Last Name:
First Name:
Instructor:

Math 151
Group Final (Fall 2009)

You are not allowed to use notes, books, calculators, personal stereos or cell phones.

You have exactly two hours.

Write clearly so that you can avoid mistakes and count on partial credits. Carry out the obvious simplifications so that you can display your answers in an easily readable manner.

The following list is for the recording of the points only. Do not write your answers on this page.

Points

- **1** /8
- **2** /8
- **3** /10
- **4** /8
- **5** /8
- **6** /8
- **7** /8
- **8** /10
- **9** /8
- **10** /8
- **11** /8
- **12** /8

Total: **/100**

1 (8 pts.) Evaluate

$$\int x^2 \ln(x)\, dx$$

2 (8 pts.) Evaluate

$$\int x \sinh(4x)\, dx$$

3 (10 pts.) Evaluate

$$\int \frac{2x-1}{(x+2)\left(x^2+1\right)} dx$$

4 (8 pts.) Determine whether the improper integral

$$\int_1^\infty \frac{x}{(x^2+9)^2}\,dx$$

converges or diverges and its value in the case of convergence.

5 (8 pts.) Use the method of cylindrical shells to calculate the volume of the solid that is obtained by revolving the region between the graph of

$$f(x) = e^{-x^2},$$

the lines $x = 0$ and $x = 1$, and the interval $[0, 1]$ about the vertical axis.

6.

a) (6 pts.) Use the technique of an integrating factor to find the general solution of the differential equaton

$$\frac{dy}{dt} = -\frac{1}{t} y(t) + \frac{1}{\sqrt{t^2+1}}$$

(Assume that $t > 0$).

b) (2 pts.) Find the solution of the initial value problem

$$\frac{dy}{dt} = -\frac{1}{t} y(t) + \frac{1}{\sqrt{t^2+1}}, \quad y(1) = 4$$

7.

a) (6 pts.) Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{y^2}{1+x^2}$$

b) (2 pts.) Find the solution of the initial value problem

$$\frac{dy}{dx} = \frac{y^2}{1+x^2},\ y\left(-1\right) = -\frac{1}{\pi}.$$

8. Let r and θ be polar coordinates.
a) (4 pts.) Sketch the graph of the equation $r = 1 + 2\sin(\theta)$ in the Cartesian θr-plane on the interval $[0, 2\pi]$. Determine the values of θ at which $f(\theta) = 0$ and the values of θ at which f attains a local maximum or minimum value. Indicate the corresponding points on the graph.

b) (6 pts.) Sketch the graph of the equation $r = 1 + 2\sin(\theta)$, where $0 \le \theta \le 2\pi$, as a polar equation in the xy-plane (i.e., $x = r\cos(\theta)$, $y = r\sin(\theta)$).

9 (8 pts.) Let S_n be the nth partial sum of the infinite series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{3^{n-1}} = 1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \cdots$$

Find a simple expression for S_n and determine the sum of the infinite series as a limit (you will not get any points by just stating what the sum is).

10 (8 pts.) Determine whether the infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$$

converges absolutely, converges conditionally or diverges.

11 (8 pts.) Determine the radius of convergence and the open interval of convergence of the power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{2^n}{n} (x-1)^n$$

(You need not investigate the series at the endpoints of the interval).

12 (8 pts.) Let

$$F(x) = \int_0^x \frac{1}{1+t^4} dt.$$

Display the first 4 nonzero terms of the Taylor series for F in powers of x.
Hint: The geometric series will be helpful.

Solutions

Remark: An arbitrary constant can be added to indefinite integrals.

1. We set $u = \ln(x)$ and $dv = x^2 dx$ so that

$$du = \frac{1}{x} dx \text{ and } v = \frac{x^3}{3}$$

Thus,

$$\begin{aligned}
\int \ln(x)\, x^2 dx = \int u dv &= uv - \int v du \\
&= \ln(x)\left(\frac{x^3}{3}\right) - \frac{1}{3}\int x^3\left(\frac{1}{x}\right) dx \\
&= \frac{1}{3}x^3 \ln(x) - \frac{1}{3}\int x^2 dx \\
&= \frac{1}{3}x^3 \ln(x) - \frac{1}{3}\left(\frac{x^3}{3}\right) = \frac{1}{3}x^3 \ln(x) - \frac{1}{9}x^3
\end{aligned}$$

2. We set $u = x$ and $dv = \sinh(4x)\, dx$ so that

$$du = dx \text{ and } v = \int \sinh(4x)\, dx = \frac{1}{4}\cosh(4x).$$

Thus,

$$\begin{aligned}
\int x \sinh(4x)\, dx = \int u dv &= uv - \int v du \\
&= x\left(\frac{1}{4}\cosh(4x)\right) - \frac{1}{4}\int \cosh(4x)\, dx \\
&= \frac{x}{4}\cosh(4x) - \frac{1}{16}\sinh(4x).
\end{aligned}$$

3. We need to determine the partial fraction decomposition:

$$\frac{2x-1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

$\Leftrightarrow$

$$\begin{aligned}
2x - 1 &= A\left(x^2+1\right) + (Bx+C)(x+2) \\
&= (A+B)x^2 + (2B+C)x + (A+2C)
\end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned}
A + B &= 0 \\
2B + C &= 2 \\
A + 2C &= -1.
\end{aligned}$$

From the first equuation $B = -A$ so that

$$\begin{aligned} -2A + C &= 2 \\ A + 2C &= -1 \end{aligned}$$

From the last equation $A = -1 - 2C$ so that

$$-2\left(-1 - 2C\right) + C = 2 \Rightarrow 5C + 2 = 2 \Rightarrow C = 0.$$

Therefore, $A = -1$ and $B = 1$. Thus,

$$\frac{2x - 1}{\left(x + 2\right)\left(x^2 + 1\right)} = -\frac{1}{x + 2} + \frac{x}{x^2 + 1}$$

Therefore,

$$\int \frac{2x - 1}{\left(x + 2\right)\left(x^2 + 1\right)} dx = -\ln\left(|x + 2|\right) + \frac{1}{2}\ln\left(x^2 + 1\right)$$

(we set $u = x + 2$ for the first integral and $u = x^2 + 1$ for the second integral.

4. We set $u = x^2 + 9$ so that $du = 2xdx$. Thus,

$$\int \frac{x}{\left(x^2 + 9\right)^2} dx = \frac{1}{2}\int \frac{1}{u^2} du = \frac{1}{2}\int u^{-2} = -\frac{1}{2u} = -\frac{1}{2\left(x^2 + 9\right)}$$

Therefore,

$$\int_1^b \frac{x}{\left(x^2 + 9\right)^2} dx = -\frac{1}{2\left(b^2 + 9\right)} + \frac{1}{20}.$$

Thus,

$$\int_1^\infty \frac{x}{\left(x^2 + 9\right)^2} dx = \lim_{b \to \infty}\left(-\frac{1}{2\left(b^2 + 9\right)} + \frac{1}{20}\right) = \frac{1}{20}.$$

5. The volume is

$$2\pi \int_0^1 xe^{-x^2} dx = -\pi \int_0^{-1} e^u du = -\pi\left(e^u\big|_0^{-1}\right) = \pi\left(1 - e^{-1}\right)$$

(we set $u = -x^2$).

6.

a) We have

$$\frac{dy}{dt} + \frac{1}{t}y\left(t\right) = \frac{1}{\sqrt{t^2 + 1}}$$

The integrating factor is

$$e^{\int \frac{1}{t}dt} = e^{\ln(t)} = t.$$

Thus,

$$t\left(\frac{dy}{dt} + \frac{1}{t}y\left(t\right)\right) = \frac{t}{\sqrt{t^2 + 1}}$$

$\Rightarrow$

$$\frac{d}{dt}\left(ty\left(t\right)\right) = \frac{t}{\sqrt{t^2 + 1}}$$

$\Rightarrow$

$$ty\left(t\right) = \int \frac{t}{\sqrt{t^2 + 1}} dt = \sqrt{t^2 + 1} + C$$

(we set $u = t^2+1$) $\Rightarrow$

$$y(t) = \frac{\sqrt{t^2+1}}{t} + \frac{C}{t}.$$

b)

$$y(1) = 4 \Leftrightarrow 4 = \sqrt{2} + C \Leftrightarrow C = 4 - \sqrt{2}.$$

Therefore

$$y(t) = \frac{\sqrt{t^2+1}}{t} + \frac{4-\sqrt{2}}{t}$$

7.
a)

$$\frac{dy}{dx} = \frac{y^2}{1+x^2}$$

$\Rightarrow$

$$\frac{1}{y^2}\frac{dy}{dx} = \frac{1}{1+x^2}$$

$\Rightarrow$

$$\int \frac{1}{y^2}\frac{dy}{dx}dx = \int \frac{1}{1+x^2}dx$$

$\Rightarrow$

$$\int \frac{1}{y^2}dy = \arctan(x) + C$$

$\Rightarrow$

$$-\frac{1}{y} = \arctan(x) + C$$

$\Rightarrow$

$$y(x) = -\frac{1}{\arctan(x) + C}$$

b)

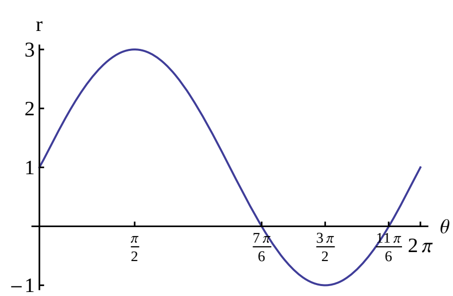
$$y(-1) = -\frac{1}{\pi} \Leftrightarrow -\frac{1}{\pi} = -\frac{1}{-\frac{\pi}{4}+C} \Leftrightarrow -\frac{\pi}{4} + C = -\pi \Leftrightarrow C = \frac{5\pi}{4}.$$

Therefore,

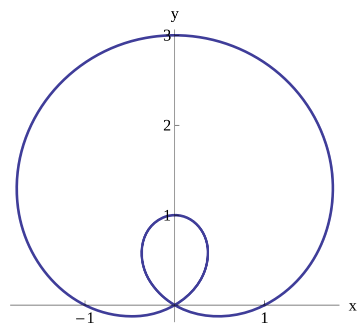
$$y(x) = -\frac{1}{\arctan(x) + \frac{5\pi}{4}}.$$

8.
a)

$$1 + 2\sin(\theta) = 0 \Leftrightarrow \sin(\theta) = -\frac{1}{2} \Leftrightarrow \theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6} \text{ or } \theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$



b)



9. We have

$$S=\lim_{n\to\infty}\frac{1-\left(-\frac{1}{3}\right)^{n}}{1-\left(-\frac{1}{3}\right)}=\lim_{n\to\infty}\frac{3}{4}\left(1-\left(-\frac{1}{3}\right)^{n}\right)=\frac{3}{4}$$

10. We have

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n^2+1}\right|=\sum_{n=1}^{\infty}\frac{1}{n^2+1}.$$

Since

$$0<\frac{1}{n^2+1}<\frac{1}{n^2}$$

and $\sum 1/n^2$ converges, the series converges absolutely.

11.

$$\lim_{n\to\infty}\left|(-1)^n\frac{2^n}{n}(x-1)^n\right|^{1/n}=|x-1|\lim_{n\to\infty}\frac{2}{n}=2\,|x-1|\,.$$

Therefore the series converges absolutely if

$$2\,|x-1|<1\Leftrightarrow|x-1|<\frac{1}{2}$$

and diverges if $|x-1|<1/2$. Thus, the radius of convergence is $1/2$ and the open interval of convergence is

$$\left(1-\frac{1}{2},1+\frac{1}{2}\right)=\left(\frac{1}{2},\frac{3}{2}\right).$$

12. We have

$$\frac{1}{1+t^4}=1-t^4+t^8-t^{12}+\cdots$$

Therefore

$$\begin{aligned}F(x)=\int_0^x\frac{1}{1+t^4}dt &= \int_0^x\left(1-t^4+t^8-t^{12}+\cdots\right)dt\\ &= x-\frac{1}{5}x^5+\frac{1}{9}x^9-\frac{1}{13}x^{13}+\cdots\end{aligned}$$