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Does Unit Analysis Help Students Construct Equations?

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Previous research has shown that students construct equations for word problems in which many of the terms have no referents. Experiment 1 attempted to eliminate some of these errors by providing instruction on canceling units. The failure of this method was attributed to the cognitive overload (Sweller, 2003) imposed by adding units to the equations and to the mechanical nature of the intervention (P. W. Thompson, 1994). Experiment 2 used contrasting pairs instruction (D. L. Schwartz & Bransford, 1998) that required students to determine which of 2 mathematical expressions referred to a real-world quantity. This brief practice at the beginning of the session significantly increased the number of correct equations relative to a control group that practiced constructing equations. The contrasting cases instruction can help prepare students for additional instruction, such as the iconic manipulations used in the Animation Tutor™ (Reed & Hoffman, 2006; see also Reed, 2005) modules.

An important unifying idea in mathematics, according to Clement and Sowder (2003), is measurement. Measurement requires assigning a value to a quantity and involves a number and usually a unit. It is a process for quantifying characteristics of our experiences such as economic indicators, a person's health, and athletic performance.

Clement and Sowder (2003) referred to P. W. Thompson's (1994) observation that students' immature use of quantitative analyses inhibits their ability to understand operations on numbers, and ultimately their ability to solve problems. P. W. Thompson proposed that a quantity is composed of (a) an object, (b) an appropriate unit or dimension, (c) a quality of the object, and (d) a process of assigning a numerical value to the quality. A subtle distinction, however, is P. W. Thompson's differentiation of a quantitative operation and a numerical operation. A quantita-

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tive operation is nonnumerical and has to do with the comprehension of a situation. The distinction is subtle because there are not separate notations for quantitative and numerical reasoning. Dividing 12 miles by 4 hr to compute 3 miles per hour provides a numerical answer, but should also remind the person of the conceptual operations that led to the appropriate arithmetic. P. W. Thompson's view of quantitative reasoning is relevant because this study is concerned with quantities that can guide the comprehension of a situation, rather than with numerical operations that are devoid of context.

Paying attention to units is an important aspect of a conceptually oriented approach to problem solving, because it helps students relate arithmetic operations to the referents described in the problem. According to A. G. Thompson, Philipp, Thompson, and Boyd (1994), conceptually oriented teachers ask the following questions:

- To what in the situation does this number refer?
- What are you trying to find when you do this calculation?
- What did this calculation give you in regard to the situation as you currently understand it?

In contrast, teachers with a *calculational orientation* have a tendency to do calculations whenever an occasion to calculate occurs regardless of the overall context.

Resnick (1986) argued that understanding quantities is necessary to bridge the two meanings of algebra. One meaning is that algebra is a formal system, abstracted from referents, in which it is possible to transform equations without thinking about the reference situations that are represented by the symbols. The second meaning is that every transformation rule can be justified in terms of the ways in which quantities behave under certain kinds of transformations. Resnick concluded her analysis with the hypothesis that the major difference between people who have a high aptitude for school mathematics and those who are poor mathematics learners is that the former routinely attempt to interpret symbol structures and rules in terms of the quantities and relationships to which the symbols refer, whereas the latter try to learn mathematics as pure symbol manipulations.

Recent research by Koedinger and Nathan (2004; also see Nathan & Koedinger, 2000) has shown that finding solutions to story problems can be easier than finding solutions to symbolic equations that model the story problems. One reason for students' success in solving easy algebra story problems is that they often use informal (nonalgebraic) strategies that allow them to bypass comprehending and manipulating symbols. An instructional implication is that students should develop competence at encoding and representing constraints in a verbal form prior to instruction on solving equations. The steps needed to solve an algebraic equation can then be related to the informal strategies that are typically more meaningful to students because they are based on grounded referents.

MOTIVATION FOR THE STUDY

Although there appears to be agreement that it is important for students to reason with quantities based on grounded referents, it is less clear how to accomplish this objective. The research reported in this article was motivated by my desire to improve the Animation Tutor™ (Reed & Hoffman, 2006) software modules, which are designed to assist students with mathematical reasoning (Reed, 2005). Although the Animation Tutor™ instruction emphasizes the simulation and manipulation of objects, it also relies on multiple forms of representation to explain and connect mathematical concepts (Reed & Jazo, 2002). Part of the instruction shows detailed written solutions for solving algebra word problems. Experiment 1 in this study investigated whether students would be more successful in constructing algebraic equations if this instruction placed a greater emphasis on the units in the equations.

The research reported in this article and previous evaluations of the Animation Tutor™ (Reed & Hoffman, 2006) software have used college students as participants. One reason for using college students is that many college students are weak in mathematical reasoning. For example, according to the California State University (CSU, 2005) Web site, 37% of the incoming CSU freshmen in 2004 required remedial instruction in mathematics during their first year of college. This is a serious problem, because the CSU system is one of the largest university systems in the United States, consisting of 405,00 students on 23 campuses.

Understanding quantities is also important in advanced college courses such as differential equations (Rasmussen, 2001). Some of the difficulties that students had in interpreting graphs of these equations resulted from their thinking with an inappropriate quantity or losing focus on the intended underlying quantity. Rasmussen's framing of how students struggle with learning differential equations built on their learning of K–12 mathematics.

Difficulties that college students have when reasoning about quantities is also evident in my own data that showed college students constructed equations for algebra word problems in which many terms did not consist of quantities. The next section documents these errors, including ones that result from the faulty use of referent–transforming operations. The following two sections describe two instructional interventions that attempt to reduce these documented errors—the canceling of units in Experiment 1 and the identification of inappropriate quantities in Experiment 2. The concluding section discusses the contributions and instructional implications of the study.

TYPES OF UNIT ERRORS IN CONSTRUCTING EQUATIONS

Referent-Transforming Operations

The correct specification of units is particularly challenging when there is a change in the referents. Arithmetic operations can either preserve or change referents, which J. L. Schwartz (1988) labeled referent–preserving and referent–transforming operations. Addition and subtraction are referent–preserving operations because they do not change the referents. Subtracting 5 marbles from 8 marbles gives 3 marbles. Multiplication and division are referent–transforming operations. Multiplying 9 ft × 12 ft to calculate the area of a room changes the dimensions from feet to square feet.

But J. L. Schwartz (1988) argued that there is a more fundamental reason why multiplication and division change referents (dimensions) and it is based on the concept of an intensive quantity. An intensive quantity typically can not be directly measured but is a ratio of two (extensive) quantities that can be measured. The price of gas, in dollars per gallon, is an intensive quantity that refers to a quality of the gas rather than to objects such as money or gasoline.

Intensive quantities are important because they are referent–transforming operations. Imagine that someone needs 12 gallons of gas and wants to calculate what it will cost. Solving the problem creates a change in dimensions from gallons to dollars. The change occurs by multiplying gallons by an intensive quantity, price per gallon. J. L. Schwartz (1988) concluded that it is incorrect to assume that children's early number knowledge leads in a continuous way to understanding the use of multiplication and division in modeling situations. An instructional strategy for teaching referent–transforming operations must, therefore, make explicit exactly how they differ from the previously learned referent–preserving operations in addition and subtraction problems.

Referent transforming operations are also problematic for college students when they construct equations for algebra word problems. Errors occur by using conflicting units, isolating intensive quantities without corresponding extensive quantities, making unnecessary calculations to calculate a quantity that is already provided, and selecting incorrect arithmetic operations that create meaningless units. The following illustrates each of these errors found in my previous research.

Conflicting Units

Conflicting units occur when arithmetic operations inappropriately combine quantities that have different units, such as adding yards and miles. Mixing units is particularly likely to occur in referent–transforming operations because of the greater complexity of this operation. An example of this error occurred during a formative evaluation of an instructional module (Reed, Hoffman, & Phares, 2001) for solving problems in which one person catches another, as in the following example:

> A little boy leaves home at 7 a.m. walking to school at 2 miles per hour (mph). Six minutes later, his big sister notices that he forgot his lunch, and she walks after him. If his big sister walks at 3 mph, how long will it take her to overtake her little brother?

The instruction shows a solution that involves equating the distance traveled by the two walkers

$$\text{Rate}_1 \times \text{Time}_1 = \text{Rate}_2 \times \text{Time}_2$$

and then substitutes the rates for the brother (2 mph) and sister (3 mph) into the equation. During the formative evaluation, 21 college students in an intermediate algebra class were then asked the best way to represent the time the brother walks if t represents the amount of time the sister walks and if she leaves 6 min after her brother. The options were (a) $t + 6$ min, (b) $t - 6$ min, (c) $t + .1$ hr, and (d) $t - .1$ hr. Eleven students selected the first option, 3 students selected the second option, 5 students selected the third option, and 2 students selected the fourth option. Two-thirds of the students, therefore, selected a measure of time that had units that differed from those used to express the rate. The module then provided feedback that it would be necessary to represent the time the brother walks as $t + .1$ hr so that rate and time have the same unit.

No Units

Another kind of error involves the use of a number that has no units, such as percentage. Interest and mixture problems use percentage to calculate quantities, such as the amount of fat in the following problem:

> One vegetable oil contains 6% saturated fats and another contains 12% saturated fats. How many quarts of the 12% solution should be added to 3 qt of the 6% solution to make a vegetable oil containing 8% saturated fats?

The amount of fat (in quarts) in the 6% and 12% solutions equals the amount of fat in the 8% solution. Similarly, the amount of antifreeze in the two parts equals the amount of antifreeze in the whole for the following problem:

> An automobile radiator contains 16 qt of a 20% solution of antifreeze. How much of the original solution must be drawn off and replaced by 80% antifreeze to make a solution of 25% antifreeze?

The amount of acid or antifreeze is found by multiplying the intensive quantity *percentage* by the extensive quantity *quarts of solution*. Errors occur when students focus on only the intensive quantity or the extensive quantity, rather than on both. Reed, Dempster, and Ettinger (1985) found 35 instances in 52 equations in which students represented the quantity of acid or antifreeze by either the quantity of the solution or by the percentage, rather than as a product of both (Table 6 in Reed et al., 1985). A particularly striking indication of the failure to represent quantities in equations is that 27 of the 35 instances contained an isolated percentage, such as representing the quantity of saturated fat in the mixture as simply .08. A percentage has no referent, and therefore no unit.

Unnecessary Calculations

A. G. Thompson et al.'s (1994) claim that one characteristic of a calculational approach is to make calculations whenever there is an opportunity can be illustrated by *unnecessary calculations*. An analysis of the quantities used to construct an equation should reveal that sometimes the quantity is stated in the problem and does not have to be calculated. For example, the following problem is similar to the saturated fat and antifreeze problems because the amount of copper in the two parts equals the amount of copper in the whole. However, the amount of copper in the whole is stated in the problem and does not have to be calculated.

> One alloy of copper is 20% pure copper and another is 12% pure copper. How much of each alloy must be melted together to obtain 60 lb of alloy containing 10.4 lb of copper?

The copper problem is one of four test problems that students were given in a transfer experiment (Reed, 1987), in which they initially studied a detailed solution to the following acid problem:

> A nurse mixes a 6% boric acid solution with a 12% boric acid solution. How many pints of each are needed to make 4.5 pt of an 8% boric acid solution?

The copper and acid problems differ in that the total amount of acid has to be calculated by finding 8% of 4.5 pt but the total amount of copper (10.4 lb) is stated in the problem. Although the copper problem is, therefore, a computationally easier problem, none of the 45 college students enrolled in an intermediate algebra class could construct a correct equation to represent it after seeing a detailed solution of the acid problem. Many students failed because they thought they needed to multiply two numbers, as was done for the acid problem. They would multiply quantities such as 60 lb × 10.4 lb, which results in an inappropriate unit ($pounds^2$). Students did quite well on some of the other transfer problems, so their inability to transfer from the acid to the copper problem did not imply a general inability to formulate equations for word problems. It did imply the failure to recognize a provided quantity that did not need be calculated.

Incorrect Arithmetic Operations

A consideration of appropriate units should also help students select a correct arithmetic operation for calculating a quantity. Students in a study by Reed, Cooke, and Jazo (2002) initially received instruction on solving a standard tank-filling problem in which two pipes combine to fill a tank. They were then asked how they would solve a problem in which there was a leak in the bottom of the tank that would cause it to lose 1/24 of its content each hour. Almost all students correctly indicated that they would need to subtract the amount of water loss to determine the amount of water in the tank. They were then asked, if the rate of loss is 1/24 tank per hour, the amount of loss after *h* hr can be expressed as which of the following equations:

Rate of Loss + *h* hr [4]
Rate of Loss – *h* hr [10]
Rate of Loss × *h* hr [16]
Rate of Loss ÷ *h* hr [4]

The distribution of responses (shown in brackets) indicates that fewer than half of the students realized that the amount of loss is calculated by multiplying the rate of loss by time, even though multiplication is the only arithmetic operation that provides an amount with the correct unit (proportion of a tank). Adding or subtracting an intensive quantity (tanks per hour) to an extensive quantity (hour) does not provide an amount. Dividing tanks per hour by hour yields tanks per $hour^2$, which is a measure of acceleration rather than a quantity with the correct dimensions.

EXPERIMENT 1: CANCELLATION OF UNITS

Experiment 1 investigated the effectiveness of an instructional intervention based on the cancellation of units. This technique is illustrated by the problem shown on page 323 of the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000):

> While driving through Canada in the late 1990s, a U.S. tourist put 60 L of gas in his car. The gas cost Can$0.50 a liter (Can$ stands for Canadian dollars). The exchange rate at that time was Can$1.49 for each U.S.$1.00 (U.S. dollar). The price for a gallon of gasoline in the Unites States was U.S.$0.99. The driver wanted to compare prices and decide whether a tank of gas was cheaper in the United States or Canada.

The provided solution illustrates how canceling units can be helpful in keeping track of conversions:

$$\frac{\text{\sout{U.S.}}\$0.99}{1\ \text{\sout{gal}}} \times \frac{1\ \text{\sout{gal}}}{3.79\ \text{\sout{L}}} \times 60\ \text{L} \times \frac{\text{Can}\$1.49}{\text{\sout{U.S.}}\$1.00} = \text{Can}\$23.35$$

Purchasing 60 L of gas in the United States would cost $23.35 in Canadian dollars, which is less than paying 30 Canadian dollars in Canada.

To determine whether this technique is helpful, I compared an instructional approach based on the cancellation of units (units condition) with a control condition in which students were not instructed to use units to construct equations. Table 1 shows the problems that were used in Experiment 1. The first three problem categories were included to evaluate whether canceling units during the construction of equations would result in a reduction in the kinds of errors that are documented in the previous section. The money problems were included because they are similar to the example just given. The first (pretest) problem in each category served as a pretest and was followed by a detailed solution that could be used to solve the other two problems in the category.

The bold text in Table 2 illustrates how solutions from my prior research were modified for students in the units condition. This information shows the use of units during the construction of the equation and provides a brief explanation of the cancellation procedure. This additional material was not printed in bold text during the study, but the bold text serves here to illustrate the differences in the solutions provided for the units and control groups. As shown in Table 2, units are included in both solutions during the initial discussion of the problem, but only in the units instruction during the equation–construction phase.

Students in both groups were encouraged to use these solutions to solve the second and third problems in each category, but were not allowed to turn back to the solutions, to discourage them from relying on a rote application. The second (test) problem was equivalent to the first problem and allowed for the application of the solution without modification. The third (transfer) problem required some modification of the solution. The transfer problem may have been a computationally easier version of the pretest and test problems. For example, it is not necessary to convert units in the third distance problem because rate and time are measured in the same units.

TABLE 1
Problems Used in the Study

Distance problems

1. A car traveling at a speed of 30 mph left a certain place. Another car departed from the same place 90 min later at 40 mph and traveled the same route. How long will it take the second car to overtake the first car?
2. A little boy leaves home at 7a.m. walking to school at 2 mph. Six minutes later his big sister notices that he forgot his lunch, and she walks after him. If his big sister walks at 3 mph, how long will it take her to overtake her little brother?
3. A jet fighter leaves an aircraft carrier and travels at 5 miles per minute. If a second fighter leaves 12 min later and travels at 7 miles per minute, how long will it take to catch up with the first fighter?

Tank problems

1. A large pipe can fill a tank in 8 hr and a small pipe can fill it in 12 hr. How long will it take to fill the tank if both pipes are used at the same time?
2. One faucet can fill a pan in 40 sec and another faucet can fill the pan in 25 sec. How long will it take to fill the pan if both faucets are used at the same time?
3. A large pipe can fill a tank in 8 hr and a small pipe can fill it in 12 hr. How long will it take to fill the tank if both pipes are used at the same time, but there is a leak in the bottom of the tank that leaks 1/24 tankful per hour?

Mixture problems

1. A nurse mixes a 6% boric acid solution with a 12% boric acid solution. How much of each solution is needed to make 4.5 pt of an 8% boric acid solution?
2. One solution of silver nitrate in water contains 18% silver nitrate, and the other contains 10% silver nitrate. How much of each solution is needed to make 8 lb of a 12% silver nitrate solution?
3. One alloy of copper is 20% pure copper and another is 12% pure copper. How much of each alloy must be melted together to obtain 60 lb of alloy containing 10.4 lb of copper?

Money problems

1. While driving through Canada in the late 1990s, a U.S. tourist put 60 L of gas in his car. The gas cost Can$0.50 a liter (Can$ stands for Canadian dollars). The exchange rate at that time was Can$1.49 for each U.S.$1.00 (U.S. dollars). How much would the tourist have to pay for the 60 L of gas in U.S. dollars?
2. While driving through Canada in the late 1990s, a U.S. tourist put 45 L of gas in her car. The gas cost Can$0.50 a liter. The exchange rate at that time was Can$1.00 for each U.S.$0.67. How much would the tourist have to pay for the 45 L of gas in United States dollars?
3. A U.S. tourist on a recent trip to Europe purchased six pieces of cut glass for 17.30 Euros a piece. If the exchange rate is 0.82 Euros for a U.S.$1.00, what did the six pieces cost in U.S. dollars?

Method

Participants. The participants were 51 students from two sections of a Statistical Methods in Psychology course that I taught. This is an introductory course for lower-division students. The study occurred during class time at the end of the 2nd week of classes. Students received extra credit for participating and were informed that the design and findings would be discussed later in the course. All students had

TABLE 2
Example Solution for Units and Control Groups

A large pipe can fill a tank in 8 hr and a small pipe can fill it in 12 hr. How long will it take to fill the tank if both pipes are used at the same time?

This problem can be solved by adding the amount of the tank filled by the large pipe (Fill1) to the amount of the tank filled by the small pipe (Fill2). Then

Fill1 + Fill2 = 1 Tank when both pipes fill the tank.

The amount filled by the large pipe is determined by the Rate of Fill (part per hour) multiplied by the Time of Fill:

Fill1 = Rate1 × Time1 = 1/8 tankful per hour × *h* hr

The amount filled by the small pipe is determined the same way:

Fill2 = Rate2 × Time2 = 1/12 tankful per hour × *h* hr

The unknown variable, *h,* in this problem is the time it takes both pipes to fill the tank.
To summarize;

Pipe	Rate of Fill (part of tank per hour)	Time of Fill (hr)	Amount Filled (part of tank)
Large	1/8	*h*	1/8 × h
Small	1/12	*h*	1/12 × h

Fill1 + Fill2 = 1 Tank

Rate1 × Time1 + Rate2 × Time2 = 1 Tank

$$\frac{1/8\ \textbf{tankful}}{\textbf{\sout{hr}}} \times\ h\ \textbf{\sout{hr}} + \frac{1/12\ \textbf{tankful}}{\textbf{\sout{hr}}} \times h\ \textbf{\sout{hr}} = 1\ \textbf{tank}$$

Notice that after canceling hour(s) in the numerator and denominator for both pipes, the two sides of the equation have an appropriate unit for volume (part of a tank). The part of the tank filled by each pipe equals 1 tank when the tank is filled. Remember to use units to construct and evaluate your equations.

completed a course on algebra, and at least 46 of the students had completed a course on intermediate algebra (5 students did not respond to the question about intermediate algebra).

Procedure. Instruction–test booklets were randomly distributed to the participants in both sections. The booklets differed in the Instructional condition and in the counterbalanced order of the four problem categories. The instructions indicated that:

> The first problem in each category evaluates how many students already know how to solve the problem before they see the instruction. Everyone will then see a detailed solution to the problem, which should be helpful for

> constructing an equation for the next two problems. You will have 3 min to work on each problem and 2 min to study the solution following the first problem in each category. I will tell you when to turn to the next page. It is important that you stay on each page and not go backward or forward in the booklet.

The first problem of each category was printed on blue paper so students could not see through the paper to view the solution on the following page. This also partitioned the problems into the four categories. Three minutes was more than sufficient time to construct an equation that students did not need to solve.

In addition to the added material in the solutions, the 26 students in the Units group received an extra paragraph in the initial instructions that emphasized the use of units in constructing equations:

> One focus of this instruction is on using units to guide your construction of equations. For example, you could calculate how far you traveled by multiplying your rate of travel (such as 65 mph) by the time you travel (such as 2.3 hr):
>
> $$\text{Distance} = \text{Rate} \times \text{Time} = \frac{65 \text{ miles}}{\cancel{\text{hr}}} \times 2.3\ \cancel{\text{hr}} = 149.5 \text{ miles}$$
>
> Notice that canceling hour(s) in the numerator and denominator provides an appropriate unit (miles) for distance.

The final page of the test booklet informed both groups about the purpose of the experiment and asked whether they thought units instruction was (or would be) beneficial.

Before presenting the results, I want to clarify my use of the term *pretest*, which refers to the first problem in each category before students have received a solution to the problem. However as noted in the summary of the experimental design shown in Table 3, students in the Units group had already received an instructional intervention about the importance of using units. Results on the pretests, therefore, measure both students' prior knowledge about solving particular problems and the effect of the instructional intervention.

Results

ANOVA. The data were analyzed in a 2 × 3 × 4 ANOVA in which Instruction (units, control) was a between-subjects variable and Test (pretest, test, transfer) and Category (distance, mixture, tank, money) were within-subjects variables. Instruction had a significant effect on performance, $F(1, 49) = 11.87$, $p < .001$, because students in the units group constructed *fewer* correct equations than the stu-

TABLE 3
Experimental Procedures

	Control Group	*Test Group*
Experiment 1 procedure[a]		
Duration (3 min)	Initial instructions: no mention of using units cancellation to construct equations. Students not instructed to use this method.	Initial instructions: included example of using units cancellation to construct equations. Students instructed to consider using this method.
Duration (44 min)	1. Work pretest problem (3 min) 2. Instruction (2 min): study explanation of correct solution to pretest problem (no emphasis placed on units cancellation in equation-construction phase of explanation) 3. Work test problem (3 min) 4. Transfer problem (3 min) Repeat Steps 1 through 4 for each of the four problem categories.	1. Work pretest problem (3 min) 2. Instruction (2 min): study explanation of correct solution to pretest problem (units cancellation technique is emphasized during and after equation-construction phase of explanation) 3. Work test problem (3 min) 4. Transfer problem (3 min) Repeat Steps 1 through 4 for each of the four problem categories.
Experiment 2 procedure[b]		
Phase 1 (10 min)	1. Pretest task (3 min): construct equation for one Tank problem. 2. Instruction (2 min): study explanation of correct solution to pretest task. 3. Posttest task (3 min): construct equation for other tank problem. 4. Instruction (2 min): study explanation of correct solution to posttest task.	1. Pretest task (3 min): select appropriate quantity for each of 5 pairs of contrasting cases. 2. Instruction (2 min): study explanation of correct answers to pretest task. 3. Posttest task (3 min): select appropriate quantity for new set of contrasting cases. 4. Instruction (2 min): study explanation of correct answers to posttest task.
Phase 2 (33 min)	Both groups follow the control-group procedure in Experiment 1 for three problem categories.	

[a]Control, $n = 25$; Test units, $n = 26$. [b]Control, $n = 32$; Test referents, $n = 34$.

dents in the control group. Students in the control group constructed correct equations for 29% of the problems and students in the units group constructed correct equations for 13% of the problems.

There was a significant effect of Test, $F(2, 98) = 30.26$, $p < .001$, and a significant Instruction × Test interaction, $F(2, 98) = 4.52$, $p < .02$. Figure 1 shows these results. The 95% confidence intervals for the two instructional groups overlap considerably for the pretest problems (0 to 9% for units and 2 to 14% for control), overlap slightly

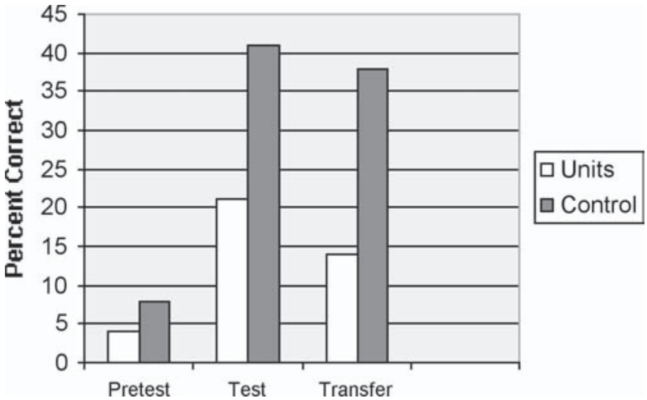

FIGURE 1 Percentage of correct equations for units and control groups.

for the test problems (11 to 32% for units and 30 to 52% for control), and do not overlap for the transfer problems (5 to 23% for units and 29 to 47% for control). There was also a main effect for Category, $F(3, 147) = 13.64, p < .001$, that did not interact with Instruction, $F(3, 147) = 1.97, p > .05$. The main effect of Category has no theoretical value, so the next section focuses on students' performance on individual problems rather than on the four categories of problems.

Representation of quantities. The problems selected for the experiment were ones that caused numerous quantity errors in my previous research. The high percentage of incorrect equations reported in Table 4 suggests that these quantity errors continued to occur.

An equation for solving the distance test problem requires that the difference in time be expressed as 0.1 hr, rather than as 6 min, because rate is expressed as miles per hour. The solution to the pretest problem illustrated this change in units by converting 90 min to 1.5 hr. However, the majority of students failed to make this conversion for the test problem. Fifteen of the 25 students in the control group repre-

TABLE 4
Percentage of Correct Equations for Units and Control Groups

	Pretest		*Test*		*Transfer*	
Problems	*Units*	*Control*	*Units*	*Control*	*Units*	*Control*
Distance	0	12	0	16	15	60
Mixture	4	0	15	36	0	0
Tank	4	0	27	56	19	36
Money	8	20	42	56	23	56

sented the time difference as 6 min, compared to four students who correctly represented the time traveled by the brother as $t + .1$ hr. Of the 26 students in the units group, 16 represented the time difference as 6 min. None of these students correctly represented the time traveled by the brother as $t + .1$ hr although two students represented this time as simply .1 hr. Students in both groups were more successful on the distance transfer problem, which did not require a units conversion.

The tank transfer problem evaluated whether an emphasis on units would help students represent the amount of water loss from the tank as a product of rate and time. Thirteen students in the control group correctly represented this amount as $1/24 \times h$ and two other students represented this amount as a product of rate and time but incorrectly represented either the rate or the time. In contrast, only seven students in the units group correctly represented this amount as $1/24 \times h$.

The mixture transfer problem was included to determine whether students could apply a problem solution in which the amount of acid in a mixture is calculated by multiplying a percentage (8%) and a quantity (4.5 pt) to a problem in which the amount of copper (10.4 lb) does not have to be calculated. None of the 51 students constructed a correct equation, replicating the previous transfer failure of 45 students in an intermediate algebra class (Reed, 1987). The most common mistake made by (7) students in the control group was to multiply 60 (lb) by 10.4 (lb), confirming the finding that students often try to reproduce the arithmetic operations in the example solution (Reed et al., 1985). Three students represented the amount of copper in the mixture as 60, and two students divided 60 by 10.4. Two students correctly represented the amount of copper as 10.4 (lb) but made some other error in the equation. The most common numerical representation for (3) students in the unit group was either 60, 10.4, or 60×10.4. The three students who correctly represented the amount of copper as 10.4 made other errors in the equation. Two students represented the amount of copper as $60 \times (10.4/60)$, which provides a correct amount but unnecessarily reproduces the multiplicative operation in the example solution.

The design of the test and transfer money problems allowed for the evaluation of inappropriate focusing on irrelevant information (Lobato, Ellis, & Munoz, 2003). If students believe from the pretest solution that the numerical value 1.00 should always go into the numerator, then they should inappropriately represent the exchange ratio as 1.00/0.67 in the test problem. Eight students in the control group and four students in the units group made this error. If students believe from the pretest example that the ratio for the exchange rate is always less than 1, they should inappropriately represent the exchange rate as .82/1.00 in the transfer problem. Eight students in the control group and seven students in the units group made this error. Overall, the control group constructed more correct equations than the units group for both the test and transfer problems.

In summary, the units instruction was clearly unsuccessful. Students in this group did significantly worse in constructing equations than students in the control group. Both groups continued to make many errors that the units instruction was designed to reduce. Although the instruction on units was brief, canceling units was not a new method for most of the students. At the end of the experiment, students were asked to list their courses that included unit analysis. Fifteen students listed no courses; 10 listed one course; 14 listed two courses; 4 listed three courses; 6 listed four courses; and 2 listed five courses. The most frequently listed courses were algebra (21 students) and chemistry (20 students) although intermediate algebra, statistics, and physics were also listed (5 each).

The last page of the booklet contained a questionnaire that asked students in the units group whether they found unit analysis to be helpful in this experiment. Students were evenly divided in their responses with 13 students responding positively, 12 students responding negatively, and one student responding indifferently. Typical positive responses included statements such as unit analysis (a) helps to keep track of what the answers were supposed to be, (b) makes the equation more clear, (c) clarifies what is what, and (d) organizes information in the problem more efficiently. Typical negative responses are included in the next section, which analyses possible causes of the failure of unit analysis to improve performance.

Discussion

One possible explanation of the ineffectiveness of unit analysis is that students in the units group often did not use units analysis. This group produced 85 answers that did not include an equation, 80 answers that included equations without any units, 88 answers that included equations with some units, and 59 answers that included equations with units for all the terms in the equation. A critic might argue that the instruction was ineffective because students did not follow directions to use units. However, the finding that the unit analysis group performed significantly worse than the control group indicates that many students in the units group did unsuccessfully attempt to follow the instructions, which proved detrimental to their performance.

The detrimental effect of the units instruction suggests that students may not have used units analyses more extensively because they realized it was not helping them. For example, Reed and Hoffman (2004) found that students who were instructed to estimate the time to fill a tank by mentally simulating the event provided estimates that were more accurate than their mental simulation times. Students therefore used additional (analytic) strategies to enhance their performance, perhaps because they realized that mental simulation was not an effective strategy.

I propose three possible, and compatible, explanations of why units analysis had a detrimental effect. The first is based on D. L. Schwartz and Bransford's (1998) proposal that students may lack the prerequisite prior knowledge to benefit from instruction. The most common negative response from students in the units group was that they were confused by the instruction on units analysis. D. L.

Schwartz and Bransford argued that prior practice on discriminating contrasting cases helps prepare students for understanding an explanation by enhancing their ability to distinguish between important ideas in the explanation. The next experiment incorporated this instructional approach by initially asking students to distinguish between cases that did or did not represent real quantities.

A second possible explanation, and consequence of lacking appropriate prerequisite knowledge, is that the instruction produced cognitive overload (Sweller, 2003). The cognitive overload explanation proposes that an excess amount of information overwhelms the capacity of short-term memory. This explanation is illustrated by those five students who complained that the units instruction (a) "provided too much information for me to absorb in the time given," (b) provides more to think about, (c) "added things to the equation that are not easy for me," (d) has too many units to choose from, and (e) makes the equation more complex. Sweller recommended partitioning the instruction into smaller components to reduce the effects of cognitive overload.

A third possible explanation of the failure of units instruction is that canceling units is a mechanical procedure that does little to enhance understanding of quantities. According to P. W. Thompson (1994),

> We should condemn dimensional analysis, at least when proposed as "arithmetic of units," and hope that it is banned from mathematics education. Its aim is to help students "get more answers," and it amounts to a formalistic substitute for comprehension. (p. 226)

P. W. Thompson's reservations are supported by the explanations of two students who claimed that unit analysis was helpful. One reported that units helped her memorize the equations and another reported that he now knew how to write the formula for some problems even though he did not understand the formula.

EXPERIMENT 2: IDENTIFYING REFERENTS

The instructional approach evaluated in Experiment 2 builds on recommendations expressed in the previous section. It utilizes contrasting cases, as proposed by D. L. Schwartz and Bransford (1998), to provide practice in distinguishing between appropriate and inappropriate quantities before students construct equations for word problems. This initial instruction required students in the referents group to (a) select the appropriate quantity for each of five pairs of contrasting cases, (b) study explanations of the correct answers, (c) select the appropriate quantities for a new set of contrasting cases, and (d) study explanations of the correct answers. Table 5 shows the instructions for one set of contrasting cases. As indicated in the instruc-

TABLE 5
Instruction on Quantities

Mathematical reasoning typically involves working with quantities consisting of both a number and a unit. Subtracting 5 apples from 9 apples involves both numbers and a unit (apples). Multiplication and division are more complex because units may change such as in the familiar formula, Distance = Rate × Time. Multiplying 60 mph by 3 hr yields 180 miles. Although the units changed from *miles per hour* and *hours* to *miles,* the new unit *miles* is an appropriate measure of distance.

The following exercise requires distinguishing between appropriate and inappropriate quantities. An appropriate quantity consists of a number and a unit that measures something in the world. For each of the following pairs of examples circle the number that lists an appropriate quantity.

1. 5%	2. 5% × 3 pt
1. 3 ft x 4 ft	2. 3 lb × 4 lb
1. $\frac{18\text{ ft}}{\text{sec}} \times 12\text{ sec}$	2. $\frac{18\text{ ft}}{\text{sec}} \times .2\text{ min}$
1. $\$20 \times \frac{.82\text{ Euros}}{\$1.00}$	2. $\$20 \times \frac{\$1.00}{.82\text{ Euros}}$
1. $\frac{.2\text{ tanks}}{\text{min}} \times 4\text{ min}$	2. $\frac{.2\text{ tanks}}{\text{min}} + 4\text{ min}$

tions, the task requires students to decide for each pair, which member of the pair refers to a number and unit that measures something in the world. Table 6 shows the explanatory feedback for the second set of contrasting cases. The two sets of contrasting cases consisted of corresponding examples that contained different units to provide generality.

The contrasting cases instruction was more consistent with Sweller's (2003) and P. W. Thompson's (1994) analyses than the canceling units instruction. Distinguishing between an appropriate and inappropriate quantity reduces cognitive load because it is a simpler task than constructing equations that include numbers, symbols, and units. The inappropriate quantity in each pair was one that had frequently occurred in students' incorrect constructions, and the goal was to encourage students to reflect on which mathematical expressions had referents. This goal supports P. W. Thompson's emphasis on understanding quantities, in contrast to the more mechanical procedure of canceling units.

The initial instruction for the control group required students to construct equations for the first two tank problems in Table 1. Students constructed an equation for one of these problems, studied its solution, constructed an equation for the other problem, and studied its solution. The order of the two problems was counterbalanced, as was the two sets of contrasting pairs given to the referents group.

TABLE 6
Feedback on Second Set of Contrasting Cases

1. $\frac{\$67}{\text{ton}} - 3\text{ tons}$	**2.** $\boldsymbol{\frac{\$67}{\text{ton}} \times 3\text{ tons}}$

$\boldsymbol{\frac{\$67}{\text{ton}} \times 3\text{ tons}}$ **equals $201, which is an appropriate currency**

$\frac{\$67}{\text{ton}} - 3\text{ tons}$ has different units and therefore is not a quantity

1. 12 yards × 15 yards	2. 12 quarts × 15 quarts

12 yards × 15 yards equals 180 square yards, which is an appropriate area

12 quarts × 15 quarts equals 180 square quarts, which is not an appropriate volume

1. $\boldsymbol{\frac{18\text{ miles}}{\text{hr}} \times .5\text{ hr}}$	2. $\frac{18\text{ miles}}{\text{hr}} \times 30\text{ min}$

$\boldsymbol{\frac{18\text{ miles}}{\text{hr}} \times .5\text{ hr}}$ **equals 9 miles, which is an appropriate distance**

$\frac{18\text{ miles}}{\text{hr}} \times 30\text{ min}$ is not an appropriate measure because rate is measured in hours and time is measured in minutes

1. $25\text{ Euros} \times \frac{.82\text{ Euros}}{\$1.00}$	**2.** $\boldsymbol{25\text{ Euros} \times \frac{\$1.00}{.82\text{ Euros}}}$

$\boldsymbol{25\text{ Euros} \times \frac{\$1.00}{.82\text{ Euros}}}$ **equals $30.50, which is an appropriate currency**

$25\text{ Euros} \times \frac{.82\text{ Euros}}{\$1.00}$ equals 20.50 Euros^2 per dollar, which is not an appropriate currency

1. 3% × 900 pieces	2. 3%

3% × 900 pieces equals 27 pieces, which is a quantity

3% by itself does not have a unit because it does not specify 3% of what

Method

Participants. The participants were 66 students from two sections of a Statistical Methods in Psychology course that I taught during the semester that followed Experiment 1. The experiment again occurred during class time at the end of the 2nd week of classes. Students received extra credit for participating and were informed that the design, analysis, and findings would be discussed later in the course. All of the students had completed a course on algebra. Sixty students responded that they had completed a course on intermediate algebra, 3 students replied that they had not taken an intermediate algebra course, and 3 students failed to answer the question.

Procedure. The instruction–test booklets were randomly distributed to the participants in both sections, resulting in 34 students in the referents group and 32 students in the control group. The second phase of the experiment was identical for both groups and consisted of students working on the three problems in the distance, mixture, and money categories, using the solutions that were given to the control group in Experiment 1. The second experiment, therefore, differed from the first experiment by including the treatment condition during the 10-min initial phase, rather than during the problem solving second phase. As in Experiment 1, students preceded through the booklets as a group, spending 3 min on solving problems and 2 min on studying solutions. Table 3 summarizes the procedure.

Results

First phase. Both groups improved from the pretest to the posttest during the initial instruction. The number of correct answers during the contrasting cases instruction for students in the referents group increased from 3.62 (72%) on the pretest to 4.68 (93%) on the posttest, $t(33) = 4.74$, $p < .001$. None of the students in the control group constructed a correct equation for a tank problem on the pretest, compared to 31% of the students on the posttest, $t(31) = 3.75$, $p < .001$. Students in both groups received solutions to the posttest problems, so they had the opportunity for additional learning.

Second phase. The data from the second phase were analyzed in an 2 × 3 × 3 ANOVA, in which instruction (referents, control) was a between-subjects variable and test (pretest, test, transfer) and category (distance, mixture, money) were within-subjects variables. Instruction had a significant effect on performance, $F(1, 64) = 4.85$, $p = .031$, because students in the referents group constructed more correct equations than the students in the control group. Students in the control group constructed correct equations for 25% of the problems and students in the referents group constructed correct equations for 34% of the problems.

There was a significant effect of Test, $F(2, 63) = 71.62$, $p < .001$, that did not interact with Instruction. Figure 2 shows these results. There was also a main effect for Category, $F(2, 63) = 9.38$, $p < .001$, that did interact with Instruction, $F(2, 63) = 4.41$, $p < .02$. Table 7 shows that the instruction on identifying referents produced a dramatic improvement on the money problems and a modest improvement on the distance and mixture problems.

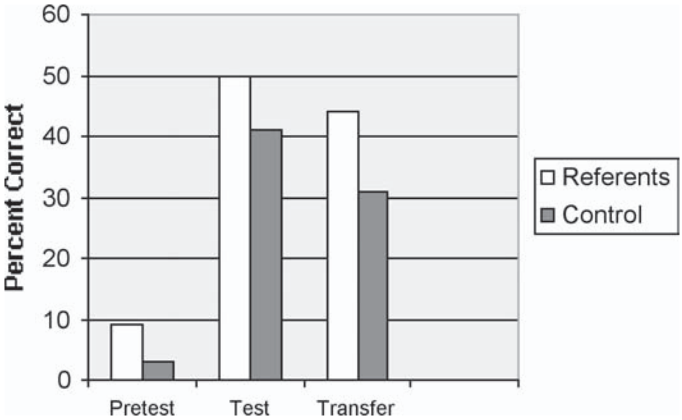

FIGURE 2 Percentage of correct equations for referents and control groups.

Discussion

The increase in performance on the money problems is impressive. The percentages of correct equations for students in the units group are 24% for the pretest problem, 76% for the test problem, and 59% for the transfer problem. The corresponding percentages are 6%, 50%, and 28% for students in the control group. The two examples of contrasting cases in the first phase illustrated the correct conversion of Euros to dollars in one case and the conversion of dollars to Euros in the other case.

Two other sets of contrasting cases were designed to reduce the number of equations that incorrectly represented the amount of copper in a mixture problem as the product of 60 lb and 10.4 lb. The contrasting cases showed that products can be formed from some extensive quantities (such as feet or yards), but not others (pounds or quarts). This instruction reduced the number of generated equations containing the product of 10.4 lb and 60 lb from 13 in the control group to 8 in the referents group.

TABLE 7
Percentage of Correct Equations for Referents and Control Groups

	Pretest		*Test*		*Transfer*	
Problems	*Referents*	*Control*	*Referents*	*Control*	*Referents*	*Control*
Distance	3	3	24	16	56	56
Mixture	0	0	50	56	18	9
Money	24	6	76	50	59	28

The least effective instruction on referents was the two sets of contrasting cases that illustrated that the same units of time are required when multiplying rate by time. Thirteen students in both the referents and control groups represented rate as $t + 6$, failing to convert 6 min to .1 hr so that it would correspond to the rate of 2 mph. The referents' instruction produced a slight gain in the number of correct equations, because 10 students in the referents group correctly represented time as $t + .1$ (hr), compared to 6 students in the control group. However, the small percentage of correct equations is troublesome, because both the contrasting cases instruction and the pretest example showed that a common unit of time is required when multiplying rate by time.

Future research could enhance these findings by conducting interviews of students to find impediments to transferring the pretest solution to the other two problems in the category. Lobato (2003) proposed an actor-oriented approach to transfer based on interviews to determine which similarities students perceive when comparing problems. Their perceived similarities often differ from the normative similarities determined by the experts who create the problems. However, part of the poor transfer is caused by procedural errors. Examples of these errors include the representation of time as $t + .06$ by 5 students, $t \times .1$ by 3 students, $t + .6$ by 1 student, and $t + .01$ by 1 student. These students seemed to realize that they needed to convert minutes to hours, but do not know the correct procedure.

INSTRUCTIONAL IMPLICATIONS

Canceling Units Instruction

My initial approach to improving quantitative reasoning was to use the cancellation of units method. I personally found this technique to be very helpful when I took an introductory physics course, and the endorsement in the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000) encouraged me to expect a successful outcome. The finding that the canceling units instruction cut the success rate in half caused me to evaluate both what might have gone wrong and how I could design a more successful intervention.

A leading candidate for what might have gone wrong is the creation of cognitive overload by adding additional components to a very demanding construction task. Students now had to construct equations that included units in addition to the mandatory numbers, symbols, and arithmetic operations. Although the cognitive overload hypothesis provides a possible explanation of why the units instruction failed, it also provides principles for predicting when canceling units might succeed (Sweller, 2003).

Canceling units might be more successful when used as a checking procedure after an equation has been constructed. The checking procedure would require adding units to the constructed equation. This allows students to initially construct equations without having to determine how to integrate the units with the rest of the equation. Dividing the task into two phases should reduce the cognitive load imposed by attempting to accomplish the entire task in a single phase.

However, canceling units might be successfully accomplished in a single phase if the students had greater expertise in solving word problems. Sweller (2003) cited several studies conducted by his research group that found an expertise reversal effect. This effect occurs when cognitive overload is eliminated after learners acquire some knowledge, thereby changing which instructional interventions are effective. My statistics students clearly had trouble constructing equations for word problems, so the results may have differed for students with greater expertise in solving mathematics problems.

I have discussed the failure of the canceling units instruction with several colleagues including an engineering professor, a chemistry professor, and an applied mathematics professor. All three were very surprised by the results. Their classroom experiences may differ because of the greater mathematics ability of their students and because the instruction utilizes familiar formulas. There is a need to further investigate the conditions in which canceling units might be effective. The cognitive load theory (Sweller, 2003) should provide a helpful framework for these investigations.

Identifying Referents Instruction

The failure of the canceling units instruction led to the design of the identifying referents instruction in Experiment 2. This instruction utilized the contrasting cases approach developed by D. L. Schwartz and Bransford (1998) to help students discriminate between important concepts that they would need to understand subsequent instruction. I implemented the contrasting cases technique by asking students to discriminate between pairs of mathematical expressions to identify which one referred to real world referents. This approach reduces cognitive load by providing instruction and practice on grounding referents before students construct equations. Identifying referents also addresses P. W. Thompson's (1994) concern that canceling units is a mechanical procedure that does not require students to consider the meaning of the terms in an equation.

The identifying-referents instruction significantly increased performance on constructing equations, relative to a control group that initially practiced constructing equations for a pair of algebra word problems. These findings become more informative if placed within the broader context of students' difficulties in making the transition from arithmetic to algebra. The difficulty is typically discussed in terms of the many differences that students encounter as they make this transition, such as the new meanings assigned to the equals sign and to arithmetic operations (Kieren, 1992). It is, therefore, important that students have mastered skills such as connecting symbols to referents before they make this transition. The errors documented in my research show that college students still have difficulty with this skill, even when it requires only arithmetic reasoning.

The burden these errors place on learning equations for algebra word problems can be illustrated by making an analogy to reading. Learning to read requires analyzing the features of letters, combining the features to identify letters, converting the letters into sounds for pronouncing words, understanding the meanings of individual words, and combining the meaning of the words to comprehend the text. LaBerge and Samuels (1974) proposed that the ability to acquire complex, multicomponent skills such as reading depends on being able to automatically carry out some component skills to avoid cognitive overload. Making the transition to algebra with weak skills in representing quantities is like making the transition to reading with weak skills in recognizing words. Although my research findings demonstrate a need for a greater emphasis on connecting symbols to referents in K–8 education, they do not address the issue of how to best accomplish this objective.

A second, more general use of the identifying referents task is its incorporation into methods of assessment. If connecting mathematical expressions to referents is important, then it needs to be assessed regardless of which instructional method is used (Pellegrino, Chudowsky, & Glaser, 2001). My development of the contrasting cases task was influenced by an assessment task designed by Anderson, Fisher, and Norman (2002) to measures students' misconceptions about natural selection. The incorrect answers (foils) in their multiple choice questions were ones that researchers had previously identified as being common misconceptions. Likewise, the incorrect answers in my contrasting cases task were mistakes that had frequently occurred when college students constructed equations for word problems. These included mixing units such as multiplying 2 mph by $t + 6$ min, creating no units such as isolating a percentage, making unnecessary calculations such as multiplying 60 lb by 10.4 lb, and selecting incorrect arithmetic operations such as adding rate and time. Contrasting cases could, therefore, be used either as a method of instruction or as a method of assessment following an alternative method of instruction.

CONCLUSION

As advocated by D. L. Schwartz and Bransford (1998), the contrasting cases instruction is intended to help students learn important conceptual distinctions that would be elaborated in subsequent instruction. The instruction investigated in Experiment 2 fits this intention. It sought to help students discriminate between mathematical expressions that either do or do not represent appropriate quantities in preparation for additional instruction that builds on this foundation.

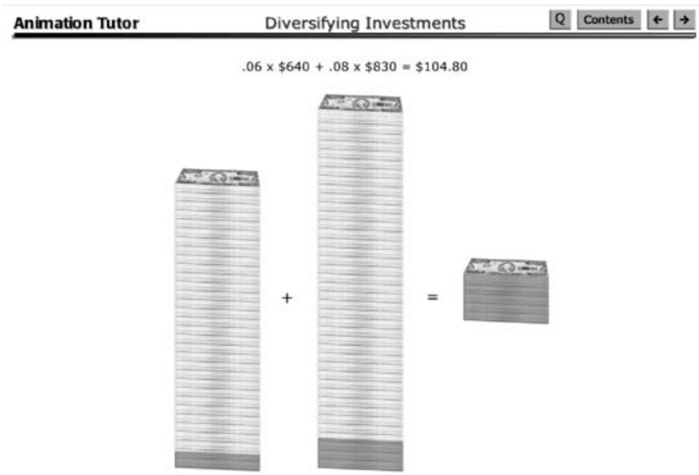

FIGURE 3 Iconic representation of earning $104.80 interest from investing $640 at 6% interest and $830 at 8% interest.

The Animation Tutor™ (Reed & Hoffman, 2006) can build on this foundation to show (a) the interrelationships among the quantities in an equation and (b) the connection of the visual representations to variables in the equation (Reed, 2005). For example, Figure 3 shows an iconic representation of an interest problem in which the total interest is the sum of the interest earned from each account (as depicted by the darker shading). The Animation Tutor™ instruction could be preceded by the contrasting cases instruction to prepare students to think about the terms of an equation as referents that can be combined to form equations. Iconic representations of these referents can then be physically manipulated by students to see how they interact in equations.

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Web site, including demonstrations and experiments, can be accessed at http://www.sci.sdsu.edu/mathtutor/

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