Math 151
Group Final (Spring 2009)
Solutions

1. We set $u = \arcsin(2x)$ and $dv = dx$ so that

$$du = \frac{2}{\sqrt{1-4x^2}}dx \text{ and } v = x.$$

Thus,

$$\begin{aligned}\int \arcsin(2x)\,dx = \int u dv &= uv - \int v du \\ &= \arcsin(2x)\,x - \int x\left(\frac{2}{\sqrt{1-4x^2}}\right)dx \\ &= x\arcsin(2x) - \int \frac{2x}{\sqrt{1-4x^2}}dx.\end{aligned}$$

We set $w = 1-4x^2$ so that $dw = -8xdx$. Thus,

$$\begin{aligned}\int \frac{2x}{\sqrt{1-4x^2}}dx &= \int \frac{1}{\sqrt{u}}\left(-\frac{1}{4}\right)du \\ &= -\frac{1}{4}\int u^{-1/2}du = -\frac{1}{4}\left(2\sqrt{u}\right) = -\frac{1}{2}\sqrt{1-4x^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int \arcsin(2x)\,dx &= x\arcsin(2x) - \int \frac{2x}{\sqrt{1-4x^2}}dx \\ &= x\arcsin(2x) + \frac{1}{2}\sqrt{1-4x^2}.\end{aligned}$$

2. We have

$$4x^2 - 8x + 13 = 4(x-1)^2 - 4 + 13 = 4(x-1)^2 + 9.$$

Therefore,

$$\begin{aligned}\int \frac{1}{4x^2-8x+13}dx &= \int \frac{1}{4(x-1)^2+9}dx \\ &= \int \frac{1}{9\left(\left(\frac{2}{3}(x-1)\right)^2+1\right)}dx\end{aligned}$$

We set

$$u = \frac{2}{3}(x-1) \Rightarrow du = \frac{2}{3}dx.$$

Thus,

$$\begin{aligned}\int \frac{1}{9\left(\left(\frac{2}{3}(x-1)\right)^2+1\right)}dx &= \int \frac{1}{9(u^2+1)}\left(\frac{3}{2}\right)du \\ &= \frac{1}{6}\int \frac{1}{u^2+1}du \\ &= \frac{1}{6}\arctan(u) = \frac{1}{6}\arctan\left(\frac{2(x-1)}{3}\right)\end{aligned}$$

3.

$$\int \sin^2\left(\frac{x}{3}\right)dx = \int \frac{1-\cos(2x/3)}{2}dx \quad = \quad \frac{x}{2} - \frac{1}{2}\left(\frac{3}{2}\right)\sin\left(\frac{2x}{3}\right)$$
$$= \quad \frac{x}{2} - \frac{3}{4}\sin\left(\frac{2x}{3}\right).$$

4.
a)We set $u = x$ and $dv = e^{-x/4}dx$ so that $du = dx$ and

$$v = \int e^{-x/4}dx = -4e^{-x/4}.$$

Thus,

$$\int xe^{-x/4}dx = \int udv \quad = \quad uv - \int vdu$$
$$= \quad x\left(-4e^{-x/4}\right) + 4\int e^{-x/4}dx$$
$$= \quad -4xe^{-x/4} - 16e^{-x/4} = -4e^{-x/4}(x+4)$$

b)

$$\int_1^b xe^{-x/4}dx = -4e^{-b/4}(b+4) + 16.$$

Therefore,

$$\int_1^\infty xe^{-x/4}dx = \lim_{b\to\infty}\left(-4e^{-b/4}(b+4)+16\right) = 20e^{-1/4}.$$

5.
a) The integral is improper since

$$\lim_{x\to 2+}\frac{1}{(x-2)^{1/3}} = +\infty$$

b)

$$\int_{2+\varepsilon}^4 \frac{1}{(x-2)^{1/3}}dx = \int_{u=\varepsilon}^2 u^{-1/3}du = \frac{3}{2}u^{2/3}\Big|_\varepsilon^2 = \frac{3\left(2^{2/3}\right)}{2} - \frac{3}{2}\varepsilon^{2/3}.$$

Therefore,

$$\int_2^4 \frac{1}{(x-2)^{1/3}}dx = \lim_{\varepsilon\to 0+}\left(\frac{3\left(2^{2/3}\right)}{2} - \frac{3}{2}\varepsilon^{2/3}\right) = \frac{3}{2^{1/3}}$$

6. The length is

$$\int_0^1 \sqrt{1+\left(\frac{d}{dx}\cosh(x)\right)^2}dx = \int_0^1\sqrt{1+\sinh^2(x)}dx = \int_0^1 \cosh(x)\,dx = \sinh(1)$$

7.
a)

$$\frac{dy}{dt} + \frac{1}{t} y(t) = \cos\left(t^2\right)$$

$$e^{\int 1/t dt} = e^{\ln(t)} = t.$$

$$t\frac{dy}{dt} + 1y(t) = t\cos\left(t^2\right)$$

$$\frac{d}{dt}\left(ty(t)\right) = t\cos\left(t^2\right)$$

$$ty(t) = \int t\cos\left(t^2\right) dt = \frac{1}{2}\sin\left(t^2\right) + C$$

$$y(t) = \frac{1}{2t}\sin\left(t^2\right) + \frac{C}{t}$$

b)

$$y\left(\sqrt{\pi}\right) = 1 \Leftrightarrow \frac{C}{\sqrt{\pi}} = 1 \Leftrightarrow C = \sqrt{\pi}.$$

$$y(t) = \frac{1}{2t}\sin\left(t^2\right) + \frac{\sqrt{\pi}}{t}$$

8.
a)

$$\frac{dy}{dx} = \frac{y^2}{\sqrt{1-x^2}}$$

$\Rightarrow$

$$\frac{1}{y^2}\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$\Rightarrow$

$$\int \frac{1}{y^2}\frac{dy}{dx}dx = \int \frac{1}{\sqrt{1-x^2}}dx$$

$\Rightarrow$

$$\int y^{-2} dy = \arcsin(x) + C$$

$\Rightarrow$

$$-\frac{1}{y} = \arcsin(x) + C$$

$\Rightarrow$

$$y(x) = -\frac{1}{\arcsin(x) + C}$$

b)

$$y(1/2) = 1 \quad \Leftrightarrow \quad -\frac{1}{\arcsin(1/2) + C}$$

$$\Leftrightarrow \quad 1 = -\frac{1}{\frac{\pi}{6} + C} \Leftrightarrow \frac{\pi}{6} + C = -1 \Leftrightarrow C = -1 - \frac{\pi}{6}$$

Therefore,

$$y(x) = -\frac{1}{\arcsin(x) - 1 - \frac{\pi}{6}}$$

9.

a)

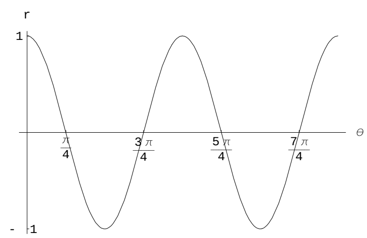
$$r = \cos(2\theta) = 0 \text{ and } \theta \in (0, 2\pi)$$

$\Leftrightarrow$

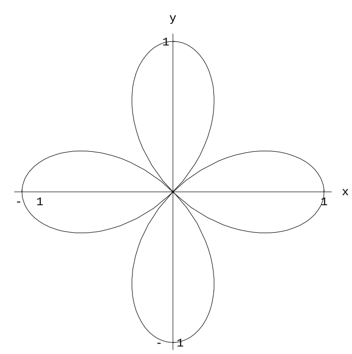
$$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

$\Leftrightarrow$

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$



b)



10.

$$\lim_{n\to\infty}\left(\frac{\dfrac{(n+1)^2\,3^{n+1}}{(n+1)!}}{\dfrac{n^2 3^n}{n!}}\right) = \lim_{n\to\infty}\left(3\left(\frac{n+1}{n}\right)^2\frac{1}{n+1}\right) = 0 < 1$$

Therefore the series converges.

11.

$$\frac{1}{\sqrt{n^2-1}} > \frac{1}{\sqrt{n^2}} = \frac{1}{n}$$

and $\sum 1/n$ diverges. Therefore, the series does not converge absolutely. But

$$\lim_{n\to\infty} \frac{1}{\sqrt{n^2-1}} = 0$$

and the sequence

$$\left\{ \frac{1}{\sqrt{n^2-1}} \right\}_{n=2}^{n=\infty}$$

is monotone decreasing. Therefore the series converges by the theorem on alternating series.

Thus, the series converges conditionally.

12.

$$\lim_{n\to\infty} \left(\frac{n}{4^n} |x-1|^n \right)^{1/n} = \lim_{n\to\infty} \frac{n^{1/n}}{4} |x-1| = \frac{1}{4} |x-1| .$$

Therefore, the series converges if

$$\frac{1}{4} |x-1| < 1 \Leftrightarrow |x-1| < 4 \Leftrightarrow x \in (-3,5)$$

Thus, the open interval of convergence is $(-3,5)$.

13.

a)

$$\begin{aligned} f(x) = \frac{1}{1-x^2} &= 1 + (x^2) + (x^2)^2 + (x^2)^3 + \cdots + (x^2)^n + \cdots \\ &= x^2 + x^4 + x^6 + \cdots + x^{2n} + \cdots \end{aligned}$$

b)

$$\begin{aligned} F(x) &= \int_0^x \frac{1}{1-t^2} dt \\ &= \int_0^1 \left(1 + t^2 + t^4 + t^6 + \cdots + t^{2n} + \cdots\right) dt \\ &= x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \cdots + \frac{1}{2n+1}x^{2n+1} + \cdots \end{aligned}$$