Finite-time Thermodynamic Analysis of Controlled Heat Integration

James D. Nulton[1] and Peter Salamon[2]

[1] Department of Mathematics, San Diego City College, San Diego, CA 92101

[2] Department of Mathematical and Computer Sciences, San Diego State University, San Diego, CA 92182-7720

A number of questions related to the efficient integration of heat in the presence of finite conductance are addressed. In particular, a calculation of the minimum exergy cost of supplying the utility demand to a heat exchange network is presented. The analysis uses an endoreversible heat engine whose working fluid undergoes a cycle custom designed to match the utility demands. The results give a simple measure of potential exergy savings from incorporating active elements into the network.

1. Introduction

The present paper considers the problem of in-principle limitations on the conversion of heat in an ideal heat engine whose steady state operation is designed to carry out a largely pre-specified rearrangement of thermal energy among many systems. Thus we treat an ideal version of the heat integration problem engineers face in the design of heat exchange networks (HEN).

The study of in-principle limitations to the operation of finite time processes has been pursued by a number of authors [1-6]. One, area of analysis that has been particularly successful, is the analysis of in-principle limitations on energy conversion imposed by the inherent irreversibility of transporting thermal energy from one system to another at a given finite rate or in a given finite time across a boundary with finite conductance. Transport processes across finite boundaries require an infinite time for reversible operation and thus a finite rate (or finite net time) constraint forces them to produce entropy. This is the rationale behind the name "finite-time thermodynamics". Given a conductance, transport modes in a system can be made reversible only in the limit of infinite time. This of course does *not* imply that all processes approach reversible operation as they proceed more slowly.

The minimum exergy losses associated with the transport of heat is known for a wide variety of situations [1-6]. The present paper extends this knowledge to the analysis of HENs. The basic problem we consider is well illustrated by the energy integration of a steady state plant. This typically involves many streams that need heating and cooling over ranges of temperatures as well as some batch reactors that need to be heated or cooled at a given rate to maintain constant temperatures. As is customary [7], we describe this situation by the heat demand functions which specify the heat needed by each system at each temperature. In pinch analysis [7, 8], one usually aggregates the demand into a positive and a negative distribution at each temperature. *Pinch analysis* is the standard engineering technique for designing such networks. In its crude form, the analysis merely tries to match up heat sources and sinks in a way which minimizes the use of additional heat sources and sinks that are known in this capacity as hot and cold utilities. The idea is to use, as much as possible, available streams that need cooling to heat streams that need heating. This results in the design of what we will call a *passive HEN*. While more refined versions of pinch analysis consider the degree of freedom that results by adding a heat pump to the system, such heat pumps operate between two temperatures and result in a modified HEN to be treated by the above methods. Pinch analysis does very well in finding economical designs.

One competing analysis which serves to characterize in-principle limits to the efficiency of a HEN is known as exergy analysis. In exergy analysis the rates do not play a role and we calculate the amount of hot and cold utility needed in a *reversible* process which supplies all the required demands. The reversible analysis must take all the conductances as infinite. One convenient way to visualize an ideal realization of exergy analysis is by means of a reversible heat engine whose working fluid is coupled in turn to each of the streams, carries out the requisite heat exchanges reversibly, and makes use of the utilities only to allow closing the cycle, i.e. to balance entropy and energy. Since we choose to focus on steady state operation and demands, we envision many copies of this working fluid traversing the cycle slightly out of phase resulting in a reversible steady state device. This visualization is particularly convenient as a starting point for our analysis below which will make use of a similar heat engine as the core element in an *active HEN*.

We consider the finite-time thermodynamic version of this problem where we now require a finite conductance between the streams representing the demand and the working fluid in our heat engine. We wish to focus on the limitations imposed on steady-state operation by thermal transport through a limited conductance. Accordingly, we assume that all other modes in the system can be controlled reversibly. Our approach is to study the set of possible operations of an *active heat-integration device* (AHD) which satisfies a given distribution of heat demands. By understanding the structure of the set of possibilities, we can ask a number of interesting questions about the optimal operation of the process.

Section 2 contains a description of our ideal device, whose optimal operation provides the basis for in-principle limits on the performance of heat-integration tasks generally. Section 3 defines the symbols and conventions used in the exposition. Sections 4 through 7 state the respectively the results of four optimization problems of interest in heat integration. In each case, a problem is specified by stated physical conditions and net effects of the operation. For the solutions, we present the optimal value of the objective, and the operating conditions which realize that value. Both the value of the objective and the operating conditions are expressed in terms of the prescribed data for the problem.

2. Description of the Active Heat-integration Device (AHD)

The operation of our ideal device, like the Carnot engine, is based on the mediation, by means of a working fluid, of required heat transfers. Consider the following general framework. The exterior, whose elements either supply or demand heat, is in thermal contact with an interior, or working system, which, in the form of a circulating fluid mediates the exchange of heat among the elements of the exterior. At each point of contact, both interior and exterior have well-defined temperatures, and heat is exchanged by a linear law across the thermal boundary. These temperatures and the associated heat flux density may vary with the point of contact between exterior and interior, but their distributions are considered to be stationary in time, so that the working fluid is in a steady state and the energy and entropy fluxes are stationary. The control of the distribution of the working fluid's temperature over the points of contact with the exterior is maintained by an active interface with the work reservoir. The AHD itself consists of the thermal boundary and the working system, which encompasses the working fluid, the work reservoir and the interface between them. Steady state conditions require, of course, that the aggregate entropy and internal energy of the working fluid not change with time. The former requirement places an important constraint on how the temperature can be controlled for steady operation. The latter, along with energy balance, requires that the net heat flow into the working system from the exterior be matched by power production in the work reservoir. Finally, we make the assumption, that heat-exchange at the thermal boundary provides the only source of entropy production.

Recall that in pinch analysis and in exergy analysis it is common to combine all demands at the same temperature and thus use the temperature as a natural parameter describing the various heat exchanges that take place. Since our analysis counts a finite conductance for each heat exchange and this conductance may be different for different streams, even though the temperatures of these streams may be the same, we do not aggregate the positive and negative heat demands. Instead we will make use of a parameter x that can be thought of as a coordinate along the boundary of contact between our AHD and the exterior. We take x to run over a finite interval for a complete circuit. We can conceive of the progression of values of x as the time sequence of the boundary seen by an element of working fluid traversing a cycle, although variations from this interpretation will also prove useful. The geometry of the exterior, apart from its thermal contact with the interior through the boundary, is irrelevant in this framework, so x becomes an effective coordinate for the exterior as well.

Finally, we expressly assume that the control of temperature in each part of the working fluid by energy exchange with the work reservoir is effectively instantaneous and reversible. A more general result that separately includes more factors limiting performance is desirable, but beyond the scope of the present discussion.

3. Table of Symbols and Conventions

The exterior is organized into m sites, indexed by the subscript j in the following symbols. Each site represents a stream of material or a batch reactor that needs to be heated ($\varepsilon_j = +1$) or cooled ($\varepsilon_j = -1$). In problems MEPR and MP only, the last two sites are designated as variable utilities to aid in the heat integration task posed by the fixed rates at the first m–2 sites. When the subscript j is replaced by an X, it refers to site m–1, which is designated as a variable heat source; the subscript Y refers to site m, a variable heat sink. In this same connection, the superscript # is used on some of the symbols below to connote fixed quantities associated with the heat integration task.

ε_j = Parity of site, i.e., +1, if the site is a heat sink, or –1, if the site is a heat source.
Q_j = Aggregate rate of heat flow from site j to the working fluid.
D_j = Aggregate rate of entropy change at site j due to heat flux Q_j.
M_j = Capacity flow rate ($\dot{m}c_P$) of material at site j.
T_j^I = Temperature of the external flow at the inlet of site j.
T_j^F = Temperature of the external flow at the outlet of site j.
$T_j^e(x)$ = Temperature of the external flow at x along the thermal boundary at site j.
$T_j(x)$ = Temperature of the working fluid at x along the thermal boundary at site j.
K_j = Aggregate conductance governing heat transfer at site j.
K = Total conductance.
u_j = Thermal conductivity at site j.
A_j = Aggregate thermal contact area at site j.
A = Total thermal contact area.
P = Power produced in the work reservoir.
D = Total entropy production rate.

General relations: $P = \sum_{j=1}^{m} Q_j$ $\quad D = \sum_{j=1}^{m} D_j$ $\quad K = \sum_{j=1}^{m} K_j$

Defining relations: $D_+ = \sum_{\varepsilon_j = +1} D_j$ $\quad D_- = \sum_{\varepsilon_j = -1} D_j$ $\quad Q^\# = P - Q_X - Q_Y$

$D_+^\# = D_+ - D_Y$ $\quad D_-^\# = D_- - D_X$ $\quad D^\# = D - D_X - D_Y$

$$\Delta^{\#} = KD^{\#} + 4D^{\#}_{-}D^{\#}_{+} \qquad \overline{T}^{e}_{j} = \frac{T^{F}_{j} - T^{I}_{j}}{ln(T^{F}_{j}/T^{I}_{j})}$$

$$\rho_j = \frac{\varepsilon_j}{\sqrt{u_j}} \qquad \xi = \frac{1}{D}\sum_{j=1}^{m} \rho_j D_j$$

A problem specified in terms of $Data1 = \{M_j, T^I_j, T^F_j\}$ can be reduced to knowledge of $Data2 = \{D_j, \overline{T}^e_j\}$ by means of the following general relations.

$$D_j = -M_j ln(T^F_j/T^I_j) \qquad Q_j = M_j(T^F_j - T^I_j)$$

$$D_j = \frac{-Q_j}{\overline{T}^e_j} \qquad \overline{T}^e_j = \frac{T^F_j - T^I_j}{ln(T^F_j/T^I_j)}$$

*Data*1 is the conventional way of specifying heat demand or supply associated with a material flow at a site in a HEN. In all of the problems treated below, the analysis is most naturally done in entropy terms. For that reason we opt for *Data*2. Furthermore, if the site is not a material flow but is instead a reservoir or a batch reactor, the same conversion between heat and entropy flux holds ($Q_j = -\overline{T}^e_j D_j$), where $\overline{T}^e_j$ is now the constant temperature of the reservoir or reactor.

4. Minimum Conductance Problem

Prescribed Quantities: $D_j,\ \varepsilon_j,\ \overline{T}^e_j,\ j = 1, \ldots, m$

Minimum Conductance: $K = \frac{4D_+|D_-|}{D}$

Associated Operating Conditions:

$$T_j = T^e_j \frac{D_+ - D_-}{D_+ - D_- - \varepsilon_j D} \qquad K_j = \frac{\varepsilon_j D_j}{D}(D_+ - D_- - \varepsilon_j D)$$

5. Minimum Entropy Production Rate Problem

Prescribed Quantities: $\overline{T}^e_j,\ j = 1, \ldots, m$

$D_j,\ \varepsilon_j,\ j = 1, \ldots, m-2$

$\varepsilon_X = -1,\ \varepsilon_Y = +1$

Case 1 (MEPR): $\Delta^{\#} \geq 0$

Minimum Entropy Production Rate: $D = \frac{4(D^{\#}_{+})^2}{K + 4D^{\#}_{+}}$

Associated Operating Conditions:

$$D_X = -\frac{\Delta^{\#}}{K + 4D^{\#}_{+}}, \qquad D_Y = 0$$

$$T_j = T^e_j \frac{K + 2D^{\#}_{+}}{K + 2D^{\#}_{+}(1 - \varepsilon_j)}, \qquad K_j = \varepsilon_j D_j \left(\frac{K}{2D^{\#}_{+}} + 1 - \varepsilon_j\right)$$

$$T_X = T^e_X \frac{K + 2D^{\#}_{+}}{K + 4D^{\#}_{+}}, \qquad K_X = \frac{\Delta^{\#}}{2D^{\#}_{+}}$$

Associated Power: $P = Q^{\#} + \overline{T}_X^e \dfrac{\Delta^{\#}}{K + 4D_+^{\#}}$

Case 2 (MEPR): $\Delta^{\#} < 0$

Minimum Entropy Production Rate: $D = \dfrac{4(D_-^{\#})^2}{K + 4D_-^{\#}}$

Associated Operating Conditions:

$$D_X = 0 \qquad D_Y = -\frac{\Delta^{\#}}{K + 4D_-^{\#}}$$

$$T_j = T_j^e \frac{K + 2D_-^{\#}}{K + 2D_-^{\#}(1 + \varepsilon_j)} \qquad K_j = \varepsilon_j D_j \left(-\frac{K}{2D_-^{\#}} - 1 - \varepsilon_j\right)$$

$$T_Y = T_Y^e \frac{K + 2D_-^{\#}}{K + 4D_-^{\#}} \qquad K_Y = \frac{\Delta^{\#}}{2D_-^{\#}}$$

Associated Power: $P = Q^{\#} + \overline{T}_Y^e \dfrac{\Delta^{\#}}{K + 4D_-^{\#}}$

6. Maximum Power Problem

Prescribed Quantities: $\overline{T}_j^e,\ j = 1, \ldots, m$

$D_j,\ \varepsilon_j,\ j = 1, \ldots, m-2$

$\varepsilon_X = -1,\ \varepsilon_Y = +1$

Case 1A (MP): $\Delta^{\#} \geq 0 \qquad \dfrac{\overline{T}_X^e}{\overline{T}_Y^e} \leq \left(1 + \dfrac{4D_+^{\#}}{K}\right)^2$

Maximum Power: $P = Q^{\#} + \overline{T}_X^e \dfrac{\Delta^{\#}}{K + 4D_+^{\#}}$

Associated Operating Conditions: same as **Case 1 (MEPR)**

Associated Entropy Production Rate: $D = \dfrac{4(D_+^{\#})^2}{K + 4D_+^{\#}}$

Case 1B (MP): $\Delta^{\#} \geq 0 \qquad \dfrac{\overline{T}_X^e}{\overline{T}_Y^e} > \left(1 + \dfrac{4D_+^{\#}}{K}\right)^2$

Maximum Power: $P = Q^{\#} + \overline{T}_X^e D_-^{\#} + \overline{T}_Y^e D_+^{\#} + \dfrac{K}{4}\left(\sqrt{\overline{T}_X^e} - \sqrt{\overline{T}_Y^e}\right)^2$

Associated Operating Conditions:

$$D_X = -D_-^{\#} - \frac{K}{4}\left(1 - \sqrt{\frac{\overline{T}_Y^e}{\overline{T}_X^e}}\right) \qquad D_Y = -D_+^{\#} + \frac{K}{4}\left(\sqrt{\frac{\overline{T}_X^e}{\overline{T}_Y^e}} - 1\right)$$

$$T_j = T_j^e\left(1 - \varepsilon_j \frac{\sqrt{\overline{T}_X^e} - \sqrt{\overline{T}_Y^e}}{\sqrt{\overline{T}_X^e} + \sqrt{\overline{T}_Y^e}}\right)^{-1} \qquad K_j = \varepsilon_j D_j \left(\frac{\sqrt{\overline{T}_X^e} + \sqrt{\overline{T}_Y^e}}{\sqrt{\overline{T}_X^e} - \sqrt{\overline{T}_Y^e}} - \varepsilon_j\right)$$

$$T_X = \frac{\overline{T}^e_X}{2}\left(1 + \sqrt{\frac{\overline{T}^e_Y}{\overline{T}^e_X}}\right) \qquad K_X = \frac{K}{2} + \frac{2D^{\#}_{-}\sqrt{\overline{T}^e_X}}{\sqrt{\overline{T}^e_X} - \sqrt{\overline{T}^e_Y}}$$

$$T_Y = \frac{\overline{T}^e_Y}{2}\left(1 + \sqrt{\frac{\overline{T}^e_X}{\overline{T}^e_Y}}\right) \qquad K_Y = \frac{K}{2} - \frac{2D^{\#}_{+}\sqrt{\overline{T}^e_Y}}{\sqrt{\overline{T}^e_X} - \sqrt{\overline{T}^e_Y}}$$

Associated Entropy Production Rate: $D = \frac{K}{4}\frac{\left(\sqrt{\overline{T}^e_X} - \sqrt{\overline{T}^e_Y}\right)^2}{\sqrt{\overline{T}^e_X\overline{T}^e_Y}}$

Case 2A (MP): $\Delta^{\#} < 0$ $\quad \frac{\overline{T}^e_X}{\overline{T}^e_Y} \le \left(1 + \frac{4D^{\#}_{-}}{K}\right)^{-2}$

Maximum Power: $P = Q^{\#} + \overline{T}^e_Y \frac{\Delta^{\#}}{K + 4D^{\#}_{-}}$

Associated Operating Conditions: same as **Case 2 (MEPR)**

Associated Entropy Production Rate: $D = \frac{4(D^{\#}_{-})^2}{K + 4D^{\#}_{-}}$

Case 2B (MP): $\Delta^{\#} < 0$ $\quad \frac{\overline{T}^e_X}{\overline{T}^e_Y} > \left(1 + \frac{4D^{\#}_{-}}{K}\right)^{-2}$

Results have the same functional form as **Case 1B (MP)**.

7. Minimum Thermal Contact Area Problem

Prescribed Quantities: $D_j,\ \varepsilon_j, u_j,\ \overline{T}^e_j,\ j = 1, \ldots, m$

Minimum Thermal contact Area[1]: $A = -\frac{1}{2D}\sum_{i=1}^{m}\sum_{j=1}^{m}(\rho_i - \rho_j)^2 D_i D_j$

Associated Operating Conditions:

$$T_j = T^e_j \frac{\xi}{\xi - \rho_j} \qquad A_j = \rho_j D_j (\xi - \rho_j)$$

So long as we require that $\xi > \rho_j$ for all j, we can show that none of the obvious physical constraints, $A_j \ge 0$, become active in the optimization, and the above description of operating conditions is valid. If that requirement is violated, the operating conditions associated with minimal thermal contact area are more subtle, and require more discussion than can be supplied here.

We remark that a notion of **minimum cost of thermal contact** can be determined by the same formalism as minimum thermal contact area. In fact, suppose that in addition to the quantities prescribed above we include a price per unit area at each site: $p_j,\ j = 1, \ldots, m$, so that the cost of thermal contact at the jth site is $C_j = p_j A_j$. Then the minimum value of the total cost, $C = \sum C_j$, is given by the same expression given above for A, where ρ_j is redefined by the formula $\rho_j = \varepsilon_j\sqrt{p_j/u_j}$. Furthermore, the optimal allocation of cost, C_j, to the jth site is given by the same expression given above for A_j.

8. Examples

[1] This can be shown to be positive despite the negative sign in front

Suppose the task is to cool a stream with M_1 = 45 Js^{-1}°K^{-1} from T_1^I = 800 °K to T_1^F = 600 °K (Q_1 = 9000 Js^{-1}) and to heat a stream with M_2 = 27.5 Js^{-1}°K^{-1} from T_2^I = 200 °K to T_2^F = 400 °K (Q_2 = –5500 Js^{-1}). The **minimum total conductance** with which this task can be accomplished is K = 161.4 Js^{-1}°K^{-1}. The associated values of the entropy production is D = 6.119 Js^{-1}°K^{-1} and the power produced is P = 3500 Js^{-1}.

Suppose we now augment the previous data by specifying the total conductance K = 200 Js^{-1}°K^{-1} and allowinging variable heat utilities: a source at $\overline{T}_X^e$ = 700 °K and a sink at $\overline{T}_Y^e$ = 300 °K. The **MEPR** is D = 5.261 Js^{-1}°K^{-1}. The associated flows at the utilities are Q_X = 598.3 Js^{-1} and Q_Y = 0.0 Js^{-1}. The associated power is P = 4098.3 Js^{-1}.

Now suppose the objective is maximizing power at the work reservoir instead. The **MP** is P = 4330.7 Js^{-1}. The associated flows at the utilities are Q_X = 3025.1 Js^{-1} and Q_Y = –2194.4 Js^{-1}. The associated entropy production rate is D = 9.109 Js^{-1}°K^{-1}.

9. Conclusions

We have presented a framework for the study of heat integration that goes beyond standard exergy analysis to include irreversibilities associated with finite heat conductance. We have stated the results of a sample of four extremal problems whose analysis our methodology permits.

Part of the motivation for this work was provided by the problem of servicing a distillation column in which the delivery and extraction of heat along the column of trays is to be orchestrated so as to maintain a prescribed temperature profile along the column. Other considerations have revealed what that profile should be to minimize the exergy loss associated with the transport of material within the column [9]. The present paper gives in-principle limits to the exergy costs associated with powering the column. How to achieve real devices approximating the idealizations in our calculations is an important problem in mechanical engineering.

10. References

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