Study Guide and Sample Problems for Math 151 Group Final

Revised: January 2009

Questions on the group final will be similar to the sample problems provided in this document. No books, notes, calculators, cell phones or personal stereos are allowed.

Questions will be selected from the following list of topics:

1. Integration by parts
2. Integration of powers of $\sin(x)$, $\cos(x)$, $\sinh(x)$, $\cosh(x)$
3. Integrals that involve $\sqrt{a^2 - x^2}$
4. Integration of rational functions via partial fraction decomposition
5. Improper integrals
6. Volumes by the method of disks or cylindrical shells
7. The length of the graph of a function and the area of a surface of revolution
8. Linear first-order differential equations by the method of an integrating factor
9. Nonlinear separable differential equations
10. Tests for convergence (absolute or conditional) of an infinite series
11. The interval and radius of convergence of a power series
12. The determination of Taylor series
13. Sketching the graph of a function of the form $r = f(\theta)$ in the Cartesian $\theta r-$plane and as a polar graph in the $xy-$plane, where $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

Students should memorize the following **differentiation formulas**:

$$\frac{d}{dx}x^r = rx^{r-1}$$

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

$$\frac{d}{dx}\sinh(x) = \cosh(x)$$

$$\frac{d}{dx}\cosh(x) = \sinh(x)$$

$$\frac{d}{dx}\tan(x) = \frac{1}{\cos^2(x)}$$

$$\frac{d}{dx}a^x = \ln(a)\,a^x$$

$$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$$

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$

Students should memorize the following **antidifferentiation formulas** (an arbitrary constant can be added to either side of the equation):

$$\int x^r\,dx = \frac{1}{r+1}x^{r+1} \quad (r \neq -1)$$

$$\int \frac{1}{x}dx = \ln\left(|x|\right)$$

$$\int \sin\left(x\right)dx = -\cos\left(x\right)$$

$$\int \cos\left(x\right)dx = \sin\left(x\right)$$

$$\int \sinh\left(x\right)dx = \cosh\left(x\right)$$

$$\int \cosh\left(x\right)dx = \sinh\left(x\right)$$

$$\int a^x\,dx = \frac{1}{\ln\left(a\right)}a^x$$

$$\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin\left(x\right)$$

$$\int \frac{1}{1+x^2}dx = \arctan\left(x\right)$$

Remark: Students should be able to compute the exact values of the trigonometric functions at special angles such as $\pi/3$, $\pi/6$, $\pi/4$, $2\pi/3$, $-3\pi/4$ and exact values of the inverse trigonometric functions such as $\arctan\left(\sqrt{3}\right)$, $\arcsin\left(-1/2\right)$, $\arccos\left(0\right)$.

SAMPLE PROBLEMS

1. Determine the following antiderivatives:

   (a) $\int x\cos(x)dx$

   (b) $\int x^2\sinh(x)dx$

   (c) $\int x^2e^{-x}dx$

   (d) $\int x^{1/3}\ln(x)dx$

   (e) $\int \sin^2\left(x\right)dx$

   (f) $\int \sqrt{9-x^2}dx$

   (g) $\int \cos^3\left(x\right)\sin^2\left(x\right)dx$

   (h) $\displaystyle\int \frac{1}{\left(x-1\right)\left(x+3\right)}dx$

   (i) $\displaystyle\int \frac{x}{x^2+9}dx$

   (j) $\displaystyle\int \frac{1}{x^2-2x+5}dx$

   (k) $\displaystyle\int \frac{x}{\left(x^2+1\right)^2}dx$

   (l) $\displaystyle\int \frac{2x^2+7x}{x^2+6x+9}$

2. Determine whether the given improper integral converges or diverges, and the value of the improper integral in case of convergence:

   (a) $\displaystyle\int_1^\infty e^{-x}dx$

   (b) $\displaystyle\int_0^\infty xe^{-x}dx$

   (c) $\displaystyle\int_0^\infty \cos\left(x\right)e^{-x}dx$

   (d) $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}}dx$

   (e) $\displaystyle\int_1^\infty \frac{x}{x^2+4}dx$

   (f) $\displaystyle\int_{-\infty}^\infty \frac{1}{x^2+9}dx$

   (g) $\displaystyle\int_1^3 \frac{1}{x-1}dx$

   (h) $\displaystyle\int_2^4 \frac{1}{\sqrt{x-2}}dx$

3. Use the method of disks to determine the volume of the solid that is formed by revolving the region between the graph of $f(x) = x^2 + 1$ and the interval $[1, 2]$ about the x-axis.

4. Use the method of cylindrical shells to determine the volume of the solid that is obtained by revolving the region between the graph of $f(x) = x^2$ and the interval $[1, 3]$ about the y-axis.

5. Let $f(x) = \sin(x)$. Express the length of the graph of f corresponding to the interval $[0, \pi/2]$ as an integral (do not evaluate the integral).

6. Let $f(x) = x^2$ and let S be the surface that is obtained by revolving the graph of $y = f(x)$ corresponding to the interval $[1, 2]$ about the x-axis. Express the area of S as an integral (do not evaluate the integral).

7. Determine $y(t)$ such that

$$\frac{dy}{dt} = \frac{1}{5}y(t) + t \text{ and } y(0) = 3.$$

8. Find the solution of the initial value problem

$$y'(t) = \frac{1}{4}y(t) - \frac{1}{100}y^2(t),\ y(0) = 5.$$

9. Determine whether the given infinite series converges or diverges (by using the indicated test):

   (a) $\sum_{n=1}^{\infty} \frac{(-2)^n}{(n+1)^n}$

   (b) $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$ (use the root test)

   (c) $\sum_{n=1}^{\infty} \frac{10^n}{n!}$ (use the ratio test)

   (d) $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

   (e) $\sum_{n=2}^{\infty} \frac{1}{n \ln^2(n)}$ (use the integral test)

10. Determine whether the given series converges absolutely or conditionally, or whether it diverges. Justify your response:

    (a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n}}$

    (b) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^{3/2}}$

    (c) $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n \ln(n)}$

    (d) $\sum_{n=2}^{\infty} e^{-n} \sin(10n)$

11. Determine the radius of convergence and the open interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{4^n n^2}.$$

(You need not investigate the series at the endpoints of the interval.)

12. Given that

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots,$$

express the first 4 terms of the MacLaurin series for $\cos\left(x^3\right)$.

13. Given that

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots,$$

express the first 4 terms of the MacLaurin series for $\arctan(x)$. Provide the necessary explanation.

14. Given that

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots,$$

and

$$F(x) = \int_0^x e^{-t^2}\,dt,$$

express the first 4 terms of the MacLaurin series for F.

15. Calculate the first four terms of the Taylor series centered at 2 for $f(x) = \dfrac{1}{\sqrt{3-x}}$. You may leave the coefficients in terms of a factorial. (You are not allowed to apply the theorem about binomial series).

16. (a) Sketch the graph of $r = f(\theta)$ in the Cartesian θr-plane on the interval $[0, 2\pi]$. Indicate the values of θ at which $f(\theta) = 0$ and at which f attains a local maximum or minimum.

    (b) Sketch the graph of $r = f(\theta)$ as a polar equation in the xy-plane, where $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

    i. $f(\theta) = 3\cos(2\theta)$

    ii. $f(\theta) = 1 - 2\sin(\theta)$

Answers

1.

a. $\int x\cos(x)dx = \cos(x) + x\sin(x) + C$

b. $\int x^2\sinh(x)dx = x^2\cosh(x) - 2x\sinh(x) + 2\cosh(x) + C$

c. $\int x^2e^{-x}dx = \left(-x^2 - 2x - 2\right)e^{-x} + C$

d. $\int x^{1/3}\ln(x)dx = \frac{3}{4}x^{4/3}\ln(x) - \frac{9}{16}x^{4/3} + C$

e. $\int \sin^2(x)\,dx = -\frac{1}{2}\cos(x)\sin(x) + \frac{1}{2}x + C$

f. $\int \sqrt{9-x^2}dx = \frac{x\sqrt{9-x^2}}{2} + \frac{9}{2}\arcsin(x/3)$

g. $\int \cos^3(x)\sin^2(x)\,dx = \frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + C$

h. $\displaystyle\int \frac{1}{(x-1)(x+3)}dx = \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+3| + C$

i. $\displaystyle\int \frac{x}{x^2+9}dx = \frac{1}{2}\ln\left(x^2+9\right) + C$

j. $\displaystyle\int \frac{1}{x^2-2x+5}dx = \frac{1}{2}\arctan\left(\frac{x-1}{2}\right) + C$

k. $\displaystyle\int \frac{x}{\left(x^2+1\right)^2}dx = -\frac{1}{2\left(x^2+1\right)} + C$

l. $\displaystyle\int \frac{2x^2+7x}{x^2+6x+9} = 2x + \frac{3}{x+3} - 5\ln|x+3| + C$

2.

a. $\displaystyle\int_1^\infty e^{-x}dx = e^{-1}$

b. $\displaystyle\int_0^\infty xe^{-x}dx = 1$

c. $\displaystyle\int_0^\infty \cos(x)\,e^{-x}dx = \frac{1}{2}$

d. $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}}dx$ diverges

e. $\displaystyle\int_1^\infty \frac{x}{x^2+4}dx$ diverges

f. $\displaystyle\int_{-\infty}^\infty \frac{1}{x^2+9}dx = \frac{\pi}{3}$

g. $\displaystyle\int_1^3 \frac{1}{x-1}dx$ diverges

h. $\displaystyle\int_2^4 \frac{1}{\sqrt{x-2}}dx = 2\sqrt{2}$

3. $\pi\int_1^2\left(x^2+1\right)^2dx = \pi\left(\frac{178}{15}\right)$

4. $2\pi\int_1^3 x\left(x^2\right)dx = 40\pi$

5. $\int_0^{\pi/2}\sqrt{1+\cos^2(x)}dx$

6. $\int_1^2 2\pi x^2\sqrt{1+4x^2}dx$

7. $y(t) = -5t - 25 + 28e^{t/5}$

8. $y(t) = \dfrac{25}{1+4e^{-t/4}}$

9.

a. $\sum_{n=1}^{\infty} \frac{(-2)^n}{(n+1)^n}$ converges

b. $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$ converges

c) $\sum_{n=1}^{\infty} \frac{10^n}{n!}$ converges

d) $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ diverges

e. $\sum_{n=2}^{\infty} \frac{1}{n\ln^2(n)}$ converges

10.

a. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n}}$ converges conditionally

b. $\sum_{n=1}^{\infty} (-1)^{n} \frac{1}{n^{3/2}}$ converges absolutely

c. $\sum_{n=2}^{\infty} (-1)^{n} \frac{1}{n\ln(n)}$ converges conditionally

d. $\sum_{n=2}^{\infty} e^{-n}\sin(10n)$ converges absolutely

11. The radius of converges is 4. The open interval of convergence is $(-1, 7)$.

12.

$$\cos\left(x^3\right) = 1 - \frac{1}{2}x^6 + \frac{1}{4!}x^{12} - \frac{1}{6!}x^{18} + \cdots$$

13.

$$\arctan(x) = \int_0^x \frac{1}{1+t^2}dt = \int_0^x \left(1 - t^2 + t^4 - t^6 + \cdots\right) dt = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

14.

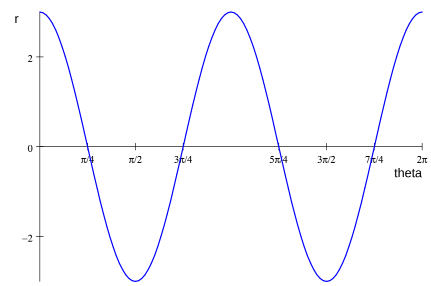
$$F(x) = \int_0^x e^{-t^2} dt = \int_0^x \left(1 - t^2 + \frac{1}{2}t^4 - \frac{1}{6}t^6 + \cdots\right) dt = x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \cdots$$

15.

$$f(2) + f'(2)(x-2) + \frac{1}{2}f''(2)(x-2)^2 + \frac{1}{3!}f^{(3)}(2)(x-2)^3$$
$$= 1 + \frac{1}{2}(x-2) + \frac{3}{8}(x-2)^2 + \frac{5}{16}(x-2)^3$$

16.

i)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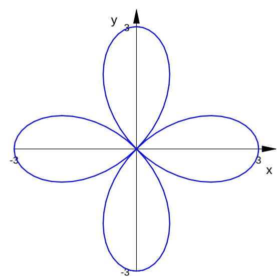
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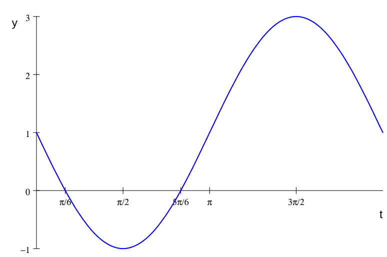
ii)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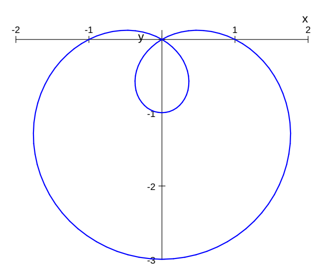
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