Math 151
Group Final (Fall 2007)
Solutions

1. We set $u = \arctan(x/4)$ and $dv = dx$ so that

$$du = \frac{1}{1+x^2/16}\left(\frac{1}{4}\right)dx = \frac{4}{16+x^2}dx \text{ and } v = x.$$

Thus,

$$\begin{aligned}\int \arctan\left(\frac{x}{4}\right)dx = \int udv &= uv - \int vdu \\ &= \arctan\left(\frac{x}{4}\right)(x) - \int x\left(\frac{4}{16+x^2}\right)dx \\ &= x\arctan\left(\frac{x}{4}\right) - 4\int \frac{x}{16+x^2}dx.\end{aligned}$$

In order to evaluate the integral on the RHS, we set $w = 16 + x^2$ so that $dw = 2x$. Thus,

$$\int \frac{x}{16+x^2}dx = \frac{1}{2}\int \frac{1}{w}dw = \frac{1}{2}\ln(|w|) = \frac{1}{2}\ln\left(16+x^2\right).$$

Therefore,

$$\begin{aligned}\int \arctan\left(\frac{x}{4}\right)dx &= x\arctan\left(\frac{x}{4}\right) - 4\int \frac{x}{16+x^2}dx \\ &= x\arctan\left(\frac{x}{4}\right) - 2\ln\left(16+x^2\right).\end{aligned}$$

2.

$$\frac{x-17}{x^2+x-12} = \frac{x-17}{(x+4)(x-3)} = \frac{A}{x+4} + \frac{B}{x-3}$$

$\Leftrightarrow$

$$x - 17 = A(x-3) + B(x+4).$$

Set $x = -4$:

$$-21 = -7A \Rightarrow A = 3.$$

Set $x = 3$:

$$-14 = 7B \Rightarrow B = -2.$$

Thus,

$$\frac{x-17}{x^2+x-12} = \frac{3}{x+4} - \frac{2}{x-3}.$$

Therefore,

$$\int \frac{x-17}{x^2+x-12}dx = 3\ln(|x+4|) = 2\ln(|x-3|).$$

3. We set $u = x$ and $dv = \cos(\pi x)$ so that

$$du = dx \text{ and } v = \int \cos(\pi x)\,dx = \frac{1}{\pi}\sin(\pi x).$$

Thus,

$$\begin{aligned}\int x\cos(\pi x)\,dx = \int u dv &= uv - \int v du \\ &= x\left(\frac{1}{\pi}\sin(\pi x)\right) - \int \left(\frac{1}{\pi}\sin(\pi x)\right) dx \\ &= \frac{1}{\pi}x\sin(\pi x) + \frac{1}{\pi^2}\cos(\pi x).\end{aligned}$$

4.
a) The integral is improper since

$$\lim_{x\to 3+} \frac{1}{(x-3)^{2/3}} = +\infty.$$

b) We set $u = x - 3$ so that $du = dx$. Thus, for any ε such that $0 < \varepsilon < 1$,

$$\int_{3+\varepsilon}^{4} \frac{1}{(x-3)^{2/3}} dx = \int_{\varepsilon}^{1} u^{-2/3} du = 3u^{1/3}\Big|_{\varepsilon}^{1} = 3 - 3\varepsilon^{1/3}.$$

Therefore,

$$\lim_{\varepsilon\to 0+} \int_{3+\varepsilon}^{4} \frac{1}{(x-3)^{2/3}} dx = \lim_{\varepsilon\to 0+} \left(3 - 3\varepsilon^{1/3}\right) = 3.$$

Thus, the given improper integral converges and

$$\int_{3}^{4} \frac{1}{(x-3)^{2/3}} dx = 3.$$

5. We have

$$0 < \frac{1}{\sqrt{x^4+1}} < \frac{1}{\sqrt{x^4}} = \frac{1}{x^2},$$

and

$$\int_1^{\infty} \frac{1}{x^2} dx$$

converges. Therefore, the given integral converges as well.
6. The length of the graph is

$$\begin{aligned}\int_0^1 \sqrt{1 + \left(\frac{d}{dx}\cosh(x)\right)^2} dx &= \int_0^1 \sqrt{1+\sinh^2(x)} dx \\ &= \int_0^1 \sqrt{\cosh^2(x)} dx \\ &= \int_0^1 \cosh(x)\,dx \\ &= \sinh(x)|_0^1 \\ &= \sinh(1) = \frac{1}{2}\left(e - e^{-1}\right).\end{aligned}$$

7. We have

$$\frac{dy}{dt} - \frac{1}{3t}y = -\frac{1}{3}$$

Thus, the integrating factor is

$$\exp\left(-\frac{1}{3}\int\frac{1}{t}dt\right) = \exp\left(-\frac{1}{3}\ln(t)\right) = (\exp(\ln t))^{-1/3} = t^{-1/3}.$$

Thus,

$$t^{-1/3}\frac{dy}{dt} + t^{-1/3}\left(-\frac{1}{3t}\right)y = -\frac{1}{3}t^{-1/3}$$

$\Rightarrow$

$$t^{-1/3}\frac{dy}{dt} - \frac{1}{3}t^{-4/3}y = -\frac{1}{3}t^{-1/3}$$

$\Rightarrow$

$$\frac{d}{dt}\left(t^{-1/3}y\right) = -\frac{1}{3}t^{-1/3}$$

$\Rightarrow$

$$t^{-1/3}y(t) = -\frac{1}{3}\left(\frac{3}{2}t^{2/3}\right) + C$$

$\Rightarrow$

$$y(t) = -\frac{1}{2}t + Ct^{1/3}$$

We have

$$y(1) = 4 \Leftrightarrow 4 = -\frac{1}{2} + C \Leftrightarrow C = 4 + \frac{1}{2} = \frac{9}{2}.$$

Therefore,

$$y(t) = -\frac{1}{2}t + \frac{9}{2}t^{1/3}$$

8.

$$\begin{aligned}\frac{1}{y^2+1}\frac{dy}{dx} = 1 \quad &\Rightarrow \quad \int\frac{1}{y^2+1}\frac{dy}{dx}dx = \int 1dx\\ &\Rightarrow \quad \int\frac{1}{y^2+1}dy = x + C\\ &\Rightarrow \quad \arctan(y) = x + C\\ &\Rightarrow \quad y(x) = \tan(x+C).\end{aligned}$$

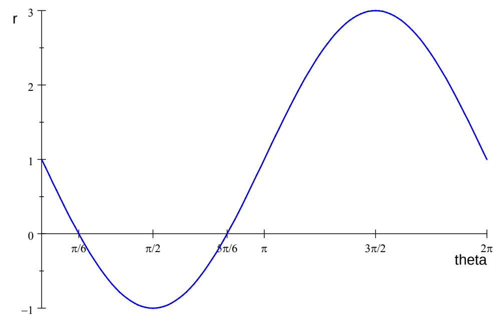
We have

$$y\left(\frac{\pi}{4}\right) = 0 \Leftrightarrow \tan\left(\frac{\pi}{4} + C\right) = 0.$$

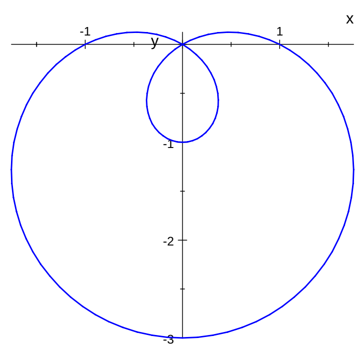
We can set $C = -\pi/4$. (any integer multiple of π may be aded). Thus,

$$y(x) = \tan\left(x - \frac{\pi}{4}\right).$$

9.
a)



b)



10. We have

$$\lim_{n\to\infty}\left(\frac{2^n}{n^{10}}\right)^{1/n} = \lim_{n\to\infty}\frac{2}{\left(n^{1/n}\right)^{10}} = \frac{2}{1} = 2 > 1.$$

The series diverges.by the root test.

11. We have

$$\left|(-1)^{n-1}\,\frac{e^{1/n}}{n^2+1}\right| = \frac{e^{1/n}}{n^2+1} \le \frac{e}{n^2+1} < \frac{e}{n^2},$$

and $e\sum 1/n^2$ converges. Therefore, the given series converges absolutely.

12. We have

$$\lim_{n\to\infty}\frac{\left|(-1)^n\,\dfrac{1}{n+1}\,(x-1)^{n+1}\right|}{\left|(-1)^{n-1}\,\frac{1}{n}\,(x-1)^n\right|} = |x-1|\lim_{n\to\infty}\left(\frac{n}{n+1}\right) = |x-1|\,.$$

Therefore, the series converges if $|x-1| < 1$ and diverges if $|x-1| > 1$. Thus, the radius of convergence of the series is 1 and the open interval of convergence is $(0, 2)$.

13. We have

$$f(x) = x^{2/3},\ f'(x) = \frac{2}{3}x^{-1/3},\ f''(x) = -\frac{2}{9}x^{-4/3},\ f^{(3)}(x) = \frac{8}{27}x^{-7/3}.$$

Thus,

$$f(1) = 1,\ f'(1) = \frac{2}{3},\ f''(1) = -\frac{2}{9},\ f^{(3)}(1) = \frac{8}{27}.$$

Therefore,

$$\begin{aligned} f(x) &= f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 + \frac{1}{3!}f^{(3)}(1)(x-1)^3 + \cdots \\ &= 1 + \frac{2}{3}(x-1) - \frac{1}{9}(x-1)^2 + \frac{4}{81}(x-1)^3 + \cdots \end{aligned}$$