Our understandings of what it is to be human—basic aspects of social life, thinking, learning, and productive activity—have historically been enmeshed with the development and use of new technologies. For example, over the history of scientific psychology (Genter & Grudin, 1985; Leary 1994) we have understood cognition by analogy to hydraulic systems, clockwork mechanisms, symbolic computation, dynamically configured networks (Thelen & Smith, 1994), or distributions across cultural and technical media (Hutchins, 1995). Our understandings of cognition and learning have also been shaped by the technologies researchers use to capture and analyze data (Danziger, 1990). The "linguistic turn" in the social sciences was largely made possible by audio recordings of human speech and conversation. Similarly, the recent growth of interest in multi-modal aspects of communication have been enabled by high quality video recording of human activity (e.g., Levine & Scollon, 2004), and claims for multi-modal mental simulation as the basis of cognitive processing have been advanced by new developments in brain imaging (e.g., Barsalou, in press; Gallese & Lakoff, 2005). In a parallel but (as yet) largely unexplored development, advances in location-aware technologies and spatial analysis are making it possible to study human activity and learning in natural settings and at larger spatial and temporal scales (Christensen, 2003; Miller, 2007). For example, wearable GPS devices can be used to capture personal space-time paths, and a growing suite of GIS software tools can be used to analyze path structures in relation to other spatially anchored data (e.g., the accessibility of cultural resources that provide children with learning opportunities).

Three of these recent developments—human communication as multi-modal interaction, human thinking as multi-modal simulation of sensory-motor activity, and embodied experience as following/producing pathways through social and cultural space—challenge a traditional view of human cognition and learning as information processing, in which the mind is a computational device designed to process a-modal symbolic structures that are separated from bodily experience in the world. But these new perspectives can also be difficult to reconcile or integrate, since they have arisen in research communities that are historically separate, or even embroiled in controversy over the nature of human cognition and learning. Finding a productive way forward is particularly important for new research in mathematics and science education, which is subject to recurring demands for the reform of traditional instruction (e.g., moving away from a didactic pedagogy of symbolic manipulation), yet has found little guidance from basic research on human cognition and learning in how to do this.

Toward a theory of embodied mathematical cognition for learning and teaching. We propose research that takes advantage of technical advances in multi-modal and spatial analysis to develop new theories of embodied mathematical cognition and learning. Three university groups will conduct a coordinated series of empirical and design studies that focus on learning the mathematics of space and motion. Studies will be conducted in professional workplaces and formal, academic settings where people learn and teach these subject matter areas.

The aim of our research is to advance understanding of these basic questions about learning and teaching through the development of a theory of embodied mathematical cognition. Such a theory, grounded in experimental studies, will help us to identify promising implications for mathematics teaching and learning as well as to articulate useful perspectives on the nature of mathematical knowledge for teachers and curriculum designers. A theory of embodied mathematical cognition will contribute to broaden the range of activities and emerging technologies that count as mathematical and to envision alternative forms of bodily engagement with mathematical problems. This broadening is particularly important at a time in which our society faces persistent achievement gaps between groups of students from different cultural backgrounds or from families with different economic standing (Ladson-Billings, 1995; Moses & Cobb, 2001; Rosebery, Warren, Ballenger & Ogonowski, 2005). Finding new ways to teach powerful mathematical ideas, including uses of these ideas that have engaging social relevance, is an urgent societal objective. Lakoff & Núñez (2000) have pioneered work toward a theory of embodied mathematical cognition. However, their analysis applies cognitive linguistics to mathematical texts; instead, the work that we propose focuses on the experimental study of subjects interacting and engaging in mathematical problem solving. We want to take into account in far greater detail the complexity and nuance of the context the participants experience as they talk, gesture, draw symbols, point at symbolic expressions, and sketch diagrams.

Various forms of individual constructivism, claiming that knowledge is constructed through purposeful activity by the learner as opposed to being "poured" into her mind by a teacher, have influenced the mathematics education reform movement of the '80s and '90s. This movement strives to shift mathematics teaching toward intuitive understanding and away from a predominate focus on symbol manipulation. While progress has been made in areas of instructional design, teaching, and assessment (e.g., English, 2002), a major weakness of this reform movement has been that researchers and designers have not, in most cases, reconsidered the nature of mathematics in relation to how human cognition is thoroughly embodied. Thus, while it is possible to create classroom participation structures for comparing alternative student strategies on a given mathematical problem, we do not learn anything new or inspiring by postulating that having a mathematical strategy indicates the possession of a corresponding mental scheme or cognitive structure. We need to re-open the fundamental question of what fluency with a mathematical strategy is about in the first place, including how learners communicate about it with peers or their teacher. To illustrate how ideas from embodied cognition cast new light on the nature of mathematical understanding we will elaborate on two of them: 1) Grounding of abstraction in perceptuo-motor activity, and 2) Cognition is for action.

Grounding of abstraction in perceptuo-motor activity. Building from Gibson's theories of active or direct perception (1979), visual perception is no longer understood as input to a central processor, but as part of a body in activity and oriented toward further, unfolding action. As Zeki (1993) argues, the science of visual perception has moved out of a narrow framing of vision as split between low-level perception/action and higher-level intelligence. Instead, vision appears to cut across these levels, using capacities specialized to different types of judgment (e.g., faces, shapes, colors, motion), where all involve diverse bodily activities, such as reaching, grasping, or avoiding. Perception does not happen in an input mode: That which one cannot understand one cannot see, and we see to the extent that we understand. Losing the capacity to see color, for example, is also losing the capacity to *understand* color (e.g., to imagine colors, to use color words properly; see (Sacks & Wasserman, 1987). What we know as principles of logic is embedded in complex ways throughout perceptual-motor activity. When we see, for instance, an ambiguous figure switching from a face to a vase (and vice versa), the switch is inherent in the visual system, as opposed to the latter "consulting" a logic module and getting back the mandate to choose one or the other in order to comply with a cognitive law of non-contradiction. This conception shifts the locus of "thinking" from a central processor to a distributed web of perceptuo-motor activity; therefore, it suggests that people ground seemingly abstract concepts in modality-specific, sensory-motor systems. Concepts, under this view, involve "modal reenactments" that are constructed on the fly for particular situations of use (see Barsalou, 2003; in press). Thus thinking with abstract concepts may involve enacting, in covert ways, sensory-motor activities that were originally part of learning whatever concept or subject one thinks about. It is as if we think (i.e., remember, imagine, anticipate, etc.) by "simulating," in embryonic form, bodily actions and affects that had been relevant to the subject matter in our past experiences using tools, symbols, and techniques available in our cultural milieu (Hostetter & Alibali, in press). This claim points at a long and substantial tradition of work in psychology, tracing back to William James, which was largely displaced by the rise of behaviorism and information-processing approaches.

This interweaving of abstraction with perceptuo-motor activity poses the need for specialized research in the case of mathematics. Perceptuo-motor activities with balls and screwdrivers are easily apparent, but what about perceptuo-motor activities with proof or limits? Many mathematical entities must be imperceptible and intangible by definition (e.g., a geometric straight line has no width). Neuroscience and behavioral studies usually investigate the imagining of situations that could occur in regular spatial, temporal, and interpersonal contexts of life. How can we imagine situations that cannot be realized in the 3-dimensional space and temporal flow characteristic of human perceptual and motor activity? This act of imagining is a common occurrence in mathematics. The uses of spaces of more than 3 dimensions, or even infinite dimensions, are frequent in mathematical work. Surfaces that cannot be smoothly realized in physical space (e.g., hyperbolic surfaces), appear in countless mathematical problems. Zeno paradoxes start by describing situations easy to imagine (e.g., a race between a fast runner and a tortoise) and then, through the introduction of an *infinite* number of operations, each one easy to imagine, lead us to conclusions that seem to contradict our common experience.

Cognition is for action. One of the central tenets of embodied cognition is that cognition is for action (Glenberg, 1997), and that most things are understood by the actions we can perform with them (Gibson, 1979) or by simulating the actions that underlie them (Glenberg and Kaschak, 2002). Recent research on language comprehension shows that comprehension is reduced when the actions made by the reader are contrary to the actions implied by a text, but enhanced when the reader's actions and those in the text are compatible. For example, Zwaan and Taylor (2006) asked readers to turn a dial clockwise or counterclockwise to advance through a text. If the meaning of a phrase (e.g., "he turned the volume down) conflicted with the required hand movement, reading of that phrase was slowed. Reading comprehension can be enhanced even when the compatible actions are merely imagined (Glenberg, in press).

These advantages extend to mathematical problem solving. In arithmetic (Glenberg et al., 2006), manipulating toy scenarios improved children's word problem solving, and benefits were evident more than a week later. Furthermore, the benefits transferred to conditions in which children simply imagined the manipulation. In geometry as well as arithmetic, early elementary students who use manipulatives for extended periods of time perform better on achievement tests than those who do not (Sowell, 1989). In various problem domains, people use different strategies when action is allowed vs. not allowed (Alibali & Kita, under review).

Inability to perform or imagine actions also leads to impoverished math problem-solving performance. In a well-established finding, children perform better on Combine and Change problems (*Lyn has 9 apples. She gives 3 apples to Jon. How many apples does Lyn have now?*) than on Compare problems (*How many more apples does Lyn have?*) (e.g., Carpenter & Moser, 1984; De Corte & Verschaffel, 1987). Performance differences have been explained by differences in semantic structure among the problems (Kintsch & Greeno, 1985). From an EC perspective, we suggest that the differences can be explained by the salience of action. Those problem types that explicitly identify actions (Change, Combine) evoke motor programs that recruit more cognitive resources than those problems that emphasize relations among problems but signal no overt actions (e.g., Compare items). In the proposed work we investigate the role of action and its relevance for understanding geometric proof and spatial reasoning.

A program of comparative research on learning and teaching the mathematics of space and motion. To study the nature of embodied mathematical cognition we will focus on the mathematics of space and motion, a domain that has wide-ranging relevance for what children need to learn in school, and that presents particularly interesting challenges for a theory of embodied cognition. We take two perspectives on this domain, arguing that we need to understand learning and teaching from both perspectives. The first is a formal or idealized perspective on the mathematics of space and motion, which includes geometry as well as ideas of calculus and differential equations as critical resources for the mathematical description of motion. The second is an informal or practical perspective on space and motion that is informed by cartography and applied problems of spatial analysis. Each perspective is linked to the other: idealized mathematical entities are used to describe processes or structures in practical application, and these applications, which are diverse and constantly changing, make new demands on mathematical idealizations.

Mathematical understanding of space and motion is likely to be grounded in our bodily-based senses of space and time, which makes it particularly rich for the study of embodied cognition. For example, Kant famously articulated the thesis that space and time are inherent in our cognitive capacities (i.e. "synthetic a-priori"). Yet the development of non-Euclidean geometries should then be taken as evidence that Kant's thesis cannot be true, since if Euclidean geometry were an articulation of human cognition itself, non-Euclidean geometry would be incomprehensible. We treat this as an empirical (not logical) question about the nature of mathematical practices, rather than assuming that properties of space and motion are "set" by our cognitive apparatus. We expect a great deal of plasticity for learner's intuitions about space and time, largely influenced by the representational technologies with which they become fluent.

For example, studies in cognitive neuroscience have shown that instead of the commonly assumed homogeneous isotropic space (e.g., metric scale is preserved in all orientations from the perceiving subject), humans appear live and think in a multiplicity of heterogeneous and anisotropic spaces (Farne & Ladavas, 2002; Làdavas, 2002; Ladavas, Di Pellegrino, Farne, & Zeloni, 1998; Maravita & Iriki, 2004; Vaishnavi, Calhoun, & Chatterjee, 2001). Under this view, our bodies constitute at least three, qualitatively different spaces: proprioceptive, peripersonal, and extrapersonal spaces. Proprioceptive space is the space of our organs, space that is sensed or *felt* through the movement of muscles and joints. Peripersonal is the space of eye-hand coordination, encompassing the region of what is *graspable* by an acting body. Extrapersonal space is more remote, constituted by things that are out of reach but to which we can *visually attend* through oculomotor activity. We would propose two other spaces as well, each qualitatively different in forms of mediation: navigable space and mapped space. Navigable space need not be visually available, but it involves locations and paths for moving between those locations—it is *reachable* in some practical fashion. In this sense, one might be able to find a familiar store in a shopping mall (or make sense of directions given as a path-ordered narrative when finding a new store), without either being able to see the desired location or its location on a spatial image drawn using map conventions. Mapped space uses cultural conventions (e.g., for precise location in metric space) to bring diverse, potentially unreachable spaces into a common medium for *comparison*. In this sense, locational data about things that cannot be grasped, seen, or even reached (e.g., patterns of human immigration, climate processes) can be drawn together on a surface for visual (or tactile) inspection (Latour, 1987). Mediating relations in navigable and mapped space are particularly important because they support shifts between qualitatively different types of space in support of human cognition and learning. For example, use of a tool like a computer mouse can extend our peripersonal space to encompass regions that otherwise would be extrapersonal. Similarly, use of conventions associated with mapped space can bring widely distributed things or processes into the realm of what is can be seen or grasped and manipulated. In this sense, a microscope can bring normally invisible regions to our extrapersonal space, or a graph of physical processes can bring processes that could not be experienced at a human scale onto a surface for tracing and re-arrangement with one's fingers (Ochs et al., 1994). These examples suggest that the plasticity of our intuitions about space and time is an extension of the plasticity of the body engaged in tool-use.

In summary, we will focus on embodied mathematical cognition in the mathematics of space and motion from two perspectives. From a formal or idealized perspective, we will ask how mathematical concepts in geometry and calculus provide resources for describing and manipulating space and motion. From a practical or applied perspective, we will ask how cartography and diverse methods of spatial analysis provide a means for using embodied cognition to understand or take action in a variety of practical domains. In both, we will be concerned with phenomena of learning and teaching. Embodied mathematical cognition under the first perspective (formal, idealized) will be studied using observational methods, supplemented with design and teaching experiments. These studies will follow academic mathematicians working with their graduate students at the university level, pre-service teachers in mathematics education courses also at the university level, and teachers working with their geometry students at the high school level. Embodied mathematical cognition under the second perspective (informal, professional or practical applications) will also be studied with observational methods (e.g., ethnographic and cognitive case studies in urban and regional planning, census and demographic research), with more limited use of design and teaching experiments in the context of high school vocational education (i.e., pre-engineering) classes. While both perspectives deal with the nature of space and time, the former is traditionally a privileged strand of pure mathematics and the latter a realm of applied mathematics critical for all kinds of human activity.

Research Questions and Prior Work

A network of researchers from three universities will collaborate to investigate 4 research questions. These questions are critical for developing a theory of embodied mathematical cognition that is relevant for mathematics educators concerned with learning and teaching.

Question 1, Grounding: *How is mathematical thinking and communication grounded in actual or simulated perceptuo-motor activity?* To address this question we will develop an empirically based account of grounding, as it pertains to learning (Barsalou, in press), the establishment of common ground (Clark, 1996), and shared intentionality (Tomasello & Carpenter, 2007).
Question 2, Action: *What is the role of action in learning and teaching the mathematics of space and motion?* To address this question we will reframe geometric thinking and spatial reasoning from an action-based account of cognition, and explore the implications this poses for learning, instruction, and professional development in schools and the work place.
Question 3, Modality and Scale: *How do changes in scale and modality provide resources for (or impede) learning and teaching the mathematics of space and motion?* With a firm analytic foundation for the concepts of grounding and action, we can then explore a third central question, how grounding and action change in response to changes in resources, modality and scale (e.g., from scaled-down drawings, to walking in spaces, to community-sized representations).
Question 4, Learning to "See:" *Within a theory of embodied mathematical cognition, how do people "learn to see" as mathematicians or as professionals working with the mathematics of space and motion?* Part of learning is "learning to see" like a professional and thereby reason like one (Goodwin, 1994; Stevens & Hall, 1999). These investigations will provide us with guiding principles to inform the design of activities and learning environments that can be used to engineer changes in mathematical reasoning and communication among nascent teachers and working professionals.

These questions were identified during collaborative work on an NSF funded "catalyst award," which we describe first, followed by a description of some of our other recent NSF awards.

Catalyst Proposal for a Center for Research on Embodied Mathematical Cognition, Technology, and Learning (NSF Award Number 0518146, 9/15/2005–8/31/2007, Rogers Hall, PI and Ricardo Nemirovsky, Co-PI). The purpose of this project was to bring together new theories and empirical studies from several different disciplines toward the development of a Science of Learning Center on Embodied Mathematical Cognition. These included:

(1) Research in cognitive science, including cognitive neuroscience, concerned with how the body is involved in thinking and learning about mathematics;

(2) Mathematics education research concerned with how mathematics teachers and learners use new representational technologies to support inquiry in classrooms; and

(3) Research in the social studies of technology and science concerned with how inquiry is accomplished in the ongoing, practical activity of professional workplaces.

This catalyst grant supported five distance seminars (video and phone conference), three face-to-face meetings (at Vanderbilt University in December 2005; San Diego State University in June

2006; and at the Center for Advanced Study in Behavioral Sciences, Palo Alto, CA, in December 2007). Two small pilot studies were conducted, one (at SDSU) on the learning of acceleration, and another (at VU) on the use of GPS technology to record human mobility. In addition, a symposium on Embodied Mathematical Cognition was organized at the 2007 Annual Meeting of the American Educational Research Association, in April, 2007, Chicago, IL. Papers presented at the meeting are currently being prepared for submission as a special journal issue on embodied mathematical cognition. Paper topics include analysis of multi-modal discourse in teaching and learning statistics (analysis led by Hall), analysis of gesture use in middle school mathematics classrooms (analysis by Alibali and Nathan), analysis of a design experiment for learning about the concept of acceleration using a novel haptic interface (analysis led by Nemirovsky), and a theoretical commentary on the role of the body in learning mathematics (Reed Stevens, University of Washington, serving as an advisor on the current proposal). The current proposal is one result of this catalyst award. No proposal for a Center of Learning Sciences was prepared because the corresponding RFP has not been renewed by the foundation.

Understanding and Cultivating the Transition from Arithmetic to Algebraic Reasoning (IERI/NSF Award Number 0115661, 2001-2006, Mitchell Nathan, PI and Martha Alibali, Co-PI). Studies completed under this award show that math teachers engage in a variety grounding acts—including the use of hand gestures, writing and speech—and the frequency of these acts increases when the ideas are novel, the representations are more abstract, and students pose questions (Alibali & Nathan, 2007a). More generally, our work suggests that speakers' gestures reveal the actions (real and simulated) and physical metaphors that ground their understanding of ideas (Hostetter & Alibali, in press). Furthermore, speakers' deictic gestures reveal the concrete and metaphorical links speakers make to the physical and imagined environment (Alibali & Nathan, 2007a). Thus, gestures manifest speakers' embodied understanding of concepts, and communicate these embodied understandings to their listeners. Of most relevance to the current proposal, teachers' gestures contribute to instructional communication in ways that matter for students' learning (e.g., Valenzeno, Alibali & Klatzky, 2003).

In any interactional setting, it is essential for speaker and listener to establish intersubjectivity, or "common ground." This is especially true in classrooms where progress on collaborative activities depend on successful communication. In a recent multi-modal analysis of whole-classroom mathematics problem solving (Nathan, et al., 2007), common ground was identified as a central influence on the structure of the discourse and the evolution of the problem representations that were publicly presented. Disagreements fostered critical dialogue and led students to articulate their disparate positions in more sophisticated ways. In a related study, Nathan & Alibali (2007b) explored the establishment of common ground when teachers introduce new, abstract representations and systems of notation in middle school algebra. Two gestural forms were critical for effective teaching. First, linking gestures guide attention to correspondences between the familiar and new representations. Second, gestural catchments repeat action features (e.g., recurrent hand shapes) to convey similarity and highlight conceptual connections across equations with different forms and at different locations within the classroom.

Proposed Studies

Comparative research on learning and teaching the mathematics of space and motion.

We propose to study learning in mathematical activities that occur in a variety of settings. Our focal settings (see Table 1) contrast (a) theoretical and applied forms of activity along one dimension, and (b) formal (i.e., school-based) versus informal (work-based) education along the other dimension. Theoretical settings typically emphasize the importance of formal representations and methods of universal proof and justification, while applied settings tend to stress circumstance-specific problem solving, reliability, and resource allocation. Formal education tends to promote learning through instruction, while informal education emphasizes learning-by-doing (Brown, Collins & Duguid, 1989) and learning through legitimate peripheral participation (Lave & Wenger, 1991).

Table 1. Settings and methods for comparative studies of embodied cognition and learning for the mathematics of space and motion.

	Focus on Theory	Focus on Application
Informal learning (graduate school, workplace)	(S1) Video recordings of working sessions and interviews with mathematics professors and graduate students (SDSU)	(S2) Connective ethnography and video recordings of work groups using mapping and spatial analysis (Vanderbilt)
Formal learning (secondary, undergraduate instruction)	(S3) Observations of teaching and video recordings of study groups with in secondary geometry classrooms. (UW) (S5) Experiments in secondary geometry classrooms comparing high/low action instruction (UW) (S7) Design experiments in teaching the mathematics of space and motion with pre-service secondary mathematics teachers (SDSU)	(S4) Observations of teaching and video recordings of study groups with in pre-engineering (vocational) classrooms. (UW) (S6) Experiments in pre-engineering (vocational) classrooms comparing high/low action instruction (UW) (S8) Design experiments in the mathematics of space and motion through mapping and spatial analysis with talented middle and high school youth (Vanderbilt, summer program)

These distinctions between theory and application, and formal and informal education are common among educational researchers and within the STEM disciplines, and provide an opportunity to see important variations and target common aspects of a broad range of mathematical activity. We appreciate that these distinctions are not always clean-cut; that, for example, applied work can and frequently has led directly to theoretical advances (e.g., Stokes, 1997), and that practices typical of informal education can be found in formal settings (e.g., Schofield & Evan-Rhodes, 1989) and vice versa.

Eight studies are conducted in the settings described in Table 1 to address the four research questions that we consider central to an embodied theory of mathematical cognition.

Studies of informal learning in academic and professional settings (Studies 1 & 2).

(S1) Studies with professional mathematicians. These studies will focus primarily on research questions 1 and 4. The data collected will be of two different types: 1) Videorecordings of conversations between a mathematician and a mathematics graduate student who will get together for a 45-minute session standing next to a large whiteboard. The mathematician will explain to the student a paper selected in advance. The paper will be close to the area of specialization of the mathematician and involve ideas that do not fit within the perceivable constraints of physical space and time (e.g. infinite sequences, n-dimension spaces). The student will spontaneously ask questions and pose comments to elucidate understanding of the paper and to draw implications for other areas of mathematics. 2) Viderecordings of a mathematician teaching graduate students. The class could be lecture-based or set as a discussion seminar.

The doctoral program in mathematics and science education offered jointly by San Diego State University and University of California–San Diego offers an ideal institutional context for the arrangement of these sessions. The program involves the Math Departments of both universities, and the program's Mathematics Education doctoral students are required to have a master's degree in mathematics.

Regarding Research Question 1, these videorecordings will allow us to investigate the issue of grounding in sensori-motor systems: How do we imagine mathematical situations that cannot be realized in the 3-dimensional space and temporal flow characteristic of human perceptual and motor activity? Since the topics of these sessions will involve mathematical entities that are by definition imperceptible and intangible, how does the use of representational technologies (e.g. algebraic expressions, geometric diagrams) enable us to bodily inspect them? With respect to Research Question 4, the recorded interactions will in all cases include an experienced mathematician and one or more graduate students in mathematics. These interactions will provide data for investigating the graduate students' process of "learning to see" as they transition toward becoming professional mathematicians.

The videotaped sessions will be transcribed and indexed. Segments will be chosen for microanalysis on the basis of their richness in the use of symbols and the gestural enactment of actions applied to the symbols. The microanalysis of the selected segments will be based on the extended annotation of the transcript, specifying the gestures, pointing acts, tones of voice, and gaze shifts that accompanied talking and listening. The analysis will draw on past work of Nemirovsky and his colleagues studying interactions among students, interviewers, and teachers (Nemirovsky & Monk, 2000; Nemirovsky, Tierney, & Wright, 1998; Noble, Nemirovsky, Wright, & Tierney, 2001; Rasmussen, Nemirovsky, Olszewski, Dost, & Johnson, 2004).

Work with professional mathematicians will be conducted during Spring of Year 1 and of Year 3. The focus of the first cycle of work will be to elucidate the use of "depictive" representations (e.g. graphs, geometric diagrams, projections of 3D shapes into 2D spaces). Depictive representations have in common that they define a continuous representational space on which entities are positioned and displaced, so that there is a type of iconic relationship between movements in this space and movements in the space they are supposed to represent (Nemirovsky, 2005). The second cycle of work (Spring of Year 3) will focus on the use of "signitive" representations (e.g. algebraic expressions, matrix notations, vector operations). Signitive representations locate signs on a space that does not have a common metric across it (e.g., There is no difference between typing an equation at the top or in the middle of a page). Complexes of signs (e.g. equations) get distributed in discrete positions and the user is allowed to navigate them by "jumping" from one to another in ways that comply with appropriate mathematical rules (Nemirovsky, 2005). By differentiating between depictive and signitive representations, these studies will allow us to investigate aspects of Research Question 3 ("How do changes in scale and modality provide resources for (or impede) learning and teaching the mathematics of space and motion?") because they involve changes in modality.

(S2) Studies of professionals engaged in mapping and spatial analysis. During the first three years of this award, Vanderbilt University (VU) investigators (Hall & Leander) will conduct a series of comparative case studies of how professionals use mapping and spatial analysis in selected domains of application. These studies will focus on Research Questions 3 and 4 (shifts in modality, learning to "see" as a professional), though field data will also provide a corpus of recordings that will be used across the research network to study grounding and action (Questions 1 and 2). These studies will use field observations, video recording, and interviews to build "connective ethnographies" (Leander & McKim, 2003; Leander, in press) of how people learn and make innovative contributions while working on projects that involve spatial modeling, inference and design. These methods combine ethnographic and cognitive analysis (Hall, 2001, 2007) with new approaches to learning trajectories and knowledge distributions in physical and information spaces. These new approaches are made possible by location-aware technologies,

including GPS handheld and wearable devices, which provide spatial data used to augment ethnographic and cognitive analyses (e.g. video recording and close analysis of physical settings of project work (Hall & Leander, 2007)). During the first year, VU investigators will negotiate access to research groups working in several organizational contexts. These include a new GIS and spatial analysis laboratory on the VU campus (led by a physical anthropologist, but with a diverse user community), a regional census research center with an active GIS consultant (also on the VU campus), and research groups in geosciences or conservation planning at universities and private foundations in the mid-south region (e.g., we have recently started discussions about this work with planning consultants at The Nature Conservancy). We plan to conduct three detailed case studies, each involving multiple participants engaged in activities of spatial analysis. Our field data collection and analysis will focus on how mathematical idealizations of space and motion are used to describe and make inferences about processes distributed over space and time (e.g., in the VU laboratory, a physical anthropologist might study hybridization of material structures and spatial practices in the Spanish colonization of Andean people; in a non-profit organization like The Nature Conservancy, a biologist mapping human use impacts on habitat/species relations and designing regulatory schemes for conservation of natural resources). We are particularly interested in forms of spatial analysis that focus on human mobility and action as part of broader, ongoing physical processes that are distributed over space (i.e., complex map surfaces) and time.

Based on research in other areas of complex modeling and inference , such as Hall's studies of learning in the context of statistical consulting across diverse client research domains (Hall, Wright & Wieckert, 2007; Hall, Wieckert & Wright; 2008), we expect that learning in these case studies will have substantial informal components, as new members of a research group learn "at the elbow" of old-timers (Lave & Wenger, 1991), and open questions in research projects lead to novel uses of existing methods (e.g., researchers regularly borrow and extend structures for modeling and inference from other groups). We expect our field materials to provide rich data for analyzing the multi-modality of thinking and communication at a micro-interactional level, on the one hand (Research Questions 1, 2 and 3), and provide manageable historical cases for analysis of learning and conceptual innovation within research groups, at a larger-grained level of analysis (Research Question 4). Selections from video recordings, documents produced and used in spatial analysis, and retrospective interviews with participants will be made available for analysis by investigators on the SDSU and UW-Madison campuses, as well. We also expect to document and understand, at a cognitive level, distinctive analytic practices (e.g., particular mapping conventions, tied to forms of analysis that have been powerful in specific application domains) that will inform design experiments conducted in other parts of the proposed research (e.g., for designing and iteratively refining a summer course on mapping and spatial analysis for middle and high school students, also at VU).

Studies of formal learning in classroom and laboratory settings (Studies 3 – 8).

(S3, S4) Observational studies in secondary geometry and vocational education classrooms. UW-Madison investigators (Nathan & Alibali) will conduct observational studies of secondary geometry and vocational education classrooms and scripted tutoring sessions. These studies will focus on grounding and action in teaching and learning, with experimental comparison of instructional conditions that are highly physically active (e.g., geometric construction or building and debugging digital circuits) versus relatively signative and low-action tasks (e.g., manipulating symbolic expressions of writing/following proofs). We seek a deeper understanding of how learners assign meanings to new representations and concepts, guided by the first research question of *what it means to ground abstract representations and novel ideas so they have meaning for participants in mathematical discourse.* To address this question, we will analyze videos of whole-classroom instruction and tutoring interactions in two contexts: high

school geometry and high school vocational education. Both contexts involve the use of teacher-led instruction, peer collaboration, technology (e.g., Geometer's Sketchpad, Fischertechnik simulation software for circuit design and analysis). In each context, we plan to analyze six classroom lessons as well as 18 tutorial interactions (six teachers each working with three students). The geometry tutorials will focus on proof, while the vocational education tutorials will focus on digital circuit design.

From this library of classroom and one-on-one instructional interactions, we will identify segments of video that involve assigning meaning to abstract representations and novel concepts. For these segments, we will conduct quantitative content analyses that examine how grounding is accomplished, with particular attention to actions and manipulation of physical objects, gestures that simulate actions and index objects in the physical environment, and drawing and writing (see Alibali & Nathan, 2007a). We will also carry out qualitative, discourse-level analyses that provide rich, contextual analyses of the multi-modal nature of the interactions and grounding behaviors (see Nathan, Eilam & Kim, 2007). We will also examine when concepts and representations are grounded, contrasting actions in early and later portions of lessons and tutorial sessions, during the introduction or review of material, and in response to student queries.

We hypothesize that careful analysis of the multimodal communication (gestures, writing/drawing, object-use) during mathematical activity (among professionals, mathematicians, teachers and students) and classroom learning (in academic and vocational ed settings) will reveal how learners and workers ground new ideas and representations in natural settings. Speakers' gestures and manipulations of objects reveal the actions (real and simulated) and physical metaphors that ground their understanding of ideas, and their deictic actions (pointing, showing) reveal concrete and metaphorical links to the physical environment (Alibali & Nathan, 2007b). We further hypothesize that ideas and representations that are grounded for people will show up in later, "off-line" communication and performance as simulated actions that mimic the on-line actions evident during the original grounding events (Hostetter & Alibali, in press). Thus, we aim to compile a set of rich, video-based exemplars and develop a coding system to operationalize grounding, and thereby advance our understanding of how people assign meaning to new concepts and abstract representations.

(S5, S6) Experimental studies in secondary geometry and vocational education classrooms. We also seek a deeper understanding of the role of action in geometry and spatial reasoning more broadly, guided by the second research question that asks about the role of action in learning. We specifically ask whether an understanding of this role can yield insights into why some aspects of geometry are harder to learn than others. To address this question, we will carry out two experiments, one focused on academic geometry, and one on geometric and spatial reasoning in pre-engineering. Both experiments will investigate the role of action, by comparing learning and performance in high-action and low-action contexts. We hypothesize that tasks that explicitly call for action (e.g., proof by construction, debugging for logic circuit reliability) facilitate both the understanding of the justification process and the mathematical concepts being proved, while those that emphasize static relations over actions (e.g., proof by deduction, logic analysis using propositions and truth tables) result in impoverished understanding of the proof process and the concepts being proved.

Each experiment will utilize a simple two-group, within-subjects design (N = 30 participants), comparing learning and performance in a high-action context and a related, low-action context. In the high-action tasks, the actions are enacted physically and perceptually (i.e., modally), while in the corresponding low-action tasks, actions are performed on logical (amodal) entities. In both experiments, we will examine amount of learning (using a pre-test/posttest design) and transfer, as well as conduct qualitative analyses of participants' multimodal communication about the tasks (i.e., uses of actions, simulated actions, and inscriptions in defending their reasoning).

(S7) Design experiments with pre-service high school mathematics teachers. These

studies will be based on SDSU courses that Nemirovsky teaches on geometry (MATH 302) and on the use of technology for high school mathematics teaching (MATH 509). The latter one is for students who major in mathematics and who intend to become certified as high school mathematics teachers. Many of the same students take the geometry course as well. The courses are taught in a special classroom (Learning Research Studio, AH 1112) equipped with state-of-the-art technology and advanced videorecording capabilities. Usually the classes are organized in sequences of small-group work and whole classroom discussions. In addition to whole-class discussions, we will videotape a selected team of students interacting in a small group setting. An important component of these courses will be the design of "tangible materials/devices" in the form of objects or installations. Students may choose to use these tangible materials and devices as exhibits for others to learn about mathematical concepts taught in class, as pieces of art, or as mathematical games. In order to support this process of design we will use a machine shop setup by the Mathematics Department at San Diego State University ("Mathematics Technology Laboratory"). This machine shop is completing its equipment with computer-controlled machines that cut or engrave materials (e.g. wood, plastic, cardboard, fabric) on the basis of drawings that students can generate with a CAD program. This proposal's budget includes a part-time technician to assist in the production of these materials.

The motivation for conducting studies based on these courses is not only that their mathematical content centers on the mathematics of space and motion, but also on the fact that they constitute a laboratory setting in which we can freely experiment with activities that engage body and tool-use. Outside of class we will interview the students who participate in the small group selected for videotaping, in order to ascertain more deeply their experiences and how they come to see the enrichment of perceptuo-motor activity towards the learning of the mathematics of space and motion. These studies will be conducted during the 1st Semester of Year 2 (Math 302), and the 1st semester of Year 4 (Math 509). The classroom activities will be informed by the past experience of Nemirovsky and his colleagues in the design of tools and devices supporting the engagement of body motion and perceptuo-motor actions in the learning of mathematics (Nemirovsky, 1994, 1996; Nemirovsky, Barros, Noble, Schnepp, & Solomon, 2005; Nemirovsky & Noble, 1997; Nemirovsky & Tinker, 1993; Noble, DiMattia, Nemirovsky , & Barros, 2006; Noble, Nemirovsky , Cara, & Wright, 2004)

(S8) Design experiments with talented middle and high school students in an elective summer course. At VU during the third and fourth years of the award, Hall & Leander will design, teach and study learning and student activity in an intensive summer course (one to three weeks, residential) for rising 8th to 12th graders. Students in Vanderbilt's Program for Talented Youth (PTY) are identified as being academically talented on the basis of GPA and PSAT scores, and there is an emphasis on recruiting talented youth from racially diverse and/or economically disadvantaged communities. VU faculty and graduate students teach these courses, which last year hosted over 360 students from across the country. Our course will be developed around the mathematics of space and motion, focusing on student projects that exemplify concepts from geometry and calculus (not a pre-requisite) in applied scientific fields using cartography and spatial analysis. For example, since maps often provide new information that could not easily have been gleaned from the environment or they reveal patterns that were previously unknown even to the cartographer (Liben, 2001), one proposed learning activity is to have students map their space-time pathways around campus or the city using GPS devices and ArcView GIS software, including time allocations (density plots) in different settings. While the production of maps representing data in three dimensions (x, y, z axis coordinates) is increasingly common in both popular media (e.g., weather reports) and STEM research, these complex spatial representations are weakly addressed in much school learning. We will draw from previous research on using GIS platforms to teach concepts in environmental and social sciences (Edelson, 2007; Enyedy & Mukhopadhyay, 2007), designing course activities and projects that have authentic overlap with our studies of learning in professional practices (described above) and that

are consistent with current proposals for best teaching practices in environmental and geosciences (Kastens & Turrin, 2006) and research on what young children understand and can learn about cartographic representation (Kastens & Liben, 2007; Lehrer, Jacobson, Kemeny & Strom, 1999; Liben, Kastens & Stevenson, 2002; NRC, 2006).

The VU summer course will be organized as a design experiment (Cobb et al., 2003), using pre- and post-assessments selected to measure changes in children's thinking about space, motion and geometry at different scales (e.g., drawing, walking, and at larger scales typical of professional analysis). Course assessments will overlap with those used in UW-Madison classroom experiments (e.g., tasks concerning geometry concepts) and with pre-service teachers in the SDSU courses. We will also collect video recordings of student and instructor conversations during course activities and in pull-out, clinical interviews as the course proceeds. Data collected during the first course offering (Summer, Year 3) will be used to refine course materials and teaching strategies during the second course offering. Data collected in these summer courses will provide evidence for our research questions about grounding, action, and how changes in modality and scale provide resources that support (or impede) learning (Research Questions 1, 2 and 3).

Plan of Work

We propose a coordinated series of studies, conducted by individual faculty research groups, but with design and analysis shared across the network of participating researchers. In the following sections, we present a coordinated plan of work over five years. In an Appendix, we describe how these studies will be organized to support synergy and collaboration across our network of investigators. Also in the Appendix, we describe plans for disseminating the findings of our research and products of our design efforts.

Table 2. Plan of work for a coordinated series of studies across three faculty research groups.

	San Diego State University (Nemirovsky)	**Vanderbilt University (Hall & Leander)**	**University of Wisconsin (Nathan & Alibali)**
Y1 Fall	Seek IRB approval Recruit mathematical research group (MRG) Comprehensive literature review *Cross-campus workshop 1 (with advisors)*	Seek IRB approval Recruit professional workgroup (PWG) Comprehensive literature review *Cross-campus workshop 1 (with advisors)*	Seek IRB approval Comprehensive literature review *Cross-campus workshop 1 (with advisors)*
Spring	Study of MRG 1 (history, interviews)	Study PWG 1 (observation, history)	Develop coding protocols for analysis of multi-modal learning and communication Recruit school sites
Summer	Study of MRG 1 (video recording, interviews) Prepare course for pre-service secondary teachers	Study PWG 1 (video recording, interviews) Analysis with SDSU & UW Recruit PWG 2	Refine coding protocols for analysis of multi-modal learning and communication Prepare for data collection
Y2 Fall	Conduct and study lab course with pre-service secondary teachers *Cross-campus workshop 2 (with advisors)*	Study PWG 1 (video recording, interviews) Analysis (with SDSU and UW) *Cross-campus workshop 2 (with advisors)*	Collect & code videos of HS geometry classes (N = 6) and tutorial sessions (N = 6 Teachers, 18 students) doing low-action and high-action *Cross-campus workshop 2 (with advisors)*
Spring	Data analysis (MRG 1 and	Study PWG 2 (observation,	Analyze HS geom. videos for

	pre-service course)	history) Begin comparative analysis (across PWG's)	on-line grounding & off-line simulated actions
Summer	Data analysis (MRG 1 and pre-service course) Recruit MRG 2	Study PWG 2 (video recording, interviews) Comparative PWG analysis Recruit PWG 3	Finalize analyses, write up and disseminate findings to scholarly community
Y3 Fall	Data analysis (MRG 1 and pre-service course)	Study PWG 2 (video recording, interviews) Comparative PWG analysis Identify activities for summer course	Conduct expt with HS geometry students (N = 30)
Spring	Study of MRG 2 (history, interviews)	Study PWG 3 (observation, history) Comparative PWG analysis Design and prepare summer course	Collect & code videos of HS voc ed classes (N = 6) and tutorial sessions (N = 6 Teachers, 18 students) doing low-action and high-action
Summer	Study of MRG 2 (video recording, interviews) Prepare course for pre-service secondary teachers	Study PWG 3 (video recording, interviews) Conduct and study summer course	Analyze voc ed videos for on-line grounding & off-line simulated actions
Y4 Fall	Conduct and study lab course with pre-service secondary teachers	Study PWG 3 (video recording, interviews) Comparative PWG analysis Analysis of summer course	Finalize analyses, write up and disseminate findings to scholarly community
Spring	Data analysis (MRG's and pre-service courses)	Analysis of summer course Iterative refinement of and preparation summer course	Conduct expt with HS voc ed students (N = 30)
Summer	Data analysis (MRG's and pre-service courses)	Conduct and study second iteration of summer course	Continue data analyses, Disseminate findings
Y5 Fall	Data analysis (MRG's and pre-service courses) *Cross-campus workshop 3 (with advisors)*	Analysis of PWG's and summer course *Cross-campus workshop 3 (with advisors)*	Finalize analyses, Disseminate findings *Cross-campus workshop 3 (with advisors)*
Spring	Writing and dissemination	Analysis of PWG's and summer course Writing and dissemination	Write up and disseminate findings to scholarly community
Summer	Writing and dissemination	Writing and dissemination	Disseminate findings to scholarly community

Senior Staff

Dr. Mitchell J. Nathan, Chair of the Leaning Sciences Program at Wisconsin, with expertise in research on classroom-based teaching and learning, experimental design, and video-based, multi-modal discourse analysis, and **Dr. Martha W. Alibali**, an expert on gesture analysis and its role in the change process during mathematical learning and intellectual development, experimental design, and spatial reasoning, will direct the research activities that focus on observational and tutorial studies of grounding in formal and informal settings (RQ1), and the experiments on the role of action in geometric reasoning and spatial cognition (RQ2).
Dr. Rogers Hall, Professor of Mathematics Education at Vanderbilt, studies mathematical activity and learning in classroom and workplace settings. He is an expert on ethnographic and cognitive studies of learning in multi-party activity and conversation, as well as on methods for capture and analysis of video recordings where interaction with material forms of representation are central to phenomena of learning. **Dr. Kevin Leanders**, Associate Professor of Language and

Literacy at Vanderbilt, studies how learners use and produce social space in classroom, as well as in out of school contexts (homes, community centers, and online). He is an expert in multi-modal discourse analysis and theories of human spatial production and mobility.
Dr. Ricardo Nemirovsky, Professor at the Department of Mathematics and Statistics, San Diego State University and Director of the Center for Research in Mathematics and Science Education. He has conducted educational projects in Argentina, Mexico, Peru, and USA. His research focuses on the learning of mathematics at all grade levels with a focus on the roles of bodily and kinesthetic activities. He will be Principal Investigator.

Advisory Board

1) Nahila Nasir. Assistant Professor. School of Education. Stanford University. Dr. Nasir's research is centrally concerned with both the theoretical and practical implications of understanding the relation between culture, learning, and development. **2) Gregg Trafton**. Navy Center for Applied Research in Artificial Intelligence (NCARAI). Dr. Trafton uses gesture analysis, experimental design, and computational modeling to study how experts and novices reason spatially, and how they use complex visualizations to make decisions, solve problems, and understand uncertainty. **3) Mike Eisenberg**. Associate Professor, Department of Computer Science. University of Colorado, Boulder. Dr. Eisenberg develops digital environments based on new fabrication technologies and advances in material science--such as laser cutters, 3D printers, and computationally-enriched objects--in order to study how computational media can extend the rich tradition of physical construction and design to enhance K-16 mathematics, science, and technology learning. **4) Reed Stevens**. Associate Professor, Cognitive Studies in Education. University of Washington. Dr. Stevens conducts research on learning in a wide range of places and situations through the detailed studies of naturally-occurring activity **5) Rafael Nunez**. associate professor department of cognitive science. University of California, San Diego. Dr. Nunez investigates cognition from the perspective of the embodied mind. **6) Marty Schnepp**. Holt High School Holt, MI. Mr. Schnepp has been teaching high school mathematics for 19 years and reflecting on the associated challenges and perspectives. **7) David Henderson** Professor of Mathematics, Cornell University. Dr. Henderson work focuses on aspects of mathematics that impinge on the teaching and learning of mathematics. **8) Andrea diSessa**. Professor of Education. University of California, Berkeley. Dr. diSessa’s research centers around conceptual and experiential knowledge in physics, and principals for designing comprehensive and usable computer systems. **9) Mei Po Kwan**. Distinguished Professor of Social and Behavioral Sciences. Department of Geography, Ohio State University. Dr. Kwan’s research addresses theoretical and substantive questions in urban, transportation, and economic geography through the application of GIS methods. **10) Noel Enyedy**. Assistant Professor. Graduate School of Education and Information Studies, University of California, Los Angeles. Dr. Enyedy conducts basic research on cognition, learning and the development of mathematical reasoning in applied settings, primarily in technology rich classrooms.

Dissemination
We will disseminate our findings at professional meetings and in the scholarly journals of our respective fields. In addition, our ideas will be translated into curriculum materials and activities for use in the VU summer program and in the pre-service teacher training classes held at SDSU. We will also hold graduate “distance seminars” on our major research questions, each one led by a specific faculty investigator from one of the three campuses, but attended by all faculty and graduate student members of the project team. In order to make our theoretical claims and empirical findings more widely visible, and to share our research methods with other educational and learning sciences researchers, we will also propose pre-conference workshops at national and international meetings (e.g., American Educational Research Association, International

Conference of the Learning Sciences).

Evaluation plan

Our goals for evaluation are to inform the on-going research activities and provide useful feedback about the value and relevance of our research project overall. Our *formative evaluation plan* is designed to strengthen the implementation of the research program and facilitate its progress by looking at reliability and construct validity. *Reliability*, the consistency of our efforts to develop analytic categories, will be established using an inductive and iterative approach to coding. For ethnographic and interview data, we will use constant comparative methods to develop, refine, and demonstrate coherent coverage for theoretical categories that describe aspects of grounding, action, shifting modalities, and learning. In cases where transcribed interactions are coded as a bounded corpus of events (e.g., utterances accompanied by gesture), we will pursue an objective accounting of the proportion of grounding acts that fit within a coding scheme developed through inductive, iterative methods. In these studies, trained coders who are not familiar with the hypotheses will be used to establish inter-rater reliability, with a threshold for acceptance set at Cohen's Kappa = 85%. *Construct validity* will be established by utilizing the outcome of earlier processes, such as coding of grounding behaviors, as input to later studies, such as the study of changes in grounding due to shifts in scale. In addition, we will seek to insure that different methods provide evidence of *discriminant construct validity*, in the sense that constructs that should not be related, theoretically, are not related in the findings of researchers using different methods (and vice versa). We will evaluate these convergent and discriminant findings carefully with input from appropriate sub-sets of outside researchers from our advisory board. In advance of each Advisory Board meeting, teams of three advisors, selected to match their areas of expertise, will receive a package of materials describing specific portions of the current project activities, including research design for each study, instruments and activities, coding schemes and any appropriate scholarly writings from team members. These Advisors, acting as external evaluators, will write critical comments regarding the extent to which we are making progress toward the goals of the project. These evaluative comments will be discussed by all members during the Advisory Board meetings. These discussions advisors will generate action items to be pursued by the project staff in order to enhance the likelihood of meeting the projects goals. Efforts related to these action items will be reviewed in subsequent evaluations.

Our research is not designed to evaluate curriculum materials, but we do want to know if activities we hypothesize to be important for learning are, in fact, important (or if not, why not). Our *summative assessment plan* focuses on this and the overall impact of our work. In all studies where we design or select instructional activities, our analysis and findings will be conducted to identify which activities are effective for which learning outcomes. For example, activities found to be particularly effective in design experiments with pre-service secondary mathematics teachers (Study 7) can be used as experimental contexts in geometry or vocational classrooms (Studies 5 and 6). In the sense of summative assessment, we plan to collect evidence across studies and research sites for instructional activities that have particular promise for creating powerful, embodied experiences for learning about the mathematics of space and motion. These evaluation efforts will focus on the *ecological validity*, or relevance, of our findings across study sites and populations. Outside the project, we will draw on the scholarly community of peer reviewers who evaluate our work. We also will draw heavily on the expertise of our advisory board members to provide a highly informed, yet external perspective. In preparation for the final workshop, we will submit to the board anonymous evaluation packets that list each study along with a concise description of its method and principal findings. Board members will rate the questions, methods, findings, contributions to the research community, and relevance to applied concerns. Ratings and advisor's written comments will be compiled by university support staff not working on the project, and the results of this compilation will be circulated among network investigators (faculty and graduate students) and conveyed to the foundation in our final report.