Last Name:
First Name:
Instructor:

Math 151
Group Final (Spring 2008)

You are not allowed to use notes, books, calculators, personal stereos or cell phones.

You have exactly two hours.

Write clearly so that you can avoid mistakes and count on partial credits. Carry out the obvious simplifications so that you can display your answers in an easily readable manner.

The following list is for the recording of the points only. Do not write your answers on this page.

Points

- **1** /8
- **2** /6
- **3** /6
- **4** /8
- **5** /8
- **6** /8
- **7** /8
- **8** /8
- **9** /10
- **10** /6
- **11** /8
- **12** /8
- **13** /8

Total: **/100**

1 (8 pts.) Evaluate

$$\int \frac{\ln(x)}{x^{2/3}}dx$$

2 (6 pts.) Evaluate

$$\int \frac{4x - 22}{x^2 - 2x - 8} dx$$

3 (6 pts.) Evaluate

$$\int \sin^2\left(\frac{x}{3}\right) dx.$$

4 (8 pts.) Determine whether the improper integral

$$\int_1^\infty e^{-x^2} x dx$$

converge or diverges, and its value in case it converges.

5 (8 pts.) Use a comparison test to determine whether the improper integral

$$\int_2^4 \frac{\cos^2(\pi x)}{\sqrt{x-2}}\,dx$$

converges or diverges. In case of convergence, you need to calculate the value of the improper integral that serves as a basis for comparison.

6 (8 pts.) Use the method of cylindrical shells to calculate the volume of the solid that is obtained by revolving the region between the graph of

$$f(x) = \frac{1}{\sqrt{x^2 + 4}}$$

and the interval $[0, 1]$ about the vertical axis.

7 (8 pts.) Use the technique of an integrating factor to find the solution of the initial value problem

$$\frac{dy}{dt} = -y + \cos(4t), \quad y(0) = 1.$$

Hint:

$$\int e^{at}\cos(bt)\,dt = \frac{a}{a^2+b^2}e^{at}\cos(bt) + \frac{b}{a^2+b^2}e^{at}\sin(bt)$$

8 (8 pts.) Find the solution of the initial value problem

$$\frac{dy}{dt} = \sinh(t)\, y^2,\ \ y(0) = 2$$

9. Let

$$r = f(\theta) = 1 + \cos(\theta).$$

a) (2 pts.) Sketch the graph of $r = f(\theta)$ in the Cartesian θr-plane on the interval $[0, 2\pi]$. Indicate the values of θ at which $f(\theta) = 0$ and the points at which f attains a local maximum or minimum value.

b) (8 pts.) Sketch the graph of $r = f(\theta)$, where $0 \leq \theta \leq 2\pi$, as a polar equation in the xy-plane (i.e., $x = r\cos(\theta)$, $y = r\sin(\theta)$).

10 (6 pts.) Determine whether the infinite series

$$\sum_{n=1}^{\infty} \frac{e^n}{n!}$$

converges or diverges.

11 (8 pts.) Determine whether the infinite series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n^3+1}}$$

converges absolutely, converges conditionally or diverges (you may have to make use of a comparison test).

12 (8 pts.) Determine the radius of convergence and the open interval of convergence of the power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{4^n}{n^2} (x-2)^n$$

(You need not investigate the series at the endpoints of the interval).

13 (8 pts.) Let $f(x) = \ln(x)$. Determine the part of the Taylor series for f in powers of $x - e$ up to the term that is a multiple of $(x-e)^4$.

Solutions

1. We set $u = \ln(x)$ and $dv = x^{-2/3}dx$ so that

$$du = \frac{1}{x}dx \text{ and } v = \int x^{-2/3}dx = \frac{x^{1/3}}{1/3} = 3x^{1/3}.$$

Thus,

$$\begin{aligned} \int \frac{\ln(x)}{x^{2/3}}dx &= \int udv \\ &= uv - \int vdu \\ &= \ln(x)\left(3x^{1/3}\right) - \int 3x^{1/3}\left(\frac{1}{x}\right)dx \\ &= 3x^{1/3}\ln(x) - 3\int x^{-2/3}dx \\ &= 3x^{1/3}\ln(x) - 3\left(3x^{1/3}\right) \\ &= 3x^{1/3}\ln(x) - 9x^{1/3}. \end{aligned}$$

2.

$$\frac{4x-22}{x^2-2x-8} = \frac{4x-22}{(x+2)(x-4)} = \frac{A}{x+2} + \frac{B}{x-4}$$

$\Leftrightarrow$

$$4x - 22 = A(x-4) + B(x+2).$$

We set $x = -2$:

$$-30 = -6A \Rightarrow A = 5.$$

We set $x = 4$:

$$-6 = 6B \Rightarrow B = -1.$$

Thus,

$$\frac{4x-22}{(x+2)(x-4)} = \frac{5}{x+2} - \frac{1}{x-4}$$

Therefore,

$$\int \frac{4x-22}{x^2-2x-8}dx = 5\ln(|x+2|) - \ln(|x-4|) + C.$$

3.

$$\begin{aligned} \int \sin^2\left(\frac{x}{3}\right)dx &= \int \frac{1-\cos(2x/3)}{2}dx \\ &= \frac{1}{2}x - \frac{1}{2}\left(\frac{3}{2}\right)\sin\left(\frac{2}{3}x\right) + C \\ &= \frac{1}{2}x - \frac{3}{4}\sin\left(\frac{2}{3}x\right) + C. \end{aligned}$$

4,. We have

$$\int e^{-x^2}xdx = -\frac{1}{2}e^{-x^2}.$$

Therefore,

$$\int_1^b e^{-x^2} x dx = -\frac{1}{2}e^{-b^2} + \frac{1}{2}e^{-1}.$$

Thus,

$$\lim_{b\to\infty} \int_1^b e^{-x^2} x dx = \frac{1}{2e}.$$

Therefore, the improper integral converges and we have

$$\int_1^\infty e^{-x^2} x dx = \frac{1}{2e}.$$

5. We have

$$0 \le \frac{\cos^2(\pi x)}{\sqrt{x-2}} \le \frac{1}{\sqrt{x-2}}.$$

Furthermore.

$$\int_{2+\varepsilon}^4 \frac{1}{\sqrt{x-2}} dx = \int_\varepsilon^2 \frac{1}{\sqrt{u}} du = 2\sqrt{u}\Big|_\varepsilon^2 = 2\sqrt{2} - 2e\sqrt{\varepsilon}.$$

Therefore,

$$\int_2^4 \frac{1}{\sqrt{x-2}} dx = 2\sqrt{2}.$$

Thus, the given integral converges as well.

6. The volume is

$$2\pi \int_0^1 \frac{x}{\sqrt{x^2+4}} dx.$$

We set $u = x^2 + 4$. Thus,

$$\begin{aligned} 2\pi \int_0^1 \frac{x}{\sqrt{x^2+4}} dx &= 2\pi \int_4^5 u^{-1/2} \left(\frac{1}{2}\right) du \\ &= \pi \left(2\sqrt{u}\Big|_4^5\right) = 2\pi\left(\sqrt{5}-2\right). \end{aligned}$$

7.

$$\frac{dy}{dt} + y = \cos(4t);$$

$$e^t\left(\frac{dy}{dt} + y\right) = e^t \cos(4t);$$

$$\frac{d}{dt}\left(e^t y\right) = e^t \cos(4t);$$

$$\begin{aligned} e^t y(t) &= \int e^t \cos(4t)\, dt \\ &= \frac{1}{17} e^t \cos(4t) + \frac{4}{17} e^t \sin(4t) + C; \end{aligned}$$

$$y(t) = \frac{1}{17}\cos(4t) + \frac{4}{17}\sin(4t) + Ce^{-t}.$$

We have $y(0) = 1$ iff

$$\frac{1}{17} + C = 1 \Leftrightarrow C = 1 - \frac{1}{17} = \frac{16}{17}.$$

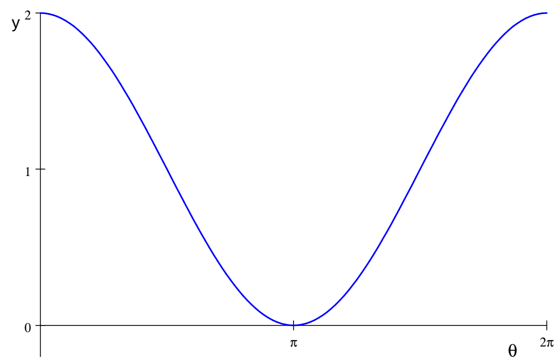
Therefore, the solution is

$$y(t) = \frac{1}{17}\cos(4t) + \frac{4}{17}\sin(4t) + \frac{16}{17}e^{-t}$$

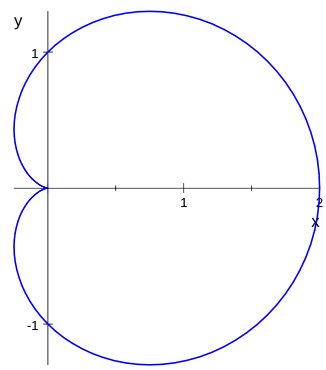
8.

$$\begin{aligned} \frac{1}{y^2}\frac{dy}{dt} &= \sinh(t); \\ \int \frac{1}{y^2} dy &= \cosh(t) + C; \\ -\frac{1}{y} &= \cosh(t) + C; \\ -\frac{1}{2} &= 1 + C;\ C = -\frac{3}{2}; \\ y &= -\frac{1}{-\frac{3}{2} + \cosh(t)} = \frac{2}{3 - 2\cosh(t)} \end{aligned}$$

9.
a)



b)



10.

$$\lim_{n\to\infty} \frac{\frac{e^{n+1}}{(n+1)!}}{\frac{e^n}{n!}} = \lim_{n\to\infty} \left(\frac{1}{n}e\right) = 0 < 1.$$

Therefore the series converges.

11. We have

$$\left|(-1)^{n-1}\frac{1}{\sqrt{n^3+1}}\right| = \frac{1}{\sqrt{n^3+1}} < \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}.$$

Since the series $\sum 1/n^{3/2}$ converges $(3/2 > 1)$, the given series converges absolutely.

12. We have

$$\lim_{n\to\infty}\left|(-1)^n\frac{4^n}{n^2}(x-2)^n\right|^{1/n}=|x-2|\lim_{n\to\infty}\frac{4}{n^{2/n}}=4|x-2|<1$$

if

$$|x-2|<\frac{1}{4}.$$

Therefore the radius of convergence of the series is $1/4$. The open interval of convergence is

$$\left(2-\frac{1}{4},2+\frac{1}{4}\right)=\left(\frac{7}{4},\frac{9}{4}\right).$$

13. We have

$$\begin{aligned} f(x) &= \ln(x)\Rightarrow f(e)=\ln(e)=1, \\ f'(x) &= \frac{1}{x}\Rightarrow f'(e)=\frac{1}{e}, \\ f''(x) &= -x^{-2}\Rightarrow f''(e)=-\frac{1}{e^2}, \\ f'''(x) &= 2x^{-3}\Rightarrow f^{(3)}(e)=\frac{2}{e^3}, \\ f^{(4)}(x) &= -6x^{-4}\Rightarrow f^{(4)}(e)=-\frac{6}{e^4}. \end{aligned}$$

Therefore,

$$\ln(x)=1+\frac{1}{e}(x-e)-\frac{1}{2e^2}(x-e)^2+\frac{1}{3e^3}(x-e)^3-\frac{1}{4e^4}(x-e)^4+\cdots$$