Last Name:
First Name:
Instructor's Name:

Math 150
Group Final (Spring 2008)

You are not allowed to use notes, books, calculators, personal stereos or cell phones.

You have exactly two hours.

Write clearly so that you can avoid mistakes and count on partial credits. Carry out the obvious simplifications so that you can display your answers in an easily readable manner.

The following list is for the recording of the points only. Do not write your answers on this page.

Points

- **1**________/4
- **2**________/4
- **3**________/4
- **4**________/4
- **5**________/4
- **6**________/8
- **7**________/10
- **8**________/6
- **9**________/8
- **10**________/6
- **11**________/10
- **12**________/4
- **13**________/6
- **14**________/4
- **15**________/6
- **16**________/4
- **17**________/8

Total: **/100**

1 (4 pts.) Evaluate the limit (the limit may be finite or infinite):

$$\lim_{x\to 4+} \frac{3-x}{x^2-5x+4}$$

2 (4 pts.) Evaluate the limit by using L'Hospital's rule:

$$\lim_{x\to 0} \frac{e^x-1-x}{x^2}$$

In problems 3-5 find the derivative (do not simplify):

3 (4 pts.)

$$\frac{d}{dx}\left(x^{5/4}\sin(x)\right)$$

4 (4 pts.)

$$\frac{d}{dx}\arctan\left(\frac{x}{3}\right)$$

5 (4 pts.)

$$\frac{d}{dx}\left(\frac{x^2}{x^2-16}\right)$$

6. Let

$$f(x) = \sqrt{x} + x$$

a) (6 pts.) Find the linearization of the function f at $a = 9$.

b) (2 pts.) Make use of the result of part a) to approximate $f(8.7)$.

7 (10 pts.) An inverted conical tank has depth of 10 m and radius at the top of 3 m. If water is filling the tank at the rate of 8 $\mathrm{m}^3/\mathrm{min}$, determine the rate at which the radius of the surface of the water is increasing at the instant when the depth of the water reaches 2 m. You need not simplify your answer.

Hint: The volume of a cone with base area A and height h is

$$\frac{1}{3}Ah$$

8. Assume that y is defined implicitly as a function of x by the equation

$$y^3 - 9y - 3x = 0$$

a) (4 pts.) Evaluate $\dfrac{dy}{dx}$.

b) (2 pts.) Calculate $\dfrac{dy}{dx}$ at $x = 0$ if $y = 3$ when $x = 0$.

9 (8 pts.) Let

$$f(x) = \frac{1}{12}x^4 - \frac{1}{6}x^3 - x^2$$

Determine the intervals on which the graph of f is concave up or concave down, and determine the x-coordinates of the points of inflection of the graph of f.

10 (6 pts.) Let

$$f(x) = x^3 - 6x^2 + 9x$$

Determine the absolute maximum and the absolute minimum of f on the interval $[0, 2]$ (you need not determine the intervals on which f is increasing or decreasing).

11 Let

$$f(x) = x + 2 + \frac{1}{x - 3}$$

a) (6 pts.) Determine the domain of f. Make use of the first derivative test to determine the intervals on which f is increasing or decreasing and the points at which f has a local maximum or local minimum.

b) (4 pts.) Determine the vertical asymptotes of the graph of f.(you need not justify your assertions). Show that

$$\lim_{x \to \pm\infty} \left(f\left(x\right) - \left(x+2\right)\right) = 0$$

Sketch the graph of f. Indicate the vertical asymptotes and the local extrema clearly. You need not find the intersections with the axes.

12 (4 pts.) Evaluate

$$\int_{\sqrt{\pi/6}}^{\sqrt{\pi/2}} \frac{d}{dx} \sin\left(x^2\right) dx$$

13 (6 pts.) Evaluate

$$\frac{d}{dx} \int_0^{1/x} e^{t^2} \, dt$$

14 (4 pts.) Evaluate

$$\int \frac{\cos(x)}{\sqrt{1+\sin(x)}}\,dx$$

15 (6 pts.) Evaluate

$$\int_0^{\sqrt{3}} \frac{1}{\sqrt{1-\frac{x^2}{4}}}\,dx$$

16 (4 pts.) Evaluate

$$\int e^{-4x}dx$$

17 (8 pts.) Assume that the velocity of an object moving along a line is

$$\frac{t}{1+t^2}$$

at time t. Let $s(t)$ be the position of the object at time t. Determine $s(t)$ for any $t > 0$ if $s(0) = 4$.

Solutions

1.

$$\begin{aligned}\lim_{x\to 4+}\frac{3-x}{x^2-5x+4} &= \lim_{x\to 4+}\frac{3-x}{(x-1)(x-4)}\\ &= \lim_{x\to 4+}\frac{3-x}{(x-1)}\left(\frac{1}{x-4}\right)=-\frac{1}{3}\lim_{x\to 4+}\frac{1}{x-4}=-\infty.\end{aligned}$$

2.

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2}=\lim_{x\to 0}\frac{e^x-1}{2x}=\lim_{x\to 0}\frac{e^x}{2}=\frac{1}{2}.$$

3.

$$\frac{d}{dx}\left(x^{5/4}\sin(x)\right)=\frac{5}{4}x^{1/4}\sin(x)+x^{5/4}\cos(x)$$

4.

$$\frac{d}{dx}\arctan\left(\frac{x}{3}\right)=\frac{1}{1+\left(\frac{x}{3}\right)^2}\left(\frac{1}{3}\right)=\frac{9}{9+x^2}\left(\frac{1}{3}\right)=\frac{3}{9+x^2}.$$

5.

$$\frac{d}{dx}\frac{x^2}{x^2-16}=\frac{2x\left(x^2-16\right)-x^2(2x)}{\left(x^2-16\right)^2}=\frac{2x^3-32x-2x^3}{\left(x^2-16\right)^2}=-\frac{32x}{\left(x^2-16\right)^2}.$$

6.

$$f'(x)=\frac{1}{2\sqrt{x}}+1$$

Therefore,

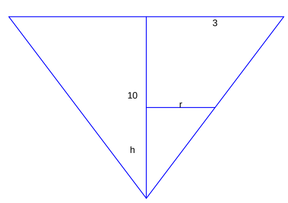
$$f'(9)=\frac{1}{6}+1=\frac{7}{6}$$

$$L_9(x)=f(9)+f'(9)(x-9)=12+\frac{7}{6}(x-9)$$

b)

$$f(8.7)\cong L_9(8.7)=12+\frac{7}{6}(-0.3)=11.65$$

7. Let h denote the depth of the water and let r denote the radius of the surface.



We have

$$\frac{h}{r}=\frac{10}{3}\Rightarrow h=\frac{10}{3}r.$$

The volume of the water in the tank is

$$V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi r^2\left(\frac{10}{3}r\right)=\frac{10\pi}{9}r^3.$$

Therefore,

$$\frac{dV}{dt}=\frac{10\pi}{9}\left(3r^2\frac{dr}{dt}\right)=\left(\frac{10\pi r^2}{3}\right)\frac{dr}{dt}.$$

Thus,

$$8=\left(\frac{10\pi r^2}{3}\right)\frac{dr}{dt}\Rightarrow\frac{dr}{dt}=\frac{24}{10\pi r^2}.$$

When $h=2$ we have

$$r=\left(\frac{3}{10}\right)h=\left(\frac{3}{10}\right)(2)=\frac{3}{5}.$$

Therefore, at that instant,

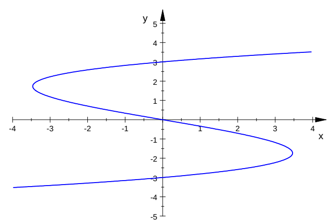
$$\frac{dr}{dt}=\frac{24}{10\pi r^2}=\frac{24}{10\pi\left(\frac{3}{5}\right)^2}=\frac{20}{3\pi}\cong 2.1$$

meters/min. The rate at which the water level is rising is

$$\frac{dh}{dt}=\left(\frac{10}{3}\right)\frac{dr}{dt}=\left(\frac{10}{3}\right)\left(\frac{20}{3\pi}\right)=\frac{200}{9\pi}\cong 7.1$$

meters/min.

8.



a)

$$3y^2\frac{dy}{dx}-9\frac{dy}{dx}-3=0$$

$\Rightarrow$

$$\left(3y^2-9\right)\frac{dy}{dx}=3\Rightarrow\frac{dy}{dx}=\frac{3}{3y^2-9}=\frac{1}{y^2-3}.$$

b) If $y(0)=3$,

$$y'(0)=\left.\frac{1}{y^2-3}\right|_{y=3}=\frac{1}{9-3}=\frac{1}{6}.$$

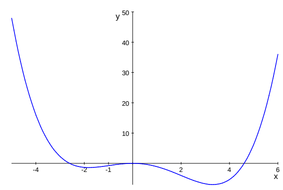
9.

$$f'(x)=\frac{d}{dx}\left(\frac{1}{12}x^4-\frac{1}{6}x^3-x^2\right)=\frac{1}{3}x^3-\frac{1}{2}x^2-2x,$$
$$f''(x)=\frac{d}{dx}\left(\frac{1}{3}x^3-\frac{1}{2}x^2-2x\right)=x^2-x-2=(x+1)(x-2)$$

Therefore, $f''(x)=0$ if $x=-1$ or $x=2$. Here is the sign chart of f'' (and the concavity of the graph of f):

x		-1		2	
$f''(x)$	$+$	0	$-$	0	$+$
concavity	concave up	infl. pt.	concave down	infl. pt.	concave up

Thus, the graph of f is concave up on $(-\infty,-1]$, concave down on $[-1,2]$, concave up on $[2,+\infty)$. The x-coordinates of the points of inflection are -1 and 2.



10. Let

$$f(x) = x^3 - 6x^2 + 9x$$

$$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$$

Thus, the only critical point of f in $[0,2]$ is 1. We have

$$f(0) = 0,\ \ f(2) = 2 \text{ and } f(1) = 4$$

Therefore, the absolute maximum of f on $[0,2]$ is $f(1) = 4$, and the absolute minimum of f on $[0,2]$ is $(f(0) = 0)$.

11.

a) The domain of f consists of all $x \neq 3$. We have

$$f'(x) = 1 - \frac{1}{(x-3)^2} = \frac{(x-3)^2 - 1}{(x-3)^2}.$$

Therefore,

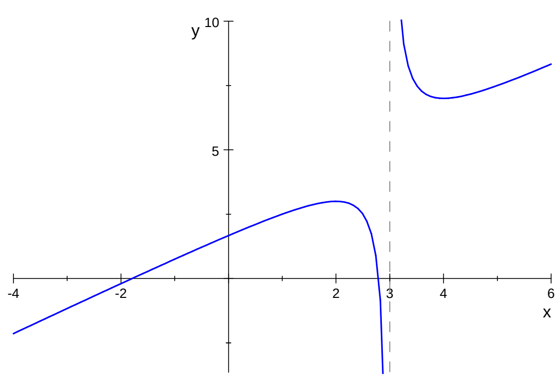
$$f'(x) = 0 \Leftrightarrow (x-3)^2 = 1 \Leftrightarrow x - 3 = \pm 1 \Leftrightarrow x = 2 \text{ or } x = 4.$$

x		2		3		4	
$f'(x)$	$+$	0	$-$	does not exist	$-$	0	$+$
f	incr.	loc. max.	decr.	does not exist	decr.	loc.min.	incr.

The function is increasing on $(-\infty, 2]$, decreasing on $[2,3)$, decreasing on $(3,4]$, increasing on $[4,+\infty)$. Thus, f has a local maximum at 2 and a local minimum at 4.

b) The line $x = 3$ is the vertical asymptote. We have

$$\lim_{x\to\pm\infty} (f(x) - (x+2)) = \lim_{x\to\pm\infty} \frac{1}{x-3} = 0.$$



12.

$$\int_{\sqrt{\pi/6}}^{\sqrt{\pi/2}} \frac{d}{dx} \sin\left(x^2\right) dx = \sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{6}\right) = 1 - \frac{1}{2} = \frac{1}{2}.$$

13.
$$\frac{d}{dx}\int_0^{1/x} e^{t^2}\,dt = e^{1/x^2}\left(-\frac{1}{x^2}\right).$$

14. We set $u = 1+\sin(x)$ so that $du = \cos(x)\,dx$. Thus,

$$\int \frac{\cos(x)}{\sqrt{1+\sin(x)}}\,dx = \int \frac{1}{\sqrt{u}du} = \int u^{-1/2}du = \frac{u^{1/2}}{1/2} = 2\sqrt{u} = 2\sqrt{1+\sin(x)}.$$

15. We set $u = x/2$ so that $du = dx/2$:

$$\begin{aligned}\int_0^{\sqrt{3}} \frac{1}{\sqrt{1-\frac{x^2}{4}}}\,dx &= 2\int_0^{\sqrt{3}/2} \frac{1}{\sqrt{1-u^2}}du \\ &= 2\left(\arcsin(u)|_0^{\sqrt{3}/2}\right) \\ &= 2\left(\arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin(0)\right) \\ &= \frac{2\pi}{3}.\end{aligned}$$

16. We set $u = -4x$ so that $du = -4dx$.

$$\int e^{-4x}dx = -\frac{1}{4}\int e^u du = -\frac{1}{4}e^{-4x}$$

17.The position of the object at t is

$$4 + \int_0^t \frac{\tau}{1+\tau^2}d\tau = \frac{1}{2}\ln\left(t^2+1\right) + 4$$