathrm{f}}^2 - C_{\mathrm{f}}^2}{\omega_{\mathrm{f}}^2}, \tag{17}$$

so

$$E_{\mathrm{f}}^2 = \frac{\omega_{\mathrm{f}}^2}{\omega_{\mathrm{i}}^2}\left(E_{\mathrm{i}}^2 - L_{\mathrm{i}}^2 - C_{\mathrm{i}}^2\right) + L_{\mathrm{f}}^2 + C_{\mathrm{f}}^2, \tag{18}$$

which shows that the lowest possible final energy will be found for $L_{\mathrm{f}} = C_{\mathrm{f}} = 0$. We thus obtain

$$E_{\mathrm{f}}^{\min} = \frac{\omega_{\mathrm{f}}}{\omega_{\mathrm{i}}}\sqrt{E_{\mathrm{i}}^2 - L_{\mathrm{i}}^2 - C_{\mathrm{i}}^2}, \tag{19}$$

which represents our solution to the minimum energy problem. While this can be formulated as a control problem, the optimal control is not determined, as in fact there are very many controls that reach this energy. For example, the adiabatic theorem guarantees that any time dependence $\omega(t)$ with $\omega(0) = \omega_{\mathrm{i}}$ and $\omega(t_{\mathrm{f}}) = \omega_{\mathrm{f}}$ will come arbitrarily close if we slow it down sufficiently, i.e., remap $t = \hat{t}/\alpha$ and let $\alpha \to \infty$. The question we ask below is how to achieve this lowest energy in minimum time.

States satisfying $L = C = 0$ are called equilibrium states. Such states are time invariant and minimize the energy on any set of states with given values of X and ω_{f}. The $L = 0$ condition can be interpreted as the equipartition of energy; the $C = 0$ condition can be interpreted as a lack of correlations at equilibrium. As we will see in Sec. VI, microcanonical and canonical ensembles have $L = C = 0$.

IV. THE MINIMUM TIME PROBLEM

Consider a collection of oscillators which have the initial frequency ω_{i} and initially vanishing Lagrangian L and correlation C. Our goal is to reach a state with frequency $\omega_{\mathrm{f}} < \omega_{\mathrm{i}}$ and again vanishing Lagrangian L and correlation C in the minimal possible time. We also require that

$$\omega_{\min} = \omega_{\mathrm{f}} \leqslant \omega(t) \leqslant \omega_{\mathrm{i}} = \omega_{\max}. \tag{20}$$

Our optimal control problem is then to minimize the time

$$\tau = \int dt \tag{21}$$

subject to the constraints represented by the dynamical equations (6)–(9), the inequalities (20), the initial state $(E_{\mathrm{i}},0,0,\omega_{\mathrm{i}})$, and the final state $(E_{\mathrm{f}},0,0,\omega_{\mathrm{f}}) = (\frac{\omega_{\mathrm{f}}}{\omega_{\mathrm{i}}}E_{\mathrm{i}},0,0,\omega_{\mathrm{f}})$.

The optimal control Hamiltonian [26] is

$$\begin{aligned} H_C = &-1 + \tilde{E}u(E-L) - \tilde{L}(u(E-L) + 2\omega C) \\ &+ \tilde{C}(2\omega L + uC) + \tilde{\omega}u\omega, \end{aligned} \tag{22}$$

where the variables $\tilde{E}$,$\tilde{L}$, and $\tilde{C}$ are conjugate variables to E, L, and C. Note that the optimal control Hamiltonian is linear in the control u. To emphasize this, we group the terms containing u and find

$$\begin{aligned} H_C &= [\tilde{\omega}\omega + (\tilde{E} - \tilde{L})(E-L)]u + [-1 + 2\omega(\tilde{C}L - \tilde{L}C)] \\ &= \sigma u + \alpha, \end{aligned} \tag{23}$$

where we have introduced the terms $\sigma = \sigma(x,\tilde{x})$ and $\alpha = \alpha(x,\tilde{x})$ for the coefficients of H_C viewed as a linear polynomial in our control u. The Pontryagin maximality principle [26] tells us that, at any instant, the value of the control must maximize H_C. Thus when the switching function σ is positive, u must be as large as possible, and when σ is negative, u must be as small as possible. Since away from the boundaries set by the inequalities $\omega_{\min} \leqslant \omega(t) \leqslant \omega_{\max}$ the value of u is not constrained, this amounts in our problem to jumps in ω. This can be seen by considering the problem with $|u(t)| \leqslant u_{\max}$ and letting $u_{\max} \to \infty$. Such jumps must terminate on the boundary arcs $\omega(t) = \omega_{\max}$ or $\omega(t) = \omega_{\min}$, which can be used as segments of the optimal trajectory. In addition to jumps and boundary arcs, the optimal control for such problems can also have singular branches along which the switching function σ vanishes identically over a time interval. The fact that this does not occur for this problem has been shown in [5,14,16,17]. The optimal control must therefore be of the bang-bang type, jumping between extreme allowed ω's interspersed with constant ω branches.

V. A THREE-JUMP OPTIMAL CONTROL FOR THE MINIMUM TIME PROBLEM

In this section we build up the optimal control including the resulting trajectories for the minimum time problem. In a first step we determine the dynamics for the $\{E,L,C,\omega\}$ variable set. As we are dealing with a bang-bang control, the frequency is either constant at one of its extreme values or jumps from one extreme value to the other. So the control consists of constant frequency arcs connected by jumps.

For constant frequency the dynamics of the system (6)–(9) simplifies considerably. The energy E is constant and the point (L,C) performs a counterclockwise rotation in the LC plane, described here by the transition from state s_ℓ to $s_{\ell+1}$:

$$s_{\ell+1} = W(\omega_\ell,t_\ell)s_\ell = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\omega_\ell t_\ell & -\sin 2\omega_\ell t_\ell & 0 \\ 0 & \sin 2\omega_\ell t_\ell & \cos 2\omega_\ell t_\ell & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_\ell \\ L_\ell \\ C_\ell \\ \omega_\ell \end{pmatrix}. \quad (24)$$

We call this a wait step.

A jump in the frequency goes beyond the dynamics given in (6)–(9). From the continuity of $\sum_n p_n^2$ and $\sum_n q_n^2$ at a jump we find, using Eqs. (3)–(5),

$$s_{\ell+1} = J(\omega_{\ell+1},\omega_\ell)s_\ell = \frac{1}{2}\begin{pmatrix} 1+r^2 & 1-r^2 & 0 & 0 \\ 1-r^2 & 1+r^2 & 0 & 0 \\ 0 & 0 & 2r & 0 \\ 0 & 0 & 0 & 2r \end{pmatrix} \begin{pmatrix} E_\ell \\ L_\ell \\ C_\ell \\ \omega_\ell \end{pmatrix}, \quad (25)$$

where $r = \omega_{\ell+1}/\omega_\ell$. The result of a bang-bang control can now be easily expressed as the action of a sequence of matrices $W(\omega_\ell,t_\ell)$ and $J(\omega_{\ell+1},\omega_\ell)$. The remaining problem is to find the optimal number of steps and the appropriate times t_ℓ.

From our work on the quantum formulation of the control problem [14,16,17] we know that a three-jump control with two appropriately chosen wait times is optimal when the frequency ratio r is not too large. For example, this is the case when the ratio is obtained from the initial and final equilibrium frequencies. However, for large frequency ratios r, controls with more than three jumps may perform better [27].

A. Equilibrium states as initial and final states

In the case of interest here we start and end in equilibrium states with $L = C = 0$. The time invariance of such states leads to the requirement that a frequency jump is required immediately at the beginning and the end of a time optimal process. The states obtained by such jumps can then be connected by a further jump separated by appropriate times t_i and t_f.

With the above introduced jump dynamics we see that the result of the initial jump from $\omega_{\max} = \omega_\mathrm{i}$ to $\omega_{\min} = \omega_\mathrm{f}$ starting from the initial state s_i is

$$\begin{aligned} s_1 &= (E_1,L_1,C_1,\omega_1)^{\mathrm{tr}} = J(\omega_\mathrm{f},\omega_\mathrm{i})(E_\mathrm{i},0,0,\omega_\mathrm{i})^{\mathrm{tr}} \\ &= (E_\mathrm{i}(1+r^2)/2, E_\mathrm{i}(1-r^2)/2, 0, \omega_\mathrm{f})^{\mathrm{tr}}. \end{aligned} \quad (26)$$

Similarly, by inverting the final jump step from $\omega_{\max} = \omega_\mathrm{i}$ to $\omega_{\min} = \omega_\mathrm{f}$ we obtain the state preceding the final state s_f,

$$\begin{aligned} s_{(\mathrm{f}-1)} &= (E_{(\mathrm{f}-1)},L_{(\mathrm{f}-1)},C_{(\mathrm{f}-1)},\omega_{(\mathrm{f}-1)})^{\mathrm{tr}} \\ &= J(\omega_\mathrm{f},\omega_{(\mathrm{f}-1)})^{-1}(E_\mathrm{f},0,0,\omega_\mathrm{f})^{\mathrm{tr}} \\ &= (E_\mathrm{f}(1+r^{-2})/2, E_\mathrm{f}(1-r^{-2})/2, 0, \omega_\mathrm{i})^{\mathrm{tr}}. \end{aligned} \quad (27)$$

B. Connecting two given states with isentropic processes

From our previous calculations on quantum oscillators we already know that we will be able to connect s_1 and s_2 by two wait steps with one intermediate jump step changing the frequency from ω_f to ω_i. Note that for finding the effectively adiabatic process that connects initial and final equilibrium states s_1 and s_2 we need to consider only states with $C = 0$. However, in this section, we generalize that problem by considering two generic states $s_1 = (E_1,L_1,C_1,\omega_1)^{\mathrm{tr}}$ and $s_2 = (E_2,L_2,C_2,\omega_2)^{\mathrm{tr}}$ and asking under what conditions can two such states be connected by two waits and a jump. This generalization gives us information about the structure of the connection problem.

First, a wait step $W(\omega_1,t_1)$ is applied for a yet undetermined time t_1. Since in a wait step L and C rotate, we note that the system has a time independent

$$A_1 = \sqrt{L_1^2 + C_1^2} = \text{const.} \quad (28)$$

The state travels along an arc of a circle with radius A_1 in the (L,C) plane. Then a jump $J(\omega_2,\omega_1)$ is applied, which leads to a state which must lie on a circle to which the final state belongs:

$$A_2 = \sqrt{L_2^2 + C_2^2} = \text{const.} \quad (29)$$

Finally, the second wait $W(\omega_2,t_2)$ automatically reaches the final state.

As we do not know yet the initial waiting time t_1, we parametrize the initial jump state by an angle $\phi_1 = 2\omega_1 t_1$:

$$s_{j1} = (E_1, A_1\cos\phi_1, A_1\sin\phi_1, \omega_1)^{\mathrm{tr}}. \quad (30)$$

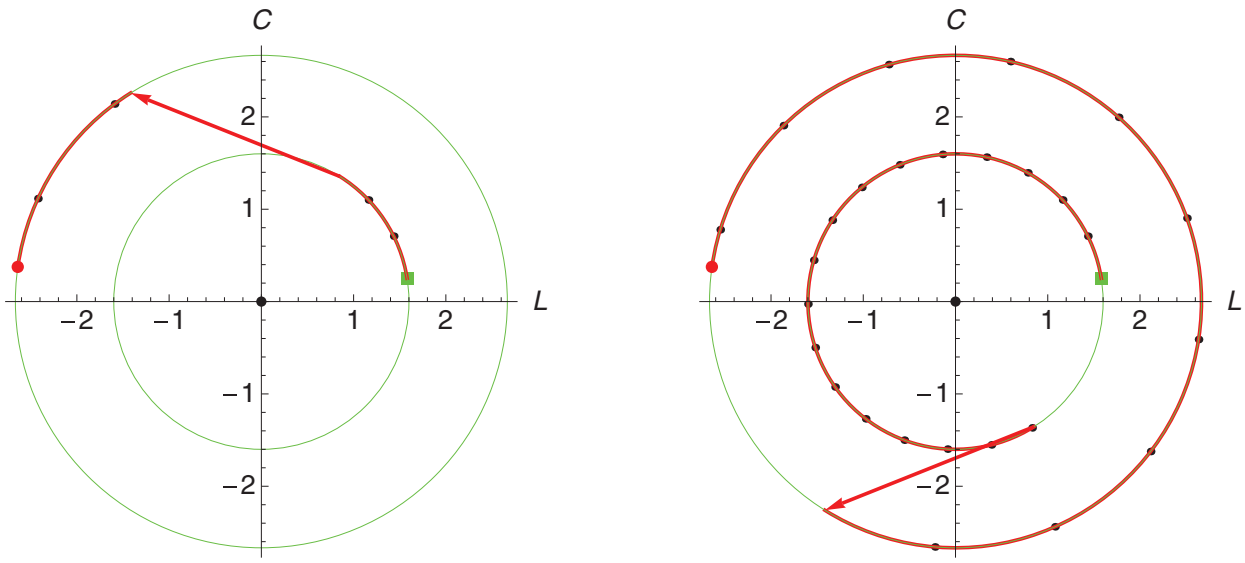

FIG. 1. (Color) A projection of the trajectory of a wait-jump-wait control in the (L,C) plane leading from the initial state (green square) with $L_1 > L_{j1}$ to the final state (red circle) with $L_{j2} > L_2$. The black ticks along the trajectory indicate equal time intervals. The left and right panels illustrate the choice between positive and negative C values for the jump point. For the values shown, $C > 0$ is clearly faster.

Likewise, the desired target state s_{j2} is parametrized as

$$s_{j2} = (E_2, A_2 \cos\phi_2, A_2 \sin\phi_2, \omega_2)^{\mathrm{tr}}. \tag{31}$$

Due to the jump conditions (25) the initial jump state s_{j1} is connected to the final jump state s_{j2} by

$$s_{j2} = J(\omega_2,\omega_1)s_{j1}. \tag{32}$$

This vector equation applies componentwise; here we look at the first element

$$E_2 = \frac{E_1}{2}\left(1 + \frac{\omega_1^2}{\omega_2^2}\right) + \frac{A_1}{2}\left(1 - \frac{\omega_1^2}{\omega_2^2}\right)\cos(\phi_1). \tag{33}$$

Solving for $\cos\phi_1$ and reinserting the result into (30) leads to only one possible solution for L_{j1}:

$$L_{j1} = \frac{2E_2\omega_1^2 - E_1(\omega_1^2 + \omega_2^2)}{\omega_1^2 - \omega_2^2}. \tag{34}$$

The corresponding $C_{j1} = \pm\sqrt{A_1^2 - L_{j1}^2}$ follows from (28). Due to the symmetry between initial and final states we find

$$L_{j2} = \frac{2E_1\omega_2^2 - E_2(\omega_2^2 + \omega_1^2)}{\omega_2^2 - \omega_1^2} \tag{35}$$

and $C_{j2} = \pm\sqrt{A_2^2 - L_{j2}^2}$. The important point to understand about this solution is that the connection of two given circles in the (L,C) plane with two given frequencies ω_1 and ω_2 can only proceed at two special points on the initial circle. Both points have the same L value with opposite signs for C.

We are now in the position to address the aforementioned connectability question: From (28) and (29) we have $|L_{j1}| \leqslant A_1$ and $|L_{j2}| \leqslant A_2$, and thus the two states s_1 and s_2 are connectable iff

$$\left|\frac{2E_2\omega_1^2 - E_1(\omega_1^2 + \omega_2^2)}{\omega_1^2 - \omega_2^2}\right| \leqslant A_1 \tag{36}$$

and

$$\left|\frac{2E_1\omega_2^2 - E_2(\omega_2^2 + \omega_1^2)}{\omega_2^2 - \omega_1^2}\right| \leqslant A_2. \tag{37}$$

We now turn to the question of which of the two jumping points should be chosen to give the shorter process time. There are first two simple cases: We note that the wait matrix W leads to counterclockwise turns and that the jump matrix J does not change the sign of C. If the initial L_1 is larger than L_{j1} and the final L_2 is smaller than the corresponding L_{j2}, then we choose the jump point with positive C_{j1}. Such a situation is shown in Fig. 1. An initial state at frequency $\omega_1 = 3$ with $L_1 = 1.58$ is rotated to $L_{j1} = 0.85$, from which a jump takes it to $\omega_2 = 5$ with $L_{j2} = -1.42$, from which the second rotation leads to $L_2 = -2.64$. Choosing the jump with $C < 0$ would have led to a much longer duration.

Correspondingly, for the situation where the initial L_1 is smaller than L_{j1} and the final L_2 is larger than L_{j2} we choose the jumping point with the negative C_{j1}.

For the cases which are not yet covered we can use either of the two principally possible jump points. In that case the question is for which of the two the total time is longer. This is decided by comparing the times spent on the two arcs between (L_{j1},C_{j1}) and $(L_{j1}, -C_{j1})$ on the initial circle and between (L_{j2},C_{j2}) and $(L_{j2}, -C_{j2})$ on the final circle. Let, for $k = 1,2$, $\phi_{k\pm} = \arg(L_{k1} \pm i|C_{k1}|)$. Then the angles of interest on the initial or final circle are $\Delta\phi_k = \phi_{k-} - \phi_{k+} = 2(\pi - \phi_{k+})$, and we find the following selection rule: If $L_1 > L_{j1}$ and $L_2 > L_{j2}$ and in addition $\Delta\phi_1/\omega_1 > \Delta\phi_2/\omega_2$, then $C_{j1} > 0$; otherwise, $C_{j1} < 0$. If, on the other hand, $L_1 < L_{j1}$ and $L_2 < L_{j2}$ and in addition $\Delta\phi_1/\omega_1 > \Delta\phi_2/\omega_2$, then $C_{j1} < 0$; otherwise, $C_{j1} > 0$. Figure 2 shows this situation. Here the negative jump point leads to a considerably shorter process time, as can be seen by counting the time ticks along the trajectory. The times can be determined from the angles turned and the corresponding frequencies.

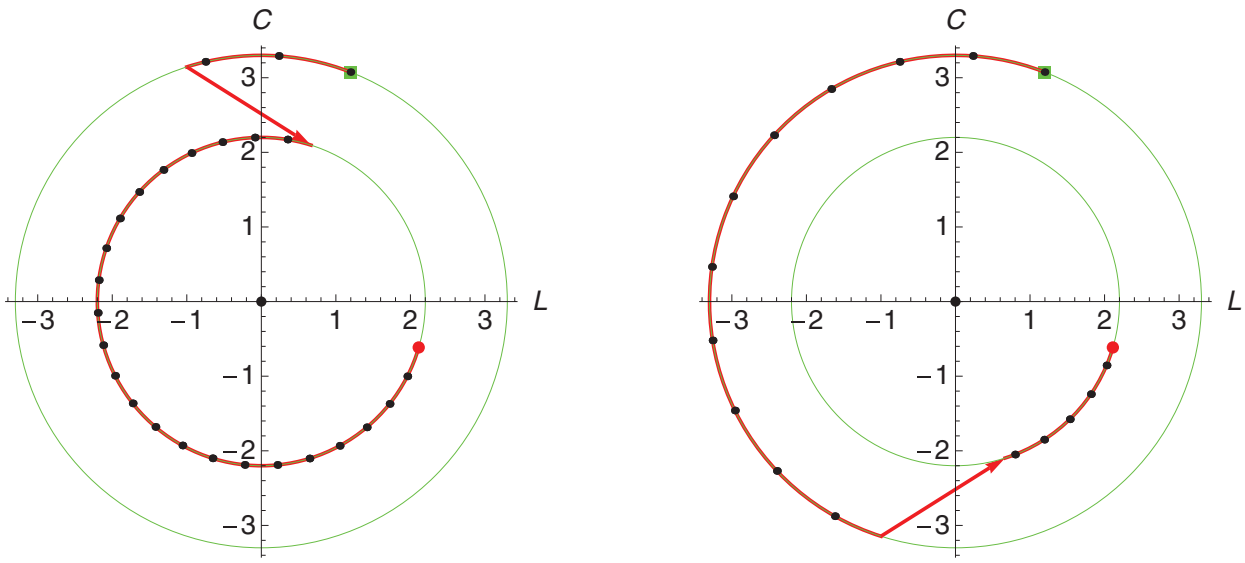

FIG. 2. (Color) A projection of the trajectory of a wait-jump-wait control in the (L,C) plane leading from the initial state (green square) to the final state (red circle). The black ticks along the trajectory indicate equal time intervals. In the left panel the jump point with positive C is used; in the right panel the negative C is used. The latter is clearly faster.

Table I presents an overview of the above decision rules.

C. Equilibrium states as initial and final states

We now apply the results of the previous section to the special case we started with, namely, s_1 and s_2 given by (26) and (27), respectively. In that case $A_1 = L_1$ and $A_2 = -L_2$, as L_2 is negative due to $\omega_\mathrm{i}/\omega_\mathrm{f} > 1$. The jump occurs for positive C_{j1} and C_{j2} as L_1 and $-L_2$ are the largest possible L on the respective circles. Then the angles ϕ_1 and ϕ_2 can be easily determined, and the times t_1 and t_2 are

$$t_1 = \frac{1}{2\omega_\mathrm{i}} \arccos\left(\frac{\omega_\mathrm{i}^2 + \omega_\mathrm{f}^2}{(\omega_\mathrm{i} + \omega_\mathrm{f})^2}\right),$$
$$t_2 = \frac{1}{2\omega_\mathrm{f}} \arccos\left(\frac{\omega_\mathrm{i}^2 + \omega_\mathrm{f}^2}{(\omega_\mathrm{i} + \omega_\mathrm{f})^2}\right), \tag{38}$$

as expected from the quantum calculation.

Figure 3 shows the resulting trajectories in the (L,C) plane. The initial and final equilibrium states are located at the origin. The first frequency jump takes the oscillator collection to an $L > 0$, $C = 0$ state, which is rotated counterclockwise to the jump states, which are connected through the intermediate jump. The second rotation then leads to an $L < 0$, $C = 0$ state, from which the final jump reaches the target equilibrium at the origin.

TABLE I. The decision rules for selecting the positive or negative jump point.

Case			Jump point selection
$L_1 < L_{j1}$,	$L_{j2} < L_2$		$C < 0$
$L_1 < L_{j1}$,	$L_{j2} > L_2$,	$\Delta\phi_1/\omega_1 < \Delta\phi_2/\omega_2$	$C > 0$
$L_1 < L_{j1}$,	$L_{j2} > L_2$,	$\Delta\phi_1/\omega_1 > \Delta\phi_2/\omega_2$	$C < 0$
$L_1 > L_{j1}$,	$L_{j2} > L_2$		$C > 0$
$L_1 > L_{j1}$,	$L_{j2} < L_2$,	$\Delta\phi_1/\omega_1 < \Delta\phi_2/\omega_2$	$C < 0$
$L_1 > L_{j1}$,	$L_{j2} < L_2$,	$\Delta\phi_1/\omega_1 > \Delta\phi_2/\omega_2$	$C > 0$

VI. SPECIAL CONFIGURATIONS AND ENSEMBLES

In this section we discuss a number of special collections of parametric harmonic oscillators sharing the same frequency. We also show that already two classical oscillators are enough

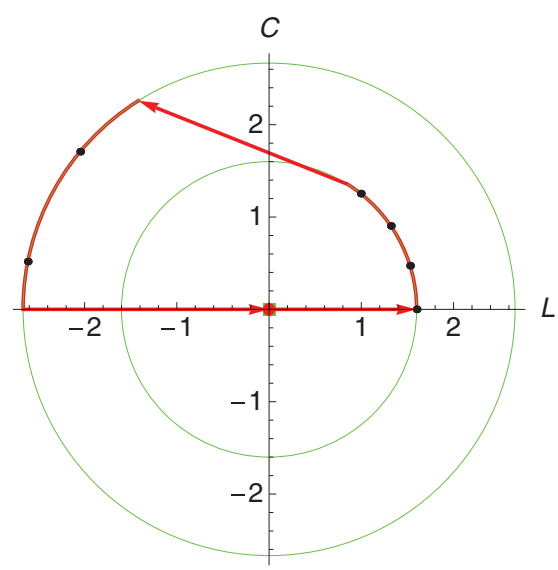

FIG. 3. (Color online) The projected trajectory of a three-jump control in the (L,C) plane. The initial and final equilibrium states are located at the origin. The first frequency jump takes the oscillator collection to an $L > 0$, $C = 0$ state, which is rotated counterclockwise to the jump states, which are connected through the intermediate jump. The second rotation then leads to an $L < 0$, $C = 0$ state, from which the final jump reaches the target equilibrium.

to represent the full complexity of the dynamics of N oscillators or of an ensemble of quantum oscillators. We also verify Hertz's theorem [28] that a microcanonical ensemble at one energy is transferred into another microcanonical ensemble at a different energy by a reversible adiabatic process. It is reassuring to see that this holds also for effectively adiabatic processes.

First, we turn to collections of few oscillators with special properties.

A. Effectively single-oscillator configurations

We define effectively single-oscillator configurations to be N oscillators with $X = 0$. Recall from Eq. (11) that X is the sum of the squares of all r_{ij}'s. Thus $X = 0$ implies that $r_{ij} = 0$ for all i and j. But $r_{ij} = 0$ means that the vectors (q_i, p_i) and (q_j, p_j) are parallel. This forces configurations of oscillators to lie on one line through the origin. They are all in phase or shifted by π. Physically, that is not surprising as the $\{E,L,C\}$ representation does not distinguish between oscillators passing through the origin from opposite sides. For these configurations the optimal control of Ref. [8] for one oscillator applies rather than the three-jump controls we have been discussing. These configurations can reach $E = 0$, albeit in infinite time.

B. Equilibrium configurations

These states are characterized by the equilibrium conditions $L = C = 0$. For one oscillator, it means sitting at $(q,p) = (0,0)$. For more oscillators the possible parameter space gets large; we restrict ourselves to configurations with oscillators all at the same energy.

Consider two oscillators with the same energy:

$$q_1 = d\sin(\omega t + \varphi_1), \tag{39}$$

$$p_1 = d\omega\cos(\omega t + \varphi_1), \tag{40}$$

$$q_2 = d\sin(\omega t + \varphi_2), \tag{41}$$

$$p_2 = d\omega\cos(\omega t + \varphi_2), \tag{42}$$

where φ_1 and φ_2 are the respective phase shifts. So both oscillators are on the same energy shell but have a relative phase shift. Using trigonometrical rules, we obtain E, L, and C as

$$E = d^2\omega^2, \tag{43}$$

$$L = d^2\omega^2\cos(\varphi_1 - \varphi_2)\cos(2\omega t + \varphi_1 + \varphi_2), \tag{44}$$

$$C = d^2\omega^2\cos(\varphi_1 - \varphi_2)\sin(2\omega t + \varphi_1 + \varphi_2). \tag{45}$$

It is now easy to see from (44) and (45) that a relative phase shift $\varphi_1 - \varphi_2 = \pi/2$ or $3\pi/2$ leads to $L = C = 0$.

If $\varphi_1 - \varphi_2 = \pi$, one sees that L and C have the same amplitude as E, which implies $X = 0$ and the effectively single-oscillator behavior we saw above. This situation of in-phase evolution is in a sense orthogonal to the equilibrium situation when $\varphi_1 - \varphi_2 = \pi/2$ or $3\pi/2$.

For three oscillators, again restricting ourselves to configurations of oscillators on the same energy shell, we have, up to an arbitrary overall phase shift, eight possible configurations of high symmetry leading to $L = C = 0$. The simplest is the equilateral triangle. All others are obtained from it by noting that the equilibrium conditions are not sensitive to rotations by π. Similarly, for N oscillators having the same energy, we have 2^N possible configurations up to an overall phase shift. These configurations are found by starting from the regular N-gon and adding π to the phase of some subset of the vertices.

Similar solutions can be found for the equilibrium conditions, dropping the same ellipse requirement for small displacements off the energy shell. Larger displacements, however, yield no solution.

C. A representation theorem

Examining Eqs. (43)–(45), it becomes apparent that for a collection of oscillators of any size one can determine a two-oscillator collection with the same $\{E,L,C\}$ behavior. To see this, we determine E_0, L_0, and C_0 at a certain time t_0 and frequency ω_0 for the collection to be represented. Then from (43)–(45) we find that with $d = \sqrt{E_0}/\omega_0$, $\cos(\varphi_1 - \varphi_2) = \sqrt{L_0^2 + C_0^2}/E_0$, and $\tan(2\omega_0 t_0 + \varphi_1 + \varphi_2) = C_0/L_0$ the same initial conditions are set at t_0. Thereafter, the two oscillators as well as the collection to be represented will always have the same E, L, and C if subjected to the same control.

D. Microcanonical ensemble

States in statistical mechanics are distributions in the phase space of one copy of the system. We now consider what effectively adiabatic moves do to a microcanonical distribution of one oscillator. By Hertz's theorem [28] a truly adiabatic process would carry this ensemble to another microcanonical distribution. Note that the special configurations discussed in the previous section could be viewed as finite realizations of such an ensemble, whereas the full distribution is what results in the thermodynamic limit as the number of oscillators tends to infinity. By the discussion in the previous section, we note that a distribution uniform around the ellipse will satisfy the equilibrium conditions $L = C = 0$.

The key observation we make here is that in effectively adiabatic moves, all the states on the initial ellipse end up at states on the same final ellipse. This follows easily from $E_\mathrm{f} = \frac{\omega_\mathrm{f}}{\omega_\mathrm{i}} E_\mathrm{i}$. The second key observation is that effectively adiabatic moves also preserve the phase shift between two oscillators on the same ellipse.

This is illustrated in Fig. 4, where we track the movement of two oscillators from the initial to the final ellipse. It can also be shown directly by evaluating r_{12} from (42):

$$r_{12} = p_1 q_2 - p_2 q_1 = -d^2\omega\sin(\varphi_1 - \varphi_2) = \frac{E}{\omega}\sin(\varphi_1 - \varphi_2), \tag{46}$$

where the last equality used (43). As r_{12} is a constant of the motion, we find

$$\frac{E_\mathrm{i}}{\omega_\mathrm{i}}\sin\left(\varphi_1^\mathrm{i} - \varphi_2^\mathrm{i}\right) = r_{12} = \frac{E_\mathrm{f}}{\omega_\mathrm{f}}\sin\left(\varphi_1^\mathrm{f} - \varphi_2^\mathrm{f}\right). \tag{47}$$

Using again $\frac{E_\mathrm{i}}{\omega_\mathrm{i}} = \frac{E_\mathrm{f}}{\omega_\mathrm{f}}$, we thus finally obtain $\sin(\varphi_1^\mathrm{i} - \varphi_2^\mathrm{i}) = \sin(\varphi_1^\mathrm{f} - \varphi_2^\mathrm{f})$.

It follows that a uniform distribution on the initial ellipse is transformed into a uniform distribution on the final ellipse;

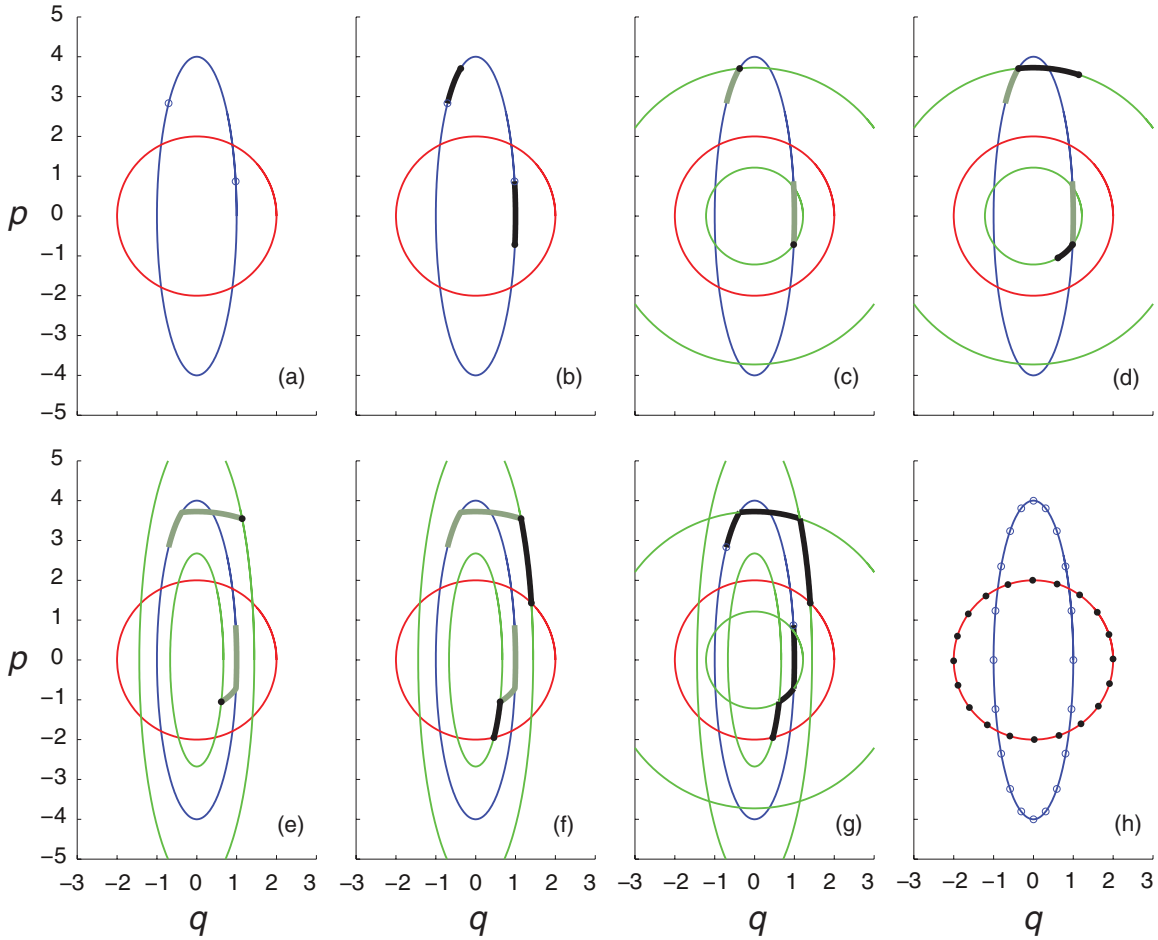

FIG. 4. (Color) Two oscillators of the same energy but with a phase difference, marked as open blue circles in (a), are followed through the three-jump move in this (q,p) phase diagram. The blue ellipse represents all possible states with the initial energy and frequency, and the red circle shows all possible final states at the final frequency. (a) Initial state. (b) Initial wait moves (thick black lines). (c) Jump to the final frequency (green circles). (d) Wait move at the final frequency (thick black lines). (e) Jump back to the initial frequency (green ellipses). (f) Wait move at the initial frequency (thick black lines). (g) Jump to the final frequency and final state on the red circle (whole trajectory marked with a thick black line). (h) A uniformly distributed set of initial points on the blue ellipse ends up as uniformly distributed final states on the red circle; i.e., a microcanonical ensemble is transformed into another microcanonical ensemble.

i.e., a microcanonical ensemble is transformed into another microcanonical ensemble.

E. Canonical ensemble

Effectively adiabatic moves carry canonical ensembles to canonical ensembles. This fact is of some significance in connection with quantum heat engines [1]. To see this, we start with the central fact of the previous section: uniform distributions on an energy slice at $E = E_\mathrm{i}$ are carried to uniform distributions at energy $E = E_\mathrm{f} = \frac{\omega_\mathrm{f}}{\omega_\mathrm{i}} E_\mathrm{i}$. Now we start with the distribution at E_i having probability mass $p(E_\mathrm{i}) \propto \exp(-E_\mathrm{i}/kT_\mathrm{i})$. Since all of this probability transfers to $E_\mathrm{f} = \frac{\omega_\mathrm{f}}{\omega_\mathrm{i}} E_\mathrm{i}$, the resulting distribution has

$$p(E_\mathrm{f}) \propto \exp(-E_\mathrm{i}/kT_\mathrm{i}) = \exp\left(-\frac{E_\mathrm{f}}{k\frac{\omega_\mathrm{f}}{\omega_i}T_\mathrm{i}}\right) = \exp(-E_\mathrm{f}/kT_\mathrm{f}), \qquad (48)$$

which is canonical with final temperature $T_\mathrm{f} = \frac{\omega_\mathrm{f}}{\omega_\mathrm{i}} T_\mathrm{i}$.

VII. CONCLUSIONS

This paper sets out to examine the classical counterpart to the optimal control of a shared frequency among an ensemble of noninteracting harmonic oscillators [1–5,8,10–12,15]. After some considerations regarding possible choices of coordinates for the problem, we are led to using the coordinates E, L, and C with the interesting feature that the same treatment applies to a classical or a quantum ensemble. We deviate from most authors on the subject by requiring our shared frequency to be in a limited range $\omega_\mathrm{min} \leqslant \omega \leqslant \omega_\mathrm{max}$ that reveals features missing in the less restrictive treatments.

Considering the problem of reaching the minimum energy in minimum time, we again find that the minimum energy is determined by the time-invariant Casimir companion X and that fast effectively adiabatic processes exist between any two states of the ensemble. We characterize the three-jump processes in this class and find conditions for two states to be connectable using a three-jump control.

We also show that such effectively adiabatic processes indeed have the features expected for infinite time adiabatic

switching in classical systems such as preserving microcanonical and canonical ensembles. We also prove a representation theorem that shows that any collection of quantum or classical harmonic oscillators can be represented by two classical oscillators.

The classical collection of oscillators does show one feature not present in the quantum case. Provided the collection of oscillators is in phase, the collection acts as effectively one oscillator whose control is quite different [8], primarily because such control can be tailored to the phase of the oscillator.

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