Math 150
Study Guide for the Group Final

Revised: September 2009

Questions on the group final will be similar to the sample problems provided in this document. The group final is multiple choice and there are 25 problems. No books, notes, calculators, cell phones or personal stereos are allowed.

Questions will be selected from the following list of topics:

- **Finite or infinite limits at a point or at infinity, L'Hospital's rule**
- **Differentiation of combinations of powers of x, trigonometric, exponential and logarithmic functions**
- **Indefinite and definite integrals, the substitution rule**
- **Local linear approximations and the differential**
- **Related rates**
- **Implicit differentiation**
- **Monotonicity, local and absolute extrema and applications**
- **Concavity**
- **The Fundamental Theorem of Calculus**
- **One-dimensional motion**
- **Area between two curves**

In preparation for the group final, students should memorize the following **differentiation formulas**.

1. $\frac{d}{dx}x^r = rx^{r-1}$
2. $\frac{d}{dx}\sin(x) = \cos(x)$
3. $\frac{d}{dx}\cos(x) = -\sin(x)$
4. $\frac{d}{dx}\tan(x) = \frac{1}{\cos^2(x)}$
5. $\frac{d}{dx}a^x = \ln(a)\, a^x$
6. $\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$
7. $\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$
8. $\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}$
9. $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$

Students should memorize the following **antidifferentiation formulas**.

1. $\int x^r dx = \frac{1}{r+1}x^{r+1} \quad (r \neq -1)$
2. $\int \frac{1}{x}dx = \ln(|x|)$
3. $\int \sin(x)\, dx = -\cos(x)$
4. $\int \cos(x)\, dx = \sin(x)$
5. $\int e^x dx = e^x$
6. $\int a^x dx = \frac{1}{\ln(a)}a^x$
7. $\int \frac{1}{1+x^2}dx = \arctan(x)$
8. $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x)$

Remark: Students should be able to compute the exact **values of the trigonometric functions at special angles** such as $\pi/3$, $\pi/6$, $\pi/4$, $2\pi/3$, $-3\pi/4$ and exact **values of the inverse trigonometric functions** such as $\arctan\left(\sqrt{3}\right)$, $\arcsin(-1/2)$, $\arccos(0)$.

SAMPLE PROBLEMS for the GROUP FINAL

1. Determine the finite or infinite limit or state that the limit does not exist in either the finite or infinite sense:

a)
$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$$

b)
$$\lim_{x\to 0} \frac{\sin(2x)}{x}$$

c)
$$\lim_{x\to 0} \frac{\cos(x)-1}{x}$$

d)
$$\lim_{x\to +\infty} \frac{x^2+3x-1}{4x^2-7}$$

e)
$$\lim_{x\to 2-} \frac{1}{x-2}$$

f)
$$\lim_{x\to -3} \frac{2}{x+3}$$

g)
$$\lim_{x\to 0+} \sqrt{x}\ln(x)$$

(using L'Hospital's rule)

h)
$$\lim_{x\to +\infty} x^2 e^{-x}$$

(using L'Hospital's rule)

i)
$$\lim_{x\to +\infty} \left(1-\frac{3}{x}\right)^x$$

(using L'Hospital's rule)

2. Determine the derivative:

a)
$$\frac{d}{dx}\left(x^2\cos(x)\right)$$

b)
$$\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

c)
$$\frac{d}{dx}\sin\left(x^2\right)$$

d)
$$\frac{d}{dx}\sqrt{4+x^2}$$

e)
$$\frac{d}{dx}\left(x^2e^{-x/10}\right)$$

f)
$$\frac{d}{dx}\left(\frac{1}{2}\sin(2x)+\frac{1}{3}\cos(3x)\right)$$

g)
$$\frac{d}{dx}\arcsin(x/4)$$

h)
$$\frac{d}{dx}\ln\left(\frac{x}{x^3+1}\right)$$

i)
$$\frac{d}{dx}\int_0^x e^{\left(t^2\right)}dt$$

j)
$$\frac{d}{dx}\sin\left(\frac{x}{2+x}\right)\Bigg|_{x=0}$$

k)
$$\frac{d}{dx}\ln\left(\sqrt{x}\right)\Big|_{x=1}$$

l)
$$\frac{d}{dx}\arccos(2x)\Big|_{x=0}$$

3. Determine the indefinite or definite integral:

a)
$$\int x\sin\left(3x^2\right)dx$$

b)
$$\int x\sqrt{x^2+4}dx$$

c)
$$\int 10^{-4x}dx$$

d)
$$\int \frac{2}{3-4x}dx$$

e)
$$\int \frac{1}{(2-3x)^2}dx$$

f)
$$\int \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}dx$$

g)
$$\int \frac{x}{x^2+9}dx$$

h)
$$\int \frac{1}{4x^2+9}dx$$

i)
$$\int e^{-x/2}dx$$

j)
$$\int \frac{1}{\sqrt{9-4x^2}}dx$$

k)
$$\int_{\pi/3}^{\pi/4} \sin(x)\,dx$$

l)
$$\int_{-3}^{3} \frac{1}{x^2+9}dx$$

m)
$$\int_{1}^{e} \frac{\ln(x)}{x}dx$$

n)
$$\int_{0}^{1} \left(\frac{d}{dx}e^{-x^2}\right)dx$$

4. Let $f(x) = x^{2/3}$. Determine the tangent line to the graph of f at $(8, f(8))$

5. Let
$$f(x) = \frac{1}{\sqrt{x}}.$$

a) Determine L, the linear approximation to f based at 4.
b) Make use of L to approximate
$$\frac{1}{\sqrt{3.9}}$$

6. Let $f(x) = \tan(x)$.
a) Determine the differential df.
b) Make use of df to approximate
$$\tan\left(\frac{\pi}{4} - 0.1\right).$$

7. Assume that helium is being pumped into a spherical balloon at a constant rate of 100 cubic centimeters per second. Also assume that the shape of the balloon is a perfect sphere as it is being inflated. Determine the rate at which the radius of the balloon is increasing at the instant its radius is 10 centimeters.

8. An airplane is flying at an altitude of 2 miles with a speed of 200 miles/hour. It is being tracked by an observer on the ground with a searchlight. Find the rate at which the angle θ between the searchlight and the vertical direction changes at the instant the horizontal distance of the plane from the observer is 10 miles.

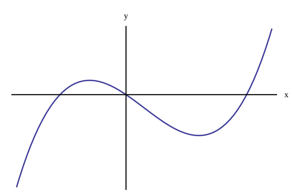
9. Assume that a ladder which is 10 feet long is leaning against a wall and its base is sliding away from the wall at the rate of 2 ft/sec. Determine the rate at which the top of the ladder is sliding down the wall at the instant the base of the ladder is 4 feet from the wall.

10. Consider the equation

$$y^5 + y^2 - y - x^2 + 1 = 0.$$

a) Assume that $y(x)$ represents a function that is defined implicitly by the given equation. Determine $y'(x)$.
b) Evaluate $y'(1)$ if $y(1) = 0$. Determine the tangent line to the graph of the equation at $(1, 0)$.

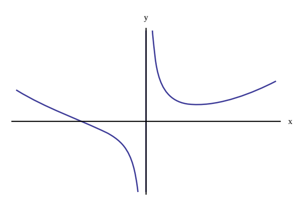
11. Let $f(x) = -2x - \frac{1}{2}x^2 + \frac{1}{3}x^3$. The figure shows the graph of f. Determine the points at which f has a local maximum or local minimum.



12. Let

$$f(x) = x^2 + \frac{1}{x}$$

The figure shows the graph of f. Determine the point at which f attains its absolute minimum on $(0, +\infty)$.



12. Let $f(x) = (x^2 - 4)^{2/5}$. Determine the absolute maximum and the absolute minimum of f on the interval $[-2, 3]$.
13. Let

$$f(x) = \frac{e^x}{x^2}.$$

Determine $\lim_{x \to 0} f(x)$ and the absolute minimum of f on $(0, +\infty)$. Determine the subintervals of $(0, +\infty)$ on which f is increasing/decreasing and the absolute maximum and the absolute minimum of f on $(0, +\infty)$, provided that such values exist. You have to justify a claim of nonexistence.

15. Let

$$f(x) = x + \frac{1}{x-2}$$

a) Determine the vertical asymptotes for the graph of f and the relevant infinite limits.
b) Show that the line $y = x$ is an oblique asymptote for the graph of f at $\pm\infty$.
c) Determine the intervals on which f is increasing/decreasing and the points at which f has a local maximum or local minimum.
d) Sketch the graph of f. Indicate the asymptotes and the local extrema.

16. Let

$$f(x) = -8x - x^2 + \frac{1}{3}x^3.$$

Use the second derivative test to determine the points at which f has a local maximum or a local minimum.

17. Let

$$f(x) = x^2 e^{-x}$$

Determine the inflection points of the graph of f.

18. Determine the dimensions of the rectangle that has the greatest area among all rectangles which are inscribed in a circle of radius 1.

19. Determine $y(x)$ if

$$\frac{dy}{dx} = \frac{1}{x^2+1} \text{ and } y(1) = \pi$$

20. Assume that $v(t) = \cos(4t)$ is the velocity at the instant t of an object in one-dimensional motion and let $f(t)$ be its position at time t. Assume that $f(\pi/8) = -2$. Determine $f(t)$.

21 Let $f(x) = x^2 - 2x - 1$ and $g(x) = -x^2 + 2x + 5$. Calculate the area of the region between the graph of f, the graph of g, the line $x = 1$ and the line $x = 5$.

Solutions

1.

a)

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}=\frac{1}{4}$$

b)

$$\lim_{x\to 0}\frac{\sin(2x)}{x}=2$$

c)

$$\lim_{x\to 0}\frac{\cos(x)-1}{x}=0$$

d)

$$\lim_{x\to +\infty}\frac{x^2+3x-1}{4x^2-7}=\frac{1}{4}$$

e)

$$\lim_{x\to 2-}\frac{1}{x-2}=-\infty$$

f)

$$\lim_{x\to -3}\frac{2}{x+3}\text{ does not exist.}$$

g)

$$\lim_{x\to 0+}\sqrt{x}\ln(x)=0$$

h)

$$\lim_{x\to +\infty}x^2e^{-x}=0$$

i)

$$\lim_{x\to +\infty}\left(1-\frac{3}{x}\right)^x=e^{-3}$$

2.

a)

$$\frac{d}{dx}\left(x^2\cos(x)\right)=2x\cos(x)-x^2\sin(x)$$

b)

$$\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)=\frac{-4x}{\left(1+x^2\right)^2}$$

c)

$$\frac{d}{dx}\sin\left(x^2\right)=2x\cos\left(x^2\right)$$

d)

$$\frac{d}{dx}\sqrt{4+x^2}=\frac{x}{\sqrt{4+x^2}}$$

e)

$$\frac{d}{dx}\left(x^2e^{-x/10}\right)=2xe^{-x/10}-\tfrac{1}{10}x^2e^{-x/10}$$

f)

$$\frac{d}{dx}\left(\frac{1}{2}\sin(2x)+\frac{1}{3}\cos(3x)\right)=\cos(2x)-\sin(3x)$$

g)

$$\frac{d}{dx}\arcsin(x/4)=\frac{1}{\sqrt{16-x^2}}$$

h)

$$\frac{d}{dx}\ln\left(\frac{x}{x^3+1}\right)=\frac{1}{x}-\frac{3x^2}{x^3+1}$$

i)

$$\frac{d}{dx}\int_0^x e^{(t^2)}dt=e^{(x^2)}$$

j)

$$\frac{d}{dx}\sin\left(\frac{x}{2+x}\right)\Big|_{x=0}=\frac{1}{2}$$

k)

$$\frac{d}{dx}\ln\left(\sqrt{x}\right)\Big|_{x=1}=\frac{1}{2}$$

l)

$$\frac{d}{dx}\arccos(2x)\Big|_{x=0}=-2$$

3. (the arbitrary constants have been omitted)

a)

$$\int x \sin\left(3x^2\right) dx = -\frac{1}{6}\cos\left(3x^2\right)$$

b)

$$\int x\sqrt{x^2+4}dx = \frac{1}{3}\left(4+x^2\right)^{3/2}$$

c)

$$\int 10^{-4x} dx = -\frac{1}{4\ln(10)}10^{-4x}$$

d)

$$\int \frac{2}{3-4x}dx = -\frac{1}{2}\ln(|3-4x|)$$

e)

$$\int \frac{1}{(2-3x)^2}dx = \frac{1}{3(2-3x)}$$

f)

$$\int \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}dx = 2\sin\left(\sqrt{x}\right)$$

g)

$$\int \frac{x}{x^2+9}dx = \frac{1}{2}\ln\left(x^2+9\right)$$

h)

$$\int \frac{1}{4x^2+9}dx = \frac{1}{6}\arctan\left(\frac{2}{3}x\right)$$

i)

$$\int e^{-x/2}dx = -2e^{-x/2}$$

j)

$$\int \frac{1}{\sqrt{9-4x^2}}dx = \frac{1}{2}\arcsin\left(\frac{2}{3}x\right)$$

k)

$$\int_{\pi/3}^{\pi/4} \sin(x)\, dx = \frac{1}{2} - \frac{\sqrt{2}}{2}$$

l)

$$\int_{-3}^{3} \frac{1}{x^2+9}dx = \frac{\pi}{6}$$

m)

$$\int_1^e \frac{\ln(x)}{x}dx = \frac{1}{2}$$

n)

$$\int_0^1 \left(\frac{d}{dx}e^{-x^2}\right) dx = e^{-1} - 1$$

4. We have

$$f(x) = x^{2/3} \text{ and } f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}.$$

Therefore,

$$f(8) = 8^{2/3} = 4 \text{ and } f'(x) = \frac{2}{3\left(8^{1/3}\right)} = \frac{1}{3}.$$

Thus, the tangent line to the graph of f at $(8, f(8))$ is the graph of the equation

$$y = f(8) + f'(8)(x-8) = 4 + \frac{1}{3}(x-8).$$

5.

a)

$$f'(x) = \frac{d}{dx}\frac{1}{\sqrt{x}} = \frac{d}{dx}x^{-1/2} = -\frac{1}{2}x^{-3/2} = -\frac{1}{2x^{3/2}}.$$

Therefore,

$$f'(4) = -\frac{1}{2\left(4^{3/2}\right)} = -\frac{1}{16}$$

Thus,

$$L_4(x) = f(4) + f'(4)\,x = \frac{1}{2} - \frac{1}{16}(x-4).$$

b)

$$\frac{1}{\sqrt{3.9}} = f(3.9) \cong L_4(3.9) = \frac{1}{2} - \frac{1}{16}(3.9-4) = \frac{1}{2} + \frac{1}{16}(0.1)$$

6.
a)

$$df = \frac{df}{dx}dx = \left(\frac{d}{dx}\tan(x)\right)dx = \sec^2(x)\,dx.$$

b)

$$\begin{aligned}\tan\left(\frac{\pi}{4} - 0.1\right) - \tan\left(\frac{\pi}{4}\right) &\cong df\left(\frac{\pi}{4}, -0.1\right)\\ &= \sec^2\left(\frac{\pi}{4}\right)(-0.1)\\ &= -\frac{0.1}{\cos^2\left(\frac{\pi}{4}\right)} = -\frac{0.1}{\left(\frac{1}{\sqrt{2}}\right)^2} = -0.2.\end{aligned}$$

Therefore,

$$\tan\left(\frac{\pi}{4} - 0.1\right) \cong \tan\left(\frac{\pi}{4}\right) - 0.2 = 1 - 0.2 = 0.8.$$

7. Let $r(t)$ denote the radius (in centimeters) and let $V(t)$ denote the volume (in cubic centimeters) of the balloon at time t (in seconds). Thus,

$$V(t) = \frac{4}{3}\pi r^3(t).$$

By the chain rule,

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt},$$

Thus,

$$\frac{dV}{dt} = \left(\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)\right)\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}.$$

The rate of change of V with respect to time is 100, i.e., $dV/dt = 100$. Therefore,

$$100 = 4\pi r^2\frac{dr}{dt}$$

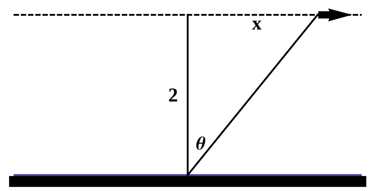
Thus, the rate of change of the radius with respect to time can be expressed as

$$\frac{dr}{dt} = \frac{100}{4\pi r^2}.$$

Therefore,

$$\left.\frac{dr}{dt}\right|_{r=10} = \frac{100}{4\pi(10)^2} = \frac{1}{4\pi} \cong 0.08\ \text{(cm/second)}.$$

8.



With reference to the figure,

$$\tan(\theta) = \frac{x}{2}.$$

Thus,

$$\frac{d}{dt}\tan(\theta) = \frac{d}{dt}\left(\frac{x}{2}\right) = \frac{1}{2}\frac{dx}{dt}.$$

By the chain rule,

$$\frac{d}{dt}\tan(\theta) = \left(\frac{d}{d\theta}\tan(\theta)\right)\frac{d\theta}{dt} = \sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{\cos^2(\theta)}\frac{d\theta}{dt}.$$

Therefore,

$$\frac{1}{\cos^2(\theta)}\frac{d\theta}{dt} = \frac{1}{2}\frac{dx}{dt}.$$

Since

$$\frac{dx}{dt} = 200,$$

$$\frac{d\theta}{dt} = \cos^2(\theta)\left(\frac{1}{2}\frac{dx}{dt}\right) = 100\cos^2(\theta).$$

At that instant $x = 10$,

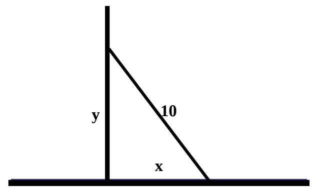
$$\cos(\theta) = \frac{2}{\sqrt{2^2 + 10^2}} = \frac{2}{\sqrt{104}}.$$

Therefore, the rate of change of the angle of elevation at that instant is

$$\left.\frac{d\theta}{dt}\right|_{\cos(\theta)=2/\sqrt{104}} = 100\left(\frac{2}{\sqrt{104}}\right)^2 = \frac{400}{104}$$

(radians/hr).

9.



With reference to the figure, we have

$$x^2 + y^2 = 10^2 = 100,$$

by Pythagoras. Both the height of the top of the ladder, y (in feet) and the distance of its base from the wall, x (in feet), are functions of time t (in seconds). Since the above relationship is an identity, we have

$$\frac{d}{dt}\left(x^2 + y^2\right) = \frac{d}{dt}(100) = 0.$$

By the chain rule,

$$\frac{d}{dt}\left(x^2 + y^2\right) = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

We are given that the rate at which the bottom of the ladder is sliding away from the wall is 2 ft/sec. Thus, $dx/dt = 2$. Therefore,

$$4x + 2y\frac{dy}{dt} = 0.$$

We are asked to compute the rate at which the height of the top of the ladder is changing at the instant the bottom of the ladder is 4 feet from the wall, i.e., at the instant $x = 4$. At that instant, $4^2 + y^2 = 10^2$, so that $y = \sqrt{84}$. Therefore,

$$4(4) + 2\left(\sqrt{84}\right)\frac{dy}{dt} = 0.$$

Thus,

$$\frac{dy}{dt} = -\frac{16}{2\sqrt{84}} \cong -0.873 \text{ (ft/sec)}.$$

The $(-)$ sign corresponds to the fact that y is decreasing as the ladder is sliding down the wall. Thus, the top of the ladder is sliding down the wall at the rate of

$$\frac{16}{2\sqrt{84}} \text{ (ft/sec)}$$

at the instant the bottom of the ladder is 4 feet from the wall.

10.
a)

$$\begin{aligned}
&\frac{d}{dx}\left(y^5+y^2-y-x^2+1\right)=0,\\
&\Rightarrow 5y^4\frac{dy}{dx}+2y\frac{dy}{dx}-\frac{dy}{dx}-2x=0\\
&\Rightarrow \left(5y^4+2y-1\right)\frac{dy}{dx}=2x\\
&\Rightarrow \frac{dy}{dx}=\frac{2x}{5y^4+2y-1}.
\end{aligned}$$

b) If $y(1)=0$, we have

$$\left.\frac{dy}{dx}\right|_{x=1}=\left.\frac{2x}{5y^4+2y-1}\right|_{x=1\text{ and }y=0}=\frac{2}{-1}=-2.$$

Therefore, the line that is tangent to the graph of the equation $y^5+y^2-y-x^2+1=0$ at $(1,0)$ is the graph of the equation

$$y=-2(x-1).$$

11. We have

$$f'(x)=-2-x+x^2.$$

Therefore, $f'(x)=0$ if $x=-1$ or $x=2$. As suggested by the picture, f has a local maximum at -1 and a local minimum at 2.

12. We have

$$f'(x)=\frac{d}{dx}\left(x^2+\frac{1}{x}\right)=2x-\frac{1}{x^2}=\frac{2x^3-1}{x^2}.$$

Therefore,

$$f'(x)=0\Leftrightarrow 2x^3-1=0\Leftrightarrow x=\frac{1}{2^{1/3}}.$$

As suggested by the picture, f attains its absolute minimum on $(0,+\infty)$ at $1/2^{1/3}$.

13.
By the chain rule,

$$f'(x)=\frac{d}{dx}(x^2-4)^{2/5}=\left(\frac{2}{5}\left(x^2-4\right)^{-3/5}\right)(2x)=\frac{4x}{\left(x^2-4\right)^{3/5}}$$

if $x^2-4\neq 0$, i.e., if $x\neq -2$ and $x\neq 2$. The function is not differentiaable at ± 2 (the graph of f has cusps at $(\pm 2,0)$). Thus, -2 and 2 are critical points of f. We have $f'(0)=0$, so that 0 is also a critical point.

We have

$$f(\pm 2)=0,\ f(0)=4^{2/5}\text{ and }f(3)=5^{2/5}.$$

Therefore, the absolute maximum of f on $[-2,3]$ is $f(3)=5^{2/5}$, and the absolute minimum of f on $[-2,3]$ is $f(\pm 2)=0$.

14. We have

$$\lim_{x\to 0} e^x = e^0 = 1 > 0 \text{ and } \lim_{x\to 0} \frac{1}{x^2} = +\infty.$$

Therefore,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \left(e^x \left(\frac{1}{x^2} \right) \right) = +\infty$$

Thus, f does not have an absolute maximum on $(0, +\infty)$. In order to determine the absolute minimum of f on $(0, +\infty)$ we will apply the derivative test for monotonicity. By the quotient rule,

$$f'(x) = \frac{d}{dx} \left(\frac{e^x}{x^2} \right) = \frac{e^x \left(x^2\right) - e^x (2x)}{x^4} = \frac{e^x x (x-2)}{x^4} = \left(\frac{e^x}{x^3} \right) (x-2)$$

if $x \neq 0$. We have $f'(x) = 0$ if $x = 2$, so that 2 is the only stationary point of f. Since $e^x > 0$ for each x and $x^3 > 0$ if $x > 0$, the sign of $f'(x)$ is determined by the sign of $x - 2$ if $x > 0$. Thus, $f'(x) < 0$ if $0 < x < 2$ and $f'(x) > 0$ if $x > 2$. Therefore, f is decreasing on the interval $(0, 2]$ and f is increasing on $[0, +\infty)$. Thus, f attains its absolute minimum on $(0, +\infty)$ at 2. We have

$$f(2) = \frac{e^2}{2^2} = \frac{e^2}{4}.$$

15.

a) The line $x = 2$ is a vertical asymptote for the graph of f. Since

$$\lim_{x\to 2} x = 2, \ \lim_{x\to 2+} \frac{1}{x-2} = +\infty \text{ and } \lim_{x\to 2-} \frac{1}{x-2} = -\infty,$$

we have $\lim_{x\to 2+} f(x) = +\infty$ and $\lim_{x\to 2-} f(x) = -\infty$.

b) Since

$$\lim_{x\to \pm\infty} (f(x) - x) = \lim_{x\to \pm\infty} \frac{1}{x-2} = 0,$$

the line $y = x$ is an oblique asymptote for the graph of f at $\pm\infty$.

c) We have

$$f'(x) = 1 - \frac{1}{(x-2)^2} = \frac{(x-2)^2 - 1}{(x-2)^2} = 0$$

if

$$(x-2)^2 = 1 \Leftrightarrow x - 2 = \pm 1 \Leftrightarrow x = 1 \text{ or } x = 3.$$

x		1		2		3	
sign of $f'(x)$	+	0	−	undef.	−	0	+
f	incr.	loc. max	decr.	undef.	decr.	loc. min.	incr.

The function f is increasing on $(-\infty, 1]$ and decreasing on $[1, 2)$. Therefore, f has a local maximum at 1.

The function f is decreasing on $(2, 3]$ and increasing on $[3, +\infty)$. Therefore, f has a local minimum at 3.

16. We have

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(-8x - x^2 + \frac{1}{3}x^3\right) = x^2 - 2x - 8, \\ f''(x) &= 2x - 2. \end{aligned}$$

Therefore,

$$f'(x) = 0 \Leftrightarrow x = -2 \text{ or } x = 4.$$

We have

$$f''(-2) = -6 < 0 \text{ and } f''(4) = 6 > 0.$$

Therefore, f has a local maximum at -2 and a local minimum at 4.

17. We have

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(x^2e^{-x}\right) = 2xe^{-x} - x^2e^{-x}, \\ f''(x) &= 2e^{-x} - 2xe^{-x} - 2xe^{-x} + x^2e^{-x} \\ &= e^{-x}\left(2 - 4x + x^2\right). \end{aligned}$$

Therefore,

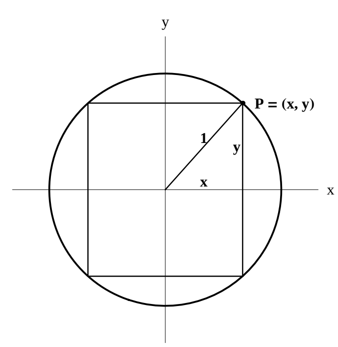
$$f''(x) = 0 \Longleftrightarrow 2 - 4x + x^2 = 0 \Leftrightarrow x = 2 \pm \sqrt{2}.$$

We have

$$\begin{aligned} f''(x) &> 0 \text{ if } x < 2 - \sqrt{2}, \\ f''(x) &< 0 \text{ if } 2 - \sqrt{2} < x < 2 + \sqrt{2}, \\ f''(x) &> 0 \text{ if } x > 2 + \sqrt{2}. \end{aligned}$$

Thus, the concavity of the graph changes at $\left(2 - \sqrt{2}, f\left(2 - \sqrt{2}\right)\right)$ and $\left(2 + \sqrt{2}.f\left(2 + \sqrt{2}\right)\right)$. Therefore, these are the inflection points of the graph of f.

18. We will consider the circle of radius 1 whose center is the origin of the xy-plane. Thus, the circle is the graph of the equation $x^2 + y^2 = 1$. Due to the symmetries, it is sufficient to consider inscribed rectangles whose sides are parallel to the coordinate axes, and we can assume that the vertices of the rectangles are on the unit circle. After all, we wish to maximize the area, and if the rectangle is strictly within the unit disk, we can enlarge the rectangle to one which has its vertices on the circle.



With reference to the figure, the inscribed rectangle is completely determined by the point $P = (x, y)$. We have $y = \sqrt{1 - x^2}$. Therefore, the area of the rectangle is

$$(2x)(2y) = 4xy = 4x\sqrt{1 - x^2}.$$

We will maximize the square of the area in order to maximize the area, since the expressions will be easier to work with, Thus, let's set

$$f(x) = \left(4x\sqrt{1 - x^2}\right)^2 = 16x^2\left(1 - x^2\right) = 16x^2 - 16x^4.$$

Since $0 < x < 1$, we would like to determine the absolute maximum of f on the interval $(0, 1)$. We have $f(0) = f(r) = 0$. The cases $x = 0$ and $x = r$ lead to the degenerate cases where the "rectangles" are intervals with 0 area. In any case, the search procedure for the determination of the absolute extrema of a continuous function on a closed and bounded interval is applicable. Since f is differentiable at any $x \in R$, the only critical points of f are its stationary points, i.e., points x such that $f'(x) = 0$. We have

$$f'(x) = \frac{d}{dx}\left(16x^2 - 16x^4\right) = 32x - 64x^3 = 32x\left(1 - 2x^2\right).$$

Therefore,

$$f'(x) = 0 \Leftrightarrow x = 0 \text{ or } x = \pm\frac{1}{\sqrt{2}}.$$

Thus, the only critical point of f in the interior of the interval $[0, 1]$ is $1/\sqrt{2}$.
We have

$$f\left(\frac{1}{\sqrt{2}}\right) = 16x^2 - 16x^4\Big|_{x=1/\sqrt{2}} = 16\left(\frac{1}{\sqrt{2}}\right)^2\left(1 - \left(\frac{1}{\sqrt{2}}\right)^2\right) = 4.$$

Therefore, $f\left(1/\sqrt{2}\right) > 0 = f(0) = f(1)$. Thus, the absolute maximum of f on $[0, 1]$ is 4, and f attains this value at $x = 1/\sqrt{2}$. Therefore, the maximum area of a rectangle that is inscribed in a circle of radius 1 is

$$\sqrt{f\left(\frac{1}{\sqrt{2}}\right)} = \sqrt{4} = 2.$$

The dimensions of a rectangle with maximum area are

$$2x = 2\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2},$$

and

$$2y = 2\sqrt{1 - \left(\frac{1}{\sqrt{2}}\right)^2} = 2\sqrt{1 - \frac{1}{2}} = 2\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}.$$

Therefore, an inscribed rectangle with maximum area is a square whose sides have length$\sqrt{2}$.

19.

$$\begin{aligned} y(x) = \pi + \int_1^x \frac{1}{t^2 + 1} dt &= \pi + \left(\arctan(x) - \frac{\pi}{4}\right) \\ &= \frac{3}{4}\pi + \arctan(x). \end{aligned}$$

20.

$$
\begin{aligned}
f(t) = -2 + \int_{\pi/8}^{t} \cos(4\tau)\, d\tau &= -2 + \left(\frac{1}{4}\sin(4t) - \frac{1}{4}\right) \\
&= \frac{1}{4}\sin(4t) - \frac{9}{4}.
\end{aligned}
$$

21.

In order to determine the x-coordinates of the points at which the graphs of f and g intersect, we need to find the solutions of the equation $f(x) = g(x)$:

$$
\begin{aligned}
x^2 - 2x - 1 = -x^2 + 2x + 5 &\Leftrightarrow 2x^2 - 4x - 6 = 0 \\
&\Leftrightarrow x - 1 \text{ or } x = 3.
\end{aligned}
$$

With reference to the figure, the area of the region is the sum of the areas of G_1 and G_2.

$$
\begin{aligned}
\text{The area of } G_1 = \int_1^3 (g(x) - f(x))\, dx &= \int_1^3 \left(-2x^2 + 4x + 6\right) dx \\
&= -\frac{2}{3}x^3 + 2x^2 + 6x\Big|_{x=1}^{3} = \frac{32}{3}.
\end{aligned}
$$

$$
\begin{aligned}
\text{The area of } G_2 = \int_3^5 (f(x) - g(x))\, dx &= \int_3^5 \left(2x^2 - 4x - 6\right) dx \\
&= \frac{2}{3}x^3 - 2x^2 - 6x\Big|_3^5 = \frac{64}{3}.
\end{aligned}
$$

Therefore, the area of G is

$$
\frac{32}{3} + \frac{64}{3} = 32.
$$