Generalized Factorization Lengths in Atomic Monoids

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Abstract

In this technical report, we introduce a method of generalization to factorization length in atomic semigroups. Using this new computation, we present a study of factorization length of iterated powers of elements through the study of several families of semigroups, including numerical semigroups, block monoids, and ACMs. Our results range from asymptotic behavior to complete, quasi-polynomial characterizations of factorization length.

Contents

Semigroup Family	Max/Min	Values of t		
		$t = 0$	$t = 1$	$t = \infty$
Numerical Semigroups	$L_t(xn)$	$\Theta(1)$	$\Theta(n)$	$\Theta(n)$
		p: 1	p: g_1	p: g_1
	$l_t(xn)$	$\Theta(1)$	$\Theta(n)$	$\Theta(n)$
		p: g^*	p: g_k	p: g
Regular ACMs & Block Monoids	$L_t(x^n)$	$\Theta(1)$	$\Theta(n)$	$\Theta(n)$
		p: 1	p $\to \infty$	
	$l_t(x^n)$	$\Theta(1)$	$\Theta(n)$	$\Theta(n)$
				p $\to \infty$
Singular ACMs	$L_t(x^n)$	$\Theta(n^{1/2})$?	$\Theta(n)$	$\Theta(n)$
	$l_t(x^n)$	$\Theta(1)$	$\Theta(n)$?	$\Theta(n)$?

Figure 1: This table summarizes our results across different monoids and t values. $\Theta(\cdot)$ describes the growth rate of L_t and l_t and p refers to the period of L_t and l_t.

1 Introduction

In the study of atomic monoids and their factorizations, the length set $\mathcal{L}(x)$ for an element x is crucial to the development of invariants such as delta sets and elasticity [7][4]. In this paper, we introduce a generalized factorization length known as $t-$length, with which the usual definition of factorization length is simply the $1-$length. To motivate this generalization, it is first useful to think of a factorization in vector form.

Let S be an atomic, cancellative, commutative monoid and let $x \in S$ be a nonunit. We may write some factorization of x as $x = a_1^{n_1} \circ a_2^{n_2} \circ \cdots \circ a_k^{n_k}$, where each $a_i \in S$ are (finitely many) distinct atoms and $n_i \in \mathbb{Z}_{\geq 0}$. We can think of this factorization as a vector $\vec{v} = (n_1, n_2, \ldots, n_k) \in \mathbb{Z}_{\geq 0}^k$.

Working in this vector notation, we can define the factorization length of x in terms of the length of the corresponding factorization vector $|\vec{v}|_t$ for a given $t \in [0, \infty]$. Formally, we define the t-length of a factorization $\vec{v}$ of x as follows:

Definition 1.0.1. *For $t \in [0, \infty)$, the t-length of a factorization $\vec{v}$ of $x \in S$ is:*

$$|\vec{v}|_t = \begin{cases} (n_1^t + \cdots + n_k^t)^{1/t}, & \text{for } t \in [1, \infty) \\ n_1^t + \cdots + n_k^t, & \text{for } t \in [0, 1] \end{cases}$$

This definition for the t-length of a factorization requires us to conventionally assume $0^0 = 0$ in order for the formula to hold when $t = 0$. In particular, when $t = 0$, this formula yields the total number of nonzero coordinates in $\vec{v}$. We also define $|\vec{v}|_\infty$ to be the maximum coordinate in $\vec{v}$.

Under this definition of a generalized norm, we may define the length set of a given element $x \in S$ with respect to t:

Definition 1.0.2. *The length set of $x \in S$ with respect to t is*

$$\mathcal{L}_t(x) = \{|\vec{v}|_t \mid \vec{v} \in \mathsf{Z}(x)\}, \tag{1}$$

where $\mathsf{Z}(x)$ is the set of all factorizations of x.

We also define the *max t-length* of x as $L_t(x) = \max\{\mathcal{L}_t(x)\}$ and the *min t-length* of x as $l_t(x) = \min\{\mathcal{L}_t(x)\}$. There currently exists sparse literature discussing $t-$lengths other than $t \neq 1$, and such literature is restricted to only numerical monoids [1] [4] . Along with introductions to several families of atomic monoids, including numerical monoids, block monoids, and ACMs [3] [2] [8] [6], we provide various results characterizing various $t-$lengths of factorizations, with a primary focus on $t = 0$ and $t = \infty$.

Definition 1.0.3. *A quasi-polynomial is a function of the form*

$$f(x) = \begin{cases} p_0(x), & \textit{if } x \equiv 0 \pmod q \\ p_1(x), & \textit{if } x \equiv 1 \pmod q \\ \vdots \\ p_{q-1}, & \textit{if } x \equiv q-1 \pmod q \end{cases} \tag{2}$$

where p_i are polynomials and $q \in \mathbb{Z}^+$. We call q the period *of f.*

We can also write a quasi-polynomial f recursively as

$$f(x) = f(x-q) + k \tag{3}$$

where q is the period. We will express our results about the periodicity of L_t and l_t in terms of this recursive formula.

Lastly, throughout the report we will denote $\mathbb{N}$ to refer to the natural numbers not including 0 and $\mathbb{N}_0$ to refer to the natural numbers including 0.

2 Background: Numerical Semigroups

Definition 2.0.1. *Given a set of natural numbers $g_1 < g_2 < ... < g_k$ whose collective gcd is 1, we define a numerical semigroup $(S, +) \subseteq (\mathbb{N}_0, +)$ to be*

$$S = \{n \in \mathbb{N}_0 \mid n = g_1a_1 + g_2a_2 + \cdots g_ka_k \textit{ for } a_i \in \mathbb{N}_0\} \tag{4}$$

Intuitively, S is comprised of all linear combinations of $g_1, \ldots, g_k$, which we will call the *generators* of S. We say a numerical semigroup S is generated by $g_1, \ldots, g_k$ and we can write S in terms of its generators as follows:

$$S = \langle g_1, \ldots, g_k \rangle \tag{5}$$

If we have that $g_i = 1$ for some g_i, it follows that $S = \mathbb{N}_0$, so we assume for the rest of this report that $g_i \geq 2$ for all g_i. Further, we assume that $g_1, \ldots, g_k$ is a minimal generating set of S.

Definition 2.0.2. *The Frobenius number of a numerical semigroup S is defined as*

$$F(S) = \max\{n \in \mathbb{N}_0 \mid n \notin S\} \tag{6}$$

Definition 2.0.3. *The Apéry Set of a numerical semigroup S with respect to $m \in \mathbb{N}_0$ is defined as*

$$Ap(S, m) = \{n \in S \mid n - m \notin S\} \tag{7}$$

Definition 2.0.4. *Let $S = \langle g_1, g_2, \ldots, g_k \rangle$ be a numerical semigroup and let $\vec{g} = (g_1, g_2, \ldots, g_k)$ be the vector of the generators of S. We define the set of all factorizations of $x \in S$ as*

$$\mathsf{Z}(x) = \{\vec{v} \in \mathbb{N}_0^k \mid x = \vec{v} \cdot \vec{g}\} \tag{8}$$

The current literature that establishes quasi-polynomial characterizations of $L_t(x)$ and $l_t(x)$ for arbitrary S is currently limited to $t = 1$ and arbitrary elements $x \in S$. In [4], Barron offers full characterizations of $L_1(x)$ and $l_1(x)$:

Theorem 2.0.5. *Given a numerical semigroup $S = \langle g_1, \ldots, g_k \rangle$, $x \in S$,*

$$L_1(x) = L_1(x - g_1) + 1 \tag{9}$$

for all $x > (g_1 - 1)g_k$.

Theorem 2.0.6. *Given a numerical semigroup $S = \langle g_1, \ldots, g_k \rangle$, $x \in S$,*

$$l_1(x) = l_1(x - g_k) + 1 \tag{10}$$

for all $x > (g_k - 1)g_{k-1}$.

This report will extend their results to different values of t, including $t = 0$ and $t = \infty$, and will include characterizations of $L_t(x^n)$ and $l_t(x^n)$ as functions of n for fixed values of t and x in an arbitrary numerical semigroup S. Note that as numerical semigroups are under addition, powers of an element x are written multiplicatively, so xn is an arbitrary n^{th} power of x.

Throughout the section on numerical semigroups, we will assume, unless otherwise stated, that $S = \langle g_1, g_2 \ldots, g_k \rangle$ for $g_i \in \mathbb{Z}_{\geq 2}$, $g_1 < g_2 < \cdots < g_k$, and that S has k generators. Also, let $g = \sum_{i=1}^{k} g_i$ be the sum of the generators of S and $g^* = \prod_{i=1}^{k} g_i$ be the product of the generators of S.

3 Results in Numerical Semigroups

3.1 Characterizing $t = 1$ in Numerical Semigroups

We will extend Theorem 2.0.5 and Theorem 2.0.6 to powers of x using a preliminary lemma defined in [4].

Lemma 3.1.1. *Let $k \geq 0$, and fix $c_1, c_2, \ldots, c_r \in \mathbb{Z}$ with $r \geq k$. There exists $T \subsetneq \{1, \ldots, r\}$ satisfying $\sum_{i \in T} c_i \equiv \sum_{i=1}^{r} c_i \pmod{k}$.*

Using this lemma, we get:

Theorem 3.1.2. *Given a numerical semigroup S, $x \in S$, and $n \in \mathbb{N}$,*

$$L_1(xn) = L_1(x(n - g_1)) + x \tag{11}$$

for all $xn > (x - 1)g_1 + (g_1 - 1)g_k$.

Proof. Fix a factorization $\vec{a}$ for xn, and suppose that $a_2 + \cdots + a_k \geq g_1$. We will show that on this condition, $\vec{a}$ is not maximal. Since $a_1g_1 + a_2g_2 + \cdots + a_kg_k = xn$, we have that $a_2g_2 + \cdots + a_kg_k \equiv xn \pmod{g_1}$. Viewing this sum as $a_2 + \cdots + a_k$ many integers selected from $\{g_2, \ldots, g_k\}$, we can apply [4, Lemma 4.1], taking $k = g_1$ and $r = a_2 + \cdots + a_k$. As such, we are guaranteed the existence of $b_2, \ldots, b_k \geq 0$ such that (i) $b_i \leq a_i$ for each $i > 1$, (ii) $\sum_{i=2}^k a_i > \sum_{i=2}^k b_i$, and (iii) $b_2g_2 + \cdots + b_kg_k \equiv xn \pmod{g_1}$. Note that (i) and (ii) imply that $b_2g_2 + \cdots + b_kg_k < a_2g_2 + \cdots + a_kg_k$, and with (iii), we can add copies of g_1 to $b_2g_2 + \cdots + b_kg_k$ to get xn. Specifically, there exists $b_1 \geq 0$ so that $\vec{b} = (b_1, b_2, \ldots, b_k) \in \mathsf{Z}(xn)$. This gives

$$(b_1 - a_1)g_1 = b_1g_1 - a_1g_1 = xn - \sum_{i=2}^k b_ig_i - \left(xn - \sum_{i=2}^k a_ig_i\right) = \sum_{i=2}^k (a_i - b_i)g_i > \sum_{i=2}^k (a_i - b_i)g_1, \tag{12}$$

and canceling g_1 from the left and right sides yields $|\vec{b}|_1 > |\vec{a}|_1$. Thus, $\vec{a}$ is not maximal.

Now, suppose that $\vec{a} \in \mathsf{Z}(xn)$ is maximal. By the contrapositive of the above argument, $a_2 + \cdots + a_k < g_1$. In particular, consider $xn > (x-1)g_1 + (g_1 - 1)g_k$. Observe that

$$\begin{aligned} (x-1)g_1 + (g_1-1)g_k < xn &= a_1g_1 + a_2g_2 + \cdots + a_kg_k && (13) \\ &\leq a_1g_1 + (a_2 + \cdots + a_k)g_k \leq a_1g_1 + (g_1 - 1)g_k, && (14) \end{aligned}$$

and manipulating the far left and right sides gives $a_1 \geq x$. Thus, we attain a factorization for $x(n - g_1)$ as follows: $\vec{a} - x\vec{e}_1 = (a_1, a_2, \ldots, a_k) - (x, 0, \ldots, 0) \in \mathsf{Z}(xn - xg_1) = \mathsf{Z}(x(n - g_1))$, and so $L_1(x(n - g_1)) \geq |\vec{a}|_1 - x$. Further, since $\vec{a}$ has maximal length, it cannot be that $L_1(x(n - g_1)) > |\vec{a}|_1 - x$, as this would imply we could construct a factorization of xn with length greater than $\vec{a}$ simply by adding back x copies of g_i. Therefore, we have that $L_1(x(n - g_1)) = |\vec{a}|_1 - x = L_1(xn) - x$, and so $L_1(xn) = L_1(x(n - g_1)) + x$. $\square$

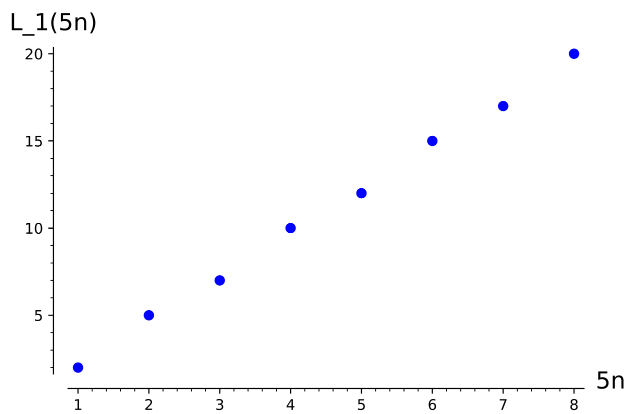

Figure 2: For an example of this periodicity, take the semigroup $\langle 2, 3\rangle$ and the element $x = 5$ as we take powers of 5, and notice that the period is $g_1 = 2$. The equation for this graph is $L_1(5n) = L_1(5(n-2)) + 5$.

Theorem 3.1.3. *Given a numerical semigroup $S = \langle g_1, \ldots, g_k\rangle$, $x \in S$, and $n \in \mathbb{N}$,*

$$l_1(xn) = l_1(x(n - g_k)) + x \tag{15}$$

for all $xn > (x-1)g_k + (g_k - 1)g_{k-1}$.

Proof. Fix a factorization $\vec{a}$ for xn, and suppose $a_1 + \cdots + a_{k-1} \geq g_k$. We will show that on this condition, $\vec{a}$ is not minimal. Since $a_1g_1 + \cdots + a_kg_k = xn$, we have that $a_1g_1 + \cdots + a_{k-1}g_{k-1} \equiv$

$xn \pmod{g_k}$. Viewing this sum as $a_1 + \cdots + a_{k-1}$ integers selected from $\{g_1, \ldots, g_{k-1}\}$, we can apply [4, Lemma 4.1], taking $k = g_k$ and $r = a_1 + \cdots + a_{k-1}$. As such, we are guaranteed the existence of $b_1, \ldots, b_{k-1} \geq 0$ such that (i) $b_i \leq a_i$ for each $i > 1$, (ii) $\sum_{i=1}^{k-1} a_i > \sum_{i=1}^{k-1} b_i$, and (iii) $b_1 g_1 + \cdots + b_{k-1} g_{k-1} \equiv xn \pmod{g_k}$. Note that (i) and (ii) imply that $b_1 g_1 + \cdots + b_{k-1} g_{k-1} < a_1 g_1 + \cdots + a_{k-1} g_{k-1}$ and with (iii), we can add copies of g_k to $b_1 g_1 + \cdots + b_{k-1} g_{k-1}$ to get xn. Specifically, there exists $b_k \geq 0$ such that $\vec{b} = (b_1, \cdots, b_{k-1}, b_k) \in \mathsf{Z}(xn)$ (in terms of trades, we can think of this as performing a sequence of trades that reduce $\{a_1, \ldots, a_{k-1}\}$ and increase a_k, thus reducing the length of $\vec{a}$, and these trades are possible because $r \geq k$). This gives

$$(b_k - a_k) g_k = b_k g_k - a_k g_k = xn - \sum_{i=1}^{k-1} b_i g_i - \left(xn - \sum_{i=1}^{k-1} a_i g_i \right) = \sum_{i=1}^{k-1} (a_i - b_i) g_i < \sum_{i=1}^{k-1} (a_i - b_i) g_k, \tag{16}$$

and canceling g_k from the left and right sides yields $|\vec{b}|_1 < |\vec{a}|_1$. Thus, $\vec{a}$ is not minimal.

Now, suppose that $\vec{a} \in \mathsf{Z}(xn)$ is minimal. By the contrapositive of the above argument, $a_1 + \cdots + a_{k-1} < g_k$. In particular, consider $xn > (x-1)g_k + (g_k - 1)g_{k-1}$. Observe that

$$(x-1)g_k + (g_k - 1)g_{k-1} < xn = a_1 g_1 + \cdots + a_{k-1} g_{k-1} + a_k g_k \tag{17}$$

$$\leq (a_1 + \cdots + a_{k-1}) g_{k-1} + a_k g_k \leq (g_k - 1) g_{k-1} + a_k g_k, \tag{18}$$

and manipulating the far left and right sides gives $a_k \geq x$. Thus, we attain a factorization for $x(n - g_k)$ as follows: $\vec{a} - x\vec{e}_k = (a_1, a_2, \ldots, a_k) - (0, \ldots, 0, x) \in \mathsf{Z}(xn - xg_k) = \mathsf{Z}(x(n - g_k))$, and so $l_1(x(n - g_k)) \leq |\vec{a}|_1 - x$. Further, since $\vec{a}$ has minimal length, it cannot be that $l_1(x(n - g_k)) < |\vec{a}|_1 - x$, as this would imply we could construct a factorization of xn with length less than $\vec{a}$ simply by adding back x copies of g_k. Therefore, we have that $l_1(x(n - g_k)) = |\vec{a}|_1 - x = l_1(xn) = l_1(x(n - g_k)) + x$. $\square$

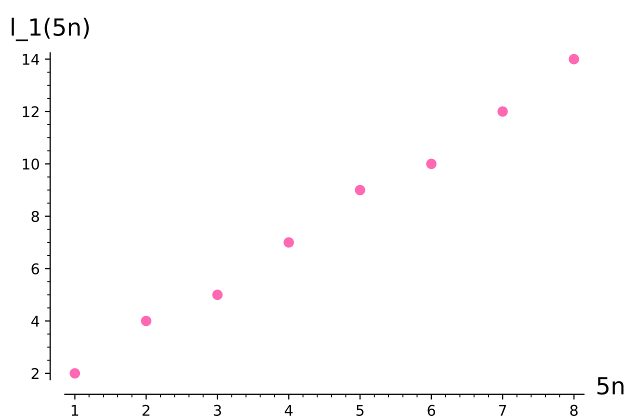

Figure 3: To illustrate the periodicity of $l_1(x^n)$, take n powers of the element $x = 5$ in the numerical semigroup $\langle 2, 3 \rangle$ and notice that the period is $g_2 = 3$. The equation for this graph is $l_1(5n) = l_1(5(n-3)) + 5$.

3.2 Characterizing $t = 0$ in Numerical Semigroups

Now that we are working with values of $t \neq 1$, we will first characterize $L_t(x)$ and $l_t(x)$ and then provide characterizations of $L_t(xn)$ and $l_t(xn)$.

Theorem 3.2.1. *Given a numerical semigroup S and $x \in S$,*

$$L_0(x) = k$$

for all $x \geq F(S) + g + 1$.

Proof. Let $s = F(S)$. It follows that $s + 1 \in S$, which has arbitrary factorization $(a_1, \ldots, a_k)$ for $a_1, \ldots, a_k \in \mathbb{Z}_{\geq 0}$. Adding g to $s + 1$ allows us to write $s + g + 1$ as $(a_1 + 1, \ldots, a_k + 1)$, which we can see is a factorization with full support. As such, for any $x \geq F(S) + g + 1, L_0(x) = k$. $\square$

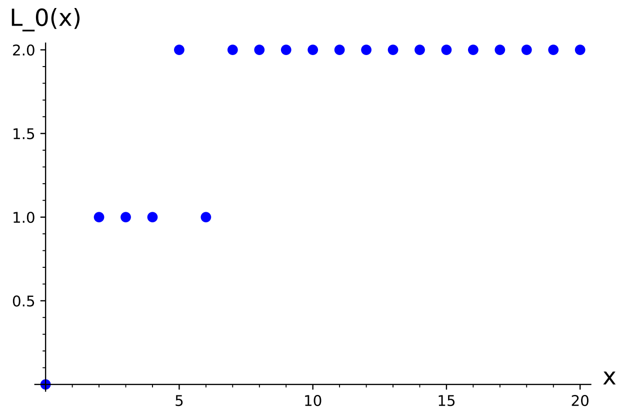

Figure 4: For large enough x, $L_0(x)$ caps out at 2, the number of generators in $\langle 2, 3\rangle$ for $x \in \langle 2, 3\rangle$.

Lemma 3.2.2. *Given a numerical semigroup S, $x \in S$ and $n \in \mathbb{N}$, $L_0(xn)$ is increasing with n. That is,*

$$L_0(xn) \leq L_0(x(n+1)).$$

Proof. Let $xn \in S$. Let $\vec{a} \in \mathsf{Z}(x)$ such that $\vec{a}$ induces the largest possible support. Then, $xn = \sum_{a_i \in \vec{a}} n a_i g_i$. Then, $x(n+1)$ has the factorization

$$x(n+1) = \sum_{a_i \in \vec{a}} (n+1) a_i g_i.$$

Therefore, $L_0(xn) \leq L_0(x(n+1))$. $\square$

Theorem 3.2.3. *Given a numerical semigroup S, $x \in S$, and $n \in \mathbb{N}$,*

$$L_0(xn) = k$$

for $xn \geq F(S) + g + 1$.

Proof. The proof is essentially the same as the proof for Theorem 3.2.1. $\square$

Definition 3.2.4. *Given a semigroup S, let $\bar{F}(S) = \max\{dn \mid n \in \mathbb{N}_0,\ dn \notin S\}$.*

Lemma 3.2.5. *Given a semigroup S and $x \in \mathbb{N}_0$ with $x > \bar{F}(S)$, $x \in S$ if and only if $d \mid x$.*

Proof. First, suppose that $x \in S$. Thus, $x = c_1 g_1 + \cdots + c_j g_j$ for $c_i \in \mathbb{N}_0$. Note that each g_i is divisible by d, and therefore $d \mid x$.

Now, suppose $d \mid x$, so $x = dm$ for some $m \in \mathbb{N}_0$. We have that $dm > \max\{dn \mid n \in \mathbb{N}_0, dn \notin S\}$, and so $x = dm \in S$. $\square$

Theorem 3.2.6. *Given a numerical semigroup S and $x \in S$,*

$$l_0(x) = l_0(x + g^*)$$

for all $x > \max\{\bar{F}(G'_S) \mid G'_S \subseteq G_S\}$.

Proof. To begin, note that since the length of any factorization under $t = 0$ is the number of generators used, $l_0(x)$ is the minimum number of generators required to sum to x. Thus, finding $l_0(x)$ is equivalent to finding the size of the smallest subset of the generators, $G''_S \subset G_S$, such that x is in the semigroup $\langle G''_S \rangle$. Since we picked $x > \max\{\bar{F}(G'_S) \mid G'_S \subseteq G_S\}$, we can apply Lemma 3.2.5 to say that this condition is equivalent to finding the size of the smallest subset of the generators, $G''_S \subset G_S$, such that $\gcd(G''_S) \mid x$. Then, $l_0(x) = |G''_S|$. Let $\alpha = |G''_S|$.

We will now argue that $l_0(x + g^*) = \alpha$. Note that $x + g^*$ can be written as $\left(c_1 + \frac{g^*}{g_1}\right) g_1 + c_2 g_2 + \cdots + c_\alpha g_\alpha$, and so $l_0(x + g^*) \leq \alpha$. However, since we can always write g^* as copies of a single generator, having $l_0(x + g^*) < \alpha$ would imply that $l_0(x) < \alpha$, a contradiction. Thus, $l_0(x) = l_0(x + g^*)$. □

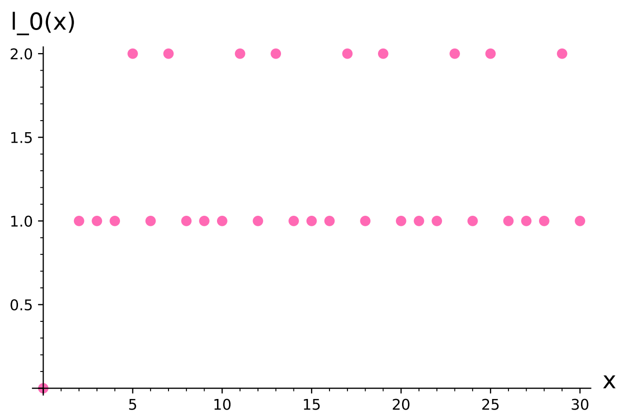

Figure 5: Let $x \in \langle 2, 3 \rangle$, this graph illustrates how $l_0(x)$ has period $6 = 2 \cdot 3$.

Theorem 3.2.7. *Given a numerical semigroup S, $x \in S$, and $n \in \mathbb{N}$,*

$$l_0(xn) = l_0(x(n + g^*))$$

for all $xn > \max\{\bar{F}(G'_S) \mid G'_S \subseteq G_S\}$.

Proof. We follow the same structure as the proof for $l_0(x)$. Since we picked

$$xn > \max\{F(G'_S) \mid G'_S \subseteq G_S\}$$

we can apply Lemma 3.2.5 in order to find the size of the smallest subset of the generators, $G''_S \subset G_S$, such that $\gcd(G''_S) \mid xn$. Then, $l_0(xn) = |G''_S|$.

Let $\alpha = |G''_S|$. We will show that $l_0(x(n + g^*)) = \alpha$. Note that $x(n + g^*)$ can be written as $\left(c_1 + x\frac{g^*}{g_1}\right) g_1 + \cdots + c_\alpha g_\alpha$, so $l_0(x(n + g^*)) \leq \alpha$. However, since we can always write xg^* as copies of a single generator, having $l_0(x(n+g^*)) < \alpha$ would imply that $l_0(xn) < \alpha$, a contradiction. Thus, $l_0(xn) = l_0(x(n + g^*))$. □

3.3 Characterizing $t = \infty$ in Numerical Semigroups

Lemma 3.3.1. *Given a numerical semigroup S, for all $x \in S$ and $c \in \mathbb{Z}^+$, if $x > cg$, then $l_\infty(x) > c$.*

Proof. We prove the statement by contrapositive. Let $x \in S$, and suppose $l_\infty(x) \leq c$. The largest factorization which satisfies this condition is the one with all of its coefficients equal to c, resulting in a factorization with value $cg_1 + cg_2 + \cdots + cg_k = cg$. So the largest possible value of x is cg. $\square$

Lemma 3.3.2. *Let S be a numerical semigroup with $0 < g_1$. Choose $x \in S$ such that $x > g_1^2 \cdot g$ and a factorization $\vec{a} \in \mathsf{Z}(x)$. If $|\vec{a}|_\infty = a_m$ for some $m \in \{2, 3, \cdots, k\}$, then there exists a factorization $\vec{b} \in \mathsf{Z}(x)$ such that $|\vec{b}|_\infty > |\vec{a}|_\infty$.*

Proof. We have $x > g_1^2 \cdot g$, so by Lemma 3.3.1, $|\vec{a}|_\infty = a_m > g_1^2$. We now write $a_m = qg_1 + r$ for $q, r \in \mathbb{Z}$ and $0 \leq r < g_1$. Now since $m \neq 1$, there is a trade between g_1 and g_m which involves exchanging g_1 copies of g_m for g_m copies of g_1. So we create a new factorization $\vec{b}$ of n by applying the trade from g_m to g_1 a total of q times. We then have $b_1 = a_1 + qg_m$ and $b_m = a_m - qg_1 = r \geq 0$. The last equation ensures that $\vec{b}$ is a valid factorization vector. We now show that $|\vec{b}|_\infty > |\vec{a}|_\infty$. First note that

$$qg_1 + g_1 > qg_1 + r > g_1^2 \Rightarrow q + 1 > g_1 \Rightarrow q \geq g_1.$$

We then get

$$a_1 + qg_m \geq qg_m \geq q(g_1 + 1) = qg_1 + q \geq qg_1 + g_1 > qg_1 + r = a_m.$$

Hence $b_1 > a_m$, and therefore $|\vec{b}|_\infty \geq b_1 > a_m = |\vec{a}|_\infty$. $\square$

Theorem 3.3.3. *Given a numerical semigroup S and $x \in S$,*

$$L_\infty(x) = L_\infty(x - g_1) + 1$$

for all $x > g_1^2 \cdot g$.

Proof. Let $\vec{a}$ be a factorization of x, and suppose that $|\vec{a}|_\infty$ is maximal. Then by the contrapositive of Lemma 3.3.2, we get $|\vec{a}|_\infty = a_1$. Now consider $\vec{a} - \vec{e}_1 \in \mathsf{Z}(x - g_1)$. Since $a_1 > a_i$ for all $i \in \{2, 3, \cdots, k\}$, we have that

$$L_\infty(x - g_1) \geq |\vec{a} - \vec{e}_1|_\infty = |\vec{a}|_\infty - 1.$$

Suppose by way of contradiction that there is a factorization $\vec{z} \in \mathsf{Z}(x - g_1)$ such that

$$|\vec{z}|_\infty > |\vec{a}|_\infty - 1.$$

If $|\vec{z}|_\infty = z_1$, then $|\vec{z} + \vec{e}_1|_\infty > |\vec{a}|_\infty$, which is impossible because $|\vec{a}|_\infty$ was maximal. If instead $|\vec{z}|_\infty = z_i$ for some $i \in \{2, 3, \cdots, k\}$, then $|\vec{z} + \vec{e}_1|_\infty = z_i \geq |\vec{a}|_\infty > g_1^2$. Then by Lemma 3.3.2, we can construct a factorization $\vec{b} \in \mathsf{Z}(x)$ with $|\vec{b}|_\infty > |\vec{z} + \vec{e}_1|_\infty \geq |\vec{a}|_\infty$. But this is also impossible, because $|\vec{a}|_\infty$ was maximal. By contradiction, we must have $L_\infty(x - g_1) = |\vec{a}|_\infty - 1 = L_\infty(x) - 1$. $\square$

Theorem 3.3.4. *Given a numerical semigroup S, let $Ap(S; g_1) = \{a_0, a_1, \ldots, a_{g_1 - 1}\}$, where $a_i \equiv i \pmod{g_1}$. Choose $x \in S$, and pick $i \in \{0, 1, \cdots, g_1 - 1\}$ such that $i \equiv x \pmod{g_1}$. Then,*

$$L_\infty(x) = \frac{x - a_i}{g_1}$$

for all $x > g_1^2 \cdot g$.

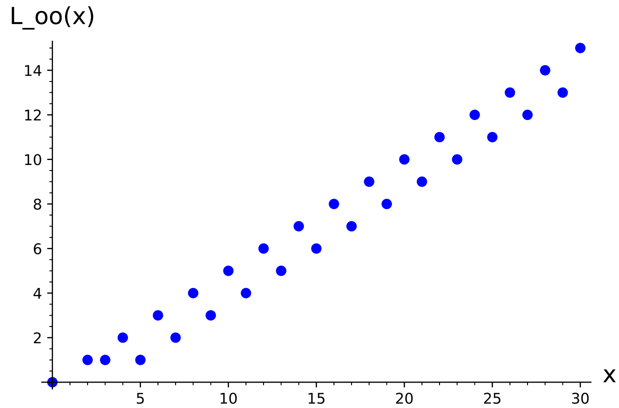

Figure 6: To illustrate the periodicity of $L_\infty(x)$, take the numerical semigroup $\langle 2,3\rangle$ and let $x \in \langle 2,3\rangle$. Notice that this graph has period $g_1 = 2$.

Proof. We have that $L_\infty(x)$ is the maximum of the set $M = \{\max\{z_i\} : \vec{z} \in \mathsf{Z}(x)\}$. Let $\vec{z} \in \mathsf{Z}(x)$ have maximal ∞-norm. Then since $x > g_1^2 \cdot g$, the contrapositive of Lemma 3.3.2 tells us that $|\vec{z}|_\infty = z_1$. So we do not need to consider any of the z_i where $i \in \{2, 3, \cdots, k\}$ when we look for the maximum element of M, since any factorization with the maximum ∞-norm will attain that maximum in its first coordinate. Hence

$$L_\infty(x) = \max(M) = \max\{z_1 : \vec{z} \in \mathsf{Z}(x)\}.$$

Now consider $g_1 \cdot x$. This has the factorization $(x, 0, 0, \cdots, 0)$. We would like to find a different factorization of $g_1 \cdot x$ where all the coefficients are multiples of g_1. To achieve this, we write x as $qg_1 + a_i$ where $q \in \mathbb{Z}^+$ and $i \equiv x \pmod{g_1}$. Then $x - a_i$ is a multiple of g_1. Now let $\vec{c}$ be a factorization of a_i. Note that because a_i is an Apéry set element, we have $c_1 = 0$. We also see that $g_1\vec{c}$ is a factorization of $g_1 a_i$, so $(qg_1, g_1c_2, g_1c_3, \cdots, g_1c_k)$ is a factorization of $g_1 \cdot x$ where all the components are multiples of g_1. Therefore

$$x = (q, c_2, c_3, \cdots, c_k)\,.$$

Finally, since a_i is an Apéry set element, we see that q is the largest possible first component in a factorization of x, and so

$$L_\infty(x) = \max\{z_1 : \vec{z} \in \mathsf{Z}(x)\} = q = \frac{x - a_i}{g_1}.$$

□

Theorem 3.3.5. *Given a numerical semigroup S, $x \in S$, and $n \in \mathbb{N}$,*

$$L_\infty(xn) = L_\infty(x(n - g_1)) + x$$

for all $xn > g_1^2 \cdot g + (x-1)g_1$.

Proof. By Theorem 3.3.3, we know that $L_\infty(x) = L_\infty(x - g_1) + 1$ for all $x > g_1^2 g$. Since $xn \geq x$ for all $n \in \mathbb{N}$, it follows that

$$L_\infty(xn) = L_\infty(xn - g_1) + 1,$$

for all $xn > g_1^2 g$ Similarly, if $xn - g_1 > g_1^2 g$, then

$$L_\infty(xn - g_1) = L_\infty(xn - 2g_1) + 1,$$

so we can write

$$L_\infty(xn) = L_\infty(xn - 2g_1) + 2.$$

Continuing this process and applying Theorem 3.3.3 to $L_\infty(xn - \alpha \cdot g_1)$, $\alpha = x - 1$ times, we find that

$$L_\infty(xn) = L_\infty(xn - xg_1) + x,$$

for all $xn - (x-1)g_1 > g_1^2 g$, as required. □

Lemma 3.3.6. *Let S be a numerical semigroup. Choose $x \in S$ such that $x > g_k \cdot g$, and a factorization $\vec{a} \in \mathsf{Z}(x)$ such that $|\vec{a}|_\infty$ is minimal. Then if $|\vec{a}|_0 < k$, there exists a factorization $\vec{b} \in \mathsf{Z}(x)$ such that $|\vec{b}|_\infty = |\vec{a}|_\infty$ and $|\vec{b}|_0 > |\vec{a}|_0$.*

Proof. Let a_m be the largest component of $\vec{a}$. We have $x > g_k \cdot g$, so by Lemma 3.3.1, $l_\infty(x) = |\vec{a}|_\infty = a_m > g_k$. Also, because $|\vec{a}|_0 < k$, there is a j in $\{0, 1, \cdots k\}$ such that $a_j = 0$. Now let $\vec{b} = \vec{a} + g_m \vec{e}_j - g_j \vec{e}_m$. Note that $b_m = a_m - g_j > a_m - g_k > 0$, so $\vec{b}$ is a valid factorization vector. Also, since the values of $g_m \vec{e}_j$ and $g_j \vec{e}_m$ are both $g_m g_j$, $\vec{b}$ is also a factorization of x. Now since $b_j = g_m > 0$, while $a_j = 0$, we have $|\vec{b}|_0 = |\vec{a}|_0 + 1$.

We now show that $|\vec{b}|_\infty = |\vec{a}|_\infty$. Note that $|\vec{b}|_\infty \geq |\vec{a}|_\infty$ because $|\vec{a}|_\infty$ is minimal. Now by the definition of the ∞-norm, we have $a_i \leq |\vec{a}|_\infty$ for all i; therefore $b_i \leq |\vec{a}|_\infty$ for all $i \neq j$. But then $b_j = a_j + g_m = g_m < g_k < |\vec{a}|_\infty$. Since $b_i \leq |\vec{a}|_\infty$ for all i, we conclude that $|\vec{b}|_\infty \leq |\vec{a}|_\infty$. Hence $|\vec{b}|_\infty = |\vec{a}|_\infty$. □

Theorem 3.3.7. *Given a numerical semigroup S and $x \in S$,*

$$l_\infty(x) = l_\infty(x - g) + 1$$

for all $x > g_k \cdot g$.

Proof. Let $x > g_k \cdot g$, and let $A(x)$ be the set of factorizations of x with minimal ∞-norm. By Lemma 3.3.6, there is a factorization $\vec{a} \in A(x)$ with full support. Let a_m be the largest component of $\vec{a}$, *i.e.* $a_m \geq a_i$ for all $i \in \{1, \cdots, k\}$. Now consider $\vec{b}$ such that $b_i = a_i - 1$ for all $i \in \{1, \cdots, k\}$. We see that $\vec{b}$ is a factorization of $x - g$. We also have $b_m = a_m - 1 \geq a_i - 1 = b_i$ for all $i \in \{1, \cdots, k\}$, so $|\vec{b}|_\infty = b_m = a_m - 1$. Since there is a factorization of x with ∞-norm $a_m - 1$, we get that $l_\infty(x - g) \leq a_m - 1$.

Suppose by way of contradiction that there is a $\vec{u} \in \mathsf{Z}(x - g)$ such that $|\vec{u}|_\infty < a_m - 1$. Then there is a $\vec{v} \in \mathsf{Z}(x)$ such that $v_i = u_i + 1$ for all $i \in \{1, \cdots, k\}$. By similar reasoning as above, we see that $|\vec{v}|_\infty < a_m$, which is impossible because $|\vec{a}|_\infty = a_m$ was minimal. By contradiction, we must have $l_\infty(x - g) = a_m - 1 = l_\infty(x) - 1$. □

Lemma 3.3.8. *If $b \in Ap(S; g)$, then $l_\infty(b) < g$.*

Proof. We prove the statement by contrapositive. Suppose $l_\infty(b) \geq g$. Then there is a factorization $\vec{z} \in \mathsf{Z}(b)$ such that $b_i \geq g$ for some $i \in \{1, 2, \cdots, k\}$. We can now construct a factorization $\vec{z}'$ of b by applying trades between b_i and every other component of $\vec{z}$. These trades are possible because they involve subtracting each of a set of distinct generators from b_i exactly once, so that the amount subtracted from b_i will not exceed g. It follows that $\vec{z}'$ has full support. But then we can subtract 1 from each component of $\vec{z}'$ to produce a factorization of $b - g$. Hence $b \notin \mathrm{Ap}(S; g)$. □

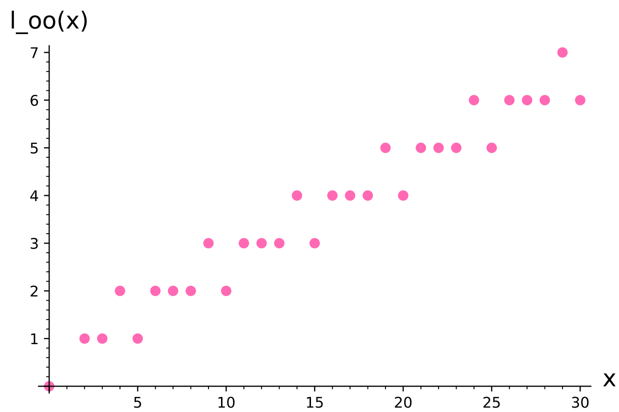

Figure 7: To illustrate the periodicity of $l_\infty(x)$, take the numerical semigroup $\langle 2, 3\rangle$ and let $x \in \langle 2, 3\rangle$. Notice that this graph has period $g = 5$.

Theorem 3.3.9. *Given a numerical semigroup S, let $Ap(S; g) = \{b_0, b_1, \ldots, b_{g-1}\}$, where $b_i \equiv i \pmod g$. Choose $x \in S$, and pick $j \in \{0, 1, \cdots, g-1\}$ such that $j \equiv -x \pmod g$. Then*

$$l_\infty(x) = \frac{x + b_j}{g}$$

for all $x > g^2$.

Proof. We first note that x can be written as $qg - b_j$ for some $q \in \mathbb{Z}^+$. We then get that $l_\infty(x+b_j) = l_\infty(qg) = q$, which is achieved with a factorization where every coefficient is q. Now consider a factorization $\vec{b}$ of b_j. Since b_j is an Apéry set element with respect to g, $\vec{b}$ will contain at least one zero component. Also, if $x > g^2$, then $q > g$, and by Lemma 3.3.8, all components of $\vec{b}$ will be less than q. Subtracting the two factorizations then gives a factorization of x with all components less than or equal to q. Hence $l_\infty(x) \leq q = l_\infty(x + b_j)$.

Suppose by way of contradiction that $l_\infty(x) < q$. Then there is a factorization $\vec{a}$ of x with all components of $\vec{a}$ less than q. Now let $\vec{q}$ be the vector with all its components equal to q. This is a factorization of qg, so $\vec{q} - \vec{a}$ is a factorization of $qg - x = b_j$. But since all components of $\vec{a}$ were less than q, we get that all components of $\vec{q} - \vec{a} \in \mathsf{Z}(b_j)$ are greater than 0, which is impossible because b_j is an Apéry set element with respect to g. By contradiction, we see that

$$l_\infty(x) = l_\infty(x + b_j) = q = \frac{x + b_j}{g}.$$

□

Theorem 3.3.10. *Given a numerical semigroup S, $x \in S$, and $n \in \mathbb{N}$,*

$$l_\infty(xn) = l_\infty(x(n - g)) + x$$

for all $xn > g_k \cdot g + (x - 1)g$.

Proof. By Theorem 3.3.7, we know that $l_\infty(x) = l_\infty(x - g) + 1$ for all $x > g_k \cdot g$, and since $xn \geq x$ for all $n \in \mathbb{N}$, this means that

$$l_\infty(xn) = l_\infty(xn - g) + 1,$$

for all $xn > g_k \cdot g$. Similarly, if $xn - g > g_k \cdot g$, then Theorem 3.3.7 implies

$$L_\infty(xn) = (L_\infty(xn - 2g) + 1) + 1 = L_\infty(xn - 2g) + 2.$$

Continuing this process and applying Theorem 3.3.7 to $l_\infty(xn - \alpha \cdot g)$, $\alpha = x - 1$ times, it follows that

$$l_\infty(xn) = l_\infty(xn - xg) + x,$$

for all $xn - (x-1)g > g_k \cdot g$, as required. □

3.4 Characterizing $t \in (0, 1)$ in Numerical Semigroups

For $t = 1/2$, while the growth rate of $L_{1/2}(x)$ can be characterized, $L_{1/2}(x)$ is believed to not be periodic. The table below illustrates this non-periodic behavior where $S = \langle 2, 3\rangle$, and $x \in S$:

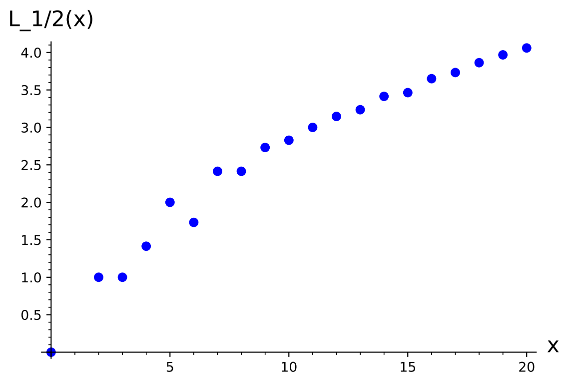

Figure 8: This graph illustrates the graph for $L_{1/2}(x)$ in $\langle 2, 3\rangle$. Notice that while the $L_{1/2}$ eventually stabilizes to a square root growth rate, the exact periodicity is not clear from the graph, nor is it clear from a table of points $(x, L_{1/2}(x))$.

Theorem 3.4.1. *Let S be a numerical semigroup and $x \in S$. Then $L_{1/2}(x) \to \sqrt{ax}$ as $x \to \infty$ such that $a = \sum_{j=1}^{k} \frac{1}{g_j}$.*

Proof. We will apply Lemma 2.14 from [5] and by letting $t = 1/2$, we get that $q = -1$, and by letting $\vec{a}$ be the factorization such that $|\vec{a}|_{1/2} = L_{1/2}(x)$, we obtain equality. Then we have that

$$L_{1/2}(x) = \left(\frac{x}{|\vec{g}|_{-1}}\right)^{1/2}.$$

Expanding the denominator gives us

$$\begin{aligned}
|\vec{g}|_{-1} &= (g_1^{-1} + \cdots + g_k^{-1})^{-1} \\
&= \frac{1}{\frac{1}{g_1} + \cdots + \frac{1}{g_k}} \\
&= \frac{g^*}{\sum\limits_{j=1}^{k} \frac{g^*}{g_j}} \\
&= \frac{\sum\limits_{j=1}^{k} \frac{g^*}{g_j}}{g^*} \\
&= \sum_{j=1}^{k} \frac{1}{g_j}
\end{aligned}$$

As such, the growth rate of $L_{1/2}(x)$ is $\sqrt{ax}$ where $a = \sum\limits_{j=1}^{k} \frac{1}{g_j}$.

□

Lemma 3.4.2. *Let $S = \langle g_1, g_2 \rangle$ be a numerical semigroup minimally generated by $g_1 < g_2$, and $x \in S$. If $\vec{a} \in \mathsf{Z}(x)$ has minimal $t = 1$ length, then $\vec{a}$ has minimal $t = \frac{1}{2}$ length.*

Proof. Let $\vec{a}$ be a factorization of $x \in S$ and suppose $\vec{a}$ has minimal $t = 1$ length. Then, for any factorization $\vec{b}$ of x, we have $a_1 + a_2 < b_1 + b_2$. We wish to show that $\sqrt{a_1} + \sqrt{a_2} < \sqrt{b_1} + \sqrt{b_2}$. Indeed, given any factorization $\vec{b}$ of x, we have $a_1g_1 + a_2g_2 = b_1g_1 + b_2g_2$, and since $\vec{a}$ has minimal $t = 1$ length, we know that $a_2 \geq b_2$ (so that, $\sqrt{a_2} \geq \sqrt{b_2}$). Combining these results, it follows that $a_1 < b_1$. Now, consider the following inequality,

$$\sqrt{a_1} + \sqrt{b_2} \leq \sqrt{a_1} + \sqrt{a_2} < \sqrt{b_1} + \sqrt{a_2}.$$

This implies

$$\sqrt{a_1} \leq \sqrt{a_1} + \sqrt{a_2} - \sqrt{b_2} < \sqrt{b_1} + \sqrt{a_2} - \sqrt{b_2} < \sqrt{b_1} + \sqrt{a_2} - \sqrt{a_2} = \sqrt{b_1}.$$

Thus,

$$0 \leq \sqrt{a_2} - \sqrt{b_2} < \sqrt{b_1} - \sqrt{a_1},$$

which means that $\sqrt{a_1} + \sqrt{a_2} < \sqrt{b_1} + \sqrt{b_2}$, as desired. □

Corollary 3.4.3. *Let $S = \langle g_1, g_2 \rangle$ be a numerical semigroup minimally generated by $g_1 < g_2$, and $x \in S$. If $\vec{a} \in \mathsf{Z}(x)$ has minimal $t = 1$ length, then $\vec{a}$ has minimal t length for all $t \in (0, 1]$.*

Theorem 3.4.4. *Let $S = \langle g_1, g_2 \rangle$ be a numerical semigroup minimally generated by $g_1 < g_2$, and $x \in S$. Denote the Apéry set of S with respect to g_2 by $Ap(S; g_2) = \{0, a_1, \ldots, a_{g_2 - 1}\}$. Then*

$$l_t(x) = \begin{cases} \left(\frac{x}{g_2}\right)^t & \text{if } x \equiv 0 \pmod{g_2} \\ \left(\frac{x - a_1}{g_2}\right)^t + \left(\frac{a_1}{g_1}\right)^t & \text{if } x \equiv 1 \pmod{g_2} \\ \vdots & \\ \left(\frac{x - a_{g_2 - 1}}{g_2}\right)^t + \left(\frac{a_{g_2 - 1}}{g_1}\right)^t & \text{if } x \equiv (g_2 - 1) \pmod{g_2}, \end{cases}$$

for all $t \in (0, 1]$.

Proof. Let $x \in S$, and suppose $x \equiv i \pmod{g_2}$. Write $x = \alpha_2 \cdot g_2 + a_i$, where $a_i \in \mathrm{Ap}(S; g_2)$. Since $g_1 \mid a_i$, we can write $a_i = \alpha_1 \cdot g_1$, for some $\alpha_1 \in \mathbb{Z}_{\geq 0}$. Substituting this into the equation $x = \alpha_2 \cdot g_2 + a_i$, gives

$$x = \alpha_2 \cdot g_2 + \alpha_1 \cdot g_1,$$

so we see that $(\alpha_1, \alpha_2) = \left(\frac{x-a_i}{g_2}, \frac{a_i}{g_1}\right)$, is a factorization of x. In particular, since $a_i \in \mathrm{Ap}(S; g_2)$, we see that $\alpha_1 \cdot g_1$ is the smallest multiple of g_1 that we can add to a multiple of g_2 to get an element of S. This means that for any factorization $\vec{\beta}$ of x, we have $\alpha_2 \geq \beta_2$, which means that $\vec{\alpha}$ has minimum $t = 1$ norm. By Corollary 3.4.3, this implies that or all $t \in (0, 1]$, $\vec{\alpha}$ also has minimum t norm. Thus,

$$l_t(x) = \left(\frac{x - a_i}{g_2}\right)^t + \left(\frac{a_i}{g_1}\right)^t.$$

Since $i \in [0, g_2 - 1]$ was arbitrary, this completes the proof. □

4 Background: Arithmetic Congruence Monoids and Block Monoids

Definition 4.0.1. *An arithmetic congruence monoid, or ACM, is a submonoid of* $(\mathbb{N}, \times)$ *of the form*

$$M_{a,b} = \{a + bk \mid k \in \mathbb{N}_0\} \cup \{1\}$$

where $a, b \in \mathbb{N}$, $0 < a \leq b$, *and* $a^2 \equiv a \pmod{b}$.

We require $a^2 \equiv a \pmod{b}$ so that this set is closed under multiplication.

Definition 4.0.2. *When* $a = 1$, *we refer to* $M_{1,b}$ *as a regular ACM.*

Definition 4.0.3. *When* $a > 1$, *we refer to* $M_{a,b}$ *as a singular ACM.*

Factoring in an ACM is comprised of taking an element x in an ACM, and first finding its prime factorization in $\mathbb{Z}$. This is,

$$x = p_1 p_2 \cdots p_k \tag{19}$$

for primes p_i. Next we can find primes that are already atoms, or combine primes to form atoms.

For example, in the ACM $M_{1,4}$ (also known as the Hilbert Monoid), take the element 441. In the integers, 441 has prime factorization $441 = 3^2 \cdot 7^2$. Neither 3 nor 7 is $1 \pmod{4}$, therefore they are not atoms. Thus we need to multiply them to obtain atoms. It follows that we can factor 441 in $M_{1,4}$ in two ways:

1. $441 = 21 \cdot 21$
2. $441 = 9 \cdot 49$

and notice that $9, 21$, and 49 are all $1 \pmod{4}$, thus they are atoms of $M_{1,4}$.

Definition 4.0.4. *Let* $M_{1,b}$ *be a regular ACM. We define the set* A *to be the set of all atoms of* $M_{1,b}$.

We will now give preliminary background knowledge about block monoids. Block monoids relate to ACMs because there is a homomorphism from any regular ACM to a block monoid over a cyclic group. Since block monoids contain only finitely many atoms, they are easier to characterize than ACMs, and so it is simpler to work with block monoids than to work with regular ACMs directly. Unfortunately, no such transformation can be applied to singular ACMs.

Definition 4.0.5. *Let G be an abelian group under the operation $+$. We define a* block *in G as a multisubset of G, whose elements sum to zero. The* block monoid *$\mathcal{B}(G)$ is the monoid of blocks over G, under the operation of multiset union. This can be thought as concatenation of blocks, with the empty set as the identity.*

For example, let $G = \mathbb{Z}_4$. Then $[1][1][2]$, $[1][1][1][3][3][3]$, and $[0][0][2][3][3]$ are blocks in $\mathcal{B}(G)$. Notice that $[1][1][1][3][3][3]$ can be factored into smaller blocks of $\mathcal{B}(G)$, *i.e.* $([1][3])^3$, while $[1][1][2]$ cannot be factored into smaller blocks.

Definition 4.0.6. *Let $\mathcal{B}(G)$ be a block monoid and let $S \subseteq G$. We define $\mathcal{B}(G,S) \subseteq \mathcal{B}(G)$ to be the set of blocks of G using only the characters in S, under the operation of multiset union.*

Definition 4.0.7. *Let $\mathcal{B}(G,S)$ be a block monoid. We define A_S to be the set of all atoms of $\mathcal{B}(G,S)$.*

5 Results in Regular ACMs and Block Monoids

5.1 Characterizing $t = 1$ in Regular ACMs and Block Monoids

It is known that $M_{1,3}$, $M_{1,4}$, and $M_{1,6}$ are half-factorial, meaning that for any of their elements, the 1-norms of every factorization of that element are all the same. It follows that for any element x in one of these ACMs, $L_1(x^n)$ and $l_1(x^n)$ are linear in n.

In this section we present a few results specific to $M_{1,5}$, namely, characterizing $L_1(x)$ with a corollary that $L_1(x^n)$ grows linearly. Then we will present some results on the growth rates for $L_t(x^n)$ and $l_t(x^n)$ for all regular ACMs and block monoids.

To provide some intuition for the following proof, consider the element $x = [1]^3[2]^7[3]^9 \in \mathcal{B}(\mathbb{Z}_4)$. We will try to construct a factorization attaining maximum 1-length. For reference, the atoms we have available to us are $\{[1]^4, [3]^4, [1][3], [2]^2, [1]^2[2], [2][3]^2\}$ (these happen to be all of the atoms in $\mathcal{B}(\mathbb{Z}_4)$ besides $[0]$). It turns out that the key to constructing the max length factorization is to maximize the copies of the $[1][3]$ atom. In particular, for this example we can write $x = ([1][3])^3([2][3]^2)([2]^2)^3([3]^4)$. This factorization has length 8, which is maximal. A formula for the max length of an arbitrary element of $\mathcal{B}(\mathbb{Z}_4)$ is now provided.

Theorem 5.1.1. *Given an element $x = [1]^a[2]^b[3]^c \in \mathcal{B}(\mathbb{Z}_4)$,*

$$L_1(x) = \begin{cases} \frac{3a+2b+c}{4} & \text{if } a \le c \\ \frac{a+2b+3c}{4} & \text{if } a \ge c. \end{cases}$$

Proof. Suppose we have a factorization $(a_1)\cdots(a_p)$ of an element $x \in \mathcal{B}(\mathbb{Z}_4)$, where $x = [1]^a[2]^b[3]^c$. Through trades that either increase the factorization length or keep it the same, emphasizing the introduction of shorter atoms, we will give a general form for the factorization of x that attains maximal length. Without loss of generality, suppose $a \le c$. If the factorization contains an atom $a_i = ([1][g_2]\cdots[g_r])$, and another atom $a_j = ([3][h_2]\cdots[h_s])$, observe that

$$([1][g_2]\cdots[g_r])([3][h_2]\cdots[h_s]) = ([1][3])([g_2]\cdots[g_r][h_2]\cdots[h_s]),$$

where the latter term on the RHS is either an atom itself or can be factored into atoms. Thus, we can create a new factorization of x, $(a_1')\cdots(a_q')$, in which we replace $([1][g_2]\cdots[g_r])([3][h_2]\cdots[h_s])$ with $([1][3])([g_2]\cdots[g_r][h_2]\cdots[h_s])$. So, $|(a_1')\cdots(a_q')|_1 \ge |(a_1)\cdots(a_p)|_1$.

We repeat this step until we reach a factorization $([1][3])^a(b_1)\cdots(b_t)$, where each b_i is an atom containing only copies of [2] and [3]. If $(b_1)\cdots(b_t)$ contains two or more copies of the atom $\left([2][3]^2\right)$, we can create a new factorization of x, $([1][3])^a(b_1')\cdots(b_u')$, by replacing $\left([2][3]^2\right)^2$ with $\left([2]^2\right)\left([3]^4\right)$. Since this trade preserves length, we still have that $|([1][3])^a(b_1')\cdots(b_u')|_1 \geq |(a_1)\cdots(a_p)|_1$.

We now repeat this step until we reach a factorization $(c_1)\cdots(c_v)$, comprised of a copies of $([1][3])$, at most one copy of $\left([2][3]^2\right)$, and completed by as many copies of $\left([2]^2\right)$ and $\left([3]^4\right)$ as necessary. Note that $|(c_1)\cdots(c_v)|_1 \geq |(a_1)\cdots(a_p)|_1$, and since $(a_1)\cdots(a_p)$ was chosen arbitrarily, $|(c_1)\cdots(c_v)|_1$ is the maximal factorization length of x. In particular, if $b \equiv 0 \pmod 2$, observe that the factorization $([1][3])^a\left([2]^2\right)^{\frac{b}{2}}\left([3]^4\right)^{\frac{c-a}{4}}$ has length

$$a + \frac{b}{2} + \frac{c-a}{4} = \frac{3a+2b+c}{4}.$$

Then, if $b \equiv 1 \pmod 2$, observe that the factorization $([1][3])^a\left([2][3]^2\right)\left([2]^2\right)^{\frac{b-1}{2}}\left([3]^4\right)^{\frac{c-a-2}{4}}$ also has length

$$a + 1 + \frac{b-1}{2} + \frac{c-a-2}{4} = \frac{3a+2b+c}{4}.$$

Thus, $(c_1)\cdots(c_v)$, which has maximal factorization length, has length $\frac{3a+2b+c}{4}$.

Finally, note that since [1] and [3], along with [2] and itself, are additive inverses, we can run a symmetric argument for the case that $a \geq c$, which completes the proof. $\square$

Corollary 5.1.2. *Let $x \in M_{1,5}$, and $\alpha = L_1(x)$. Then $L_1(x^n) = \alpha n$.*

Proof. Since $M_{1,5}$ is homomorphic to $\mathcal{B}(\mathbb{Z}_4)$, any arbitrary element $x \in M_{1,5}$ corresponds to an element $x' \in \mathcal{B}(\mathbb{Z}_4)$ where $x' = [1]^a[2]^b[3]^c$ for arbitrary $a, b, c \in \mathbb{Z}_{\geq 0}$ (where a, b, c are under the constraint that they must produce an element in the block monoid). We now apply Theorem 5.1.1 to x'. Suppose $a \leq c$, then $\alpha = L_1(x') = \frac{3a+2b+c}{4}$. Taking arbitrary n powers of x' gives

$$\begin{aligned} L_1\left(x'^n\right) &= L_1\left([1]^{an}[2]^{bn}[3]^{cn}\right) \\ &= \frac{3an+2bn+cn}{4} \\ &= L_1\left(x'\right)\cdot n \\ &= \alpha n \end{aligned}$$

By homomorphism, we thus conclude $x^n \in M_{1,5}$ has maximum factorization length $L_1(x^n) = \alpha n$. By the same justification in Theorem 5.1.1, we can apply a symmetric argument for the case $a \geq c$ to show that $L_1(x^n) = \left(\frac{a+2b+3c}{4}\right)n = \alpha n$. $\square$

As it turns out, this strictly linear behaviour does not generalize. However, the behaviour does seem to be quasi-linear, and we are able to determine linear bounds for the max and min factorization lengths as functions of n, so can claim that these are asymptotically linear. To illustrate the idea behind the following result, consider the example of $x = [1][3][4]^2 \in \mathcal{B}(\mathbb{Z}_6)$. We note that x contains 4 characters, and in general, x^n will contain $4n$ characters. To create a factorization with minimal factorization length, we might imagine splitting these $4n$ characters between many atoms, each with a relatively low character count. Then, to create a factorization with maximal factorization length, we might imagine splitting these $4n$ characters between fewer atoms, each with a relatively high character count.

Since, based on our choice of x, the atom available to us with the highest character count is $[1]^6$ (having 6 characters) and the one with the lowest is $[3]^2$ (having 2 characters), we can argue that that the length of any factorization of x^n is bounded above by $\frac{4n}{2}$ (if we were to only use the atom with the lowest character count) and bounded below by $\frac{4n}{6}$ (if we were to only use the atom with the highest character count). These bounds are shown in Figure 9, which plots them alongside the actual maximum and minimum factorizations of $260 \in M_{1,7}$, a corresponding element in the corresponding ACM.

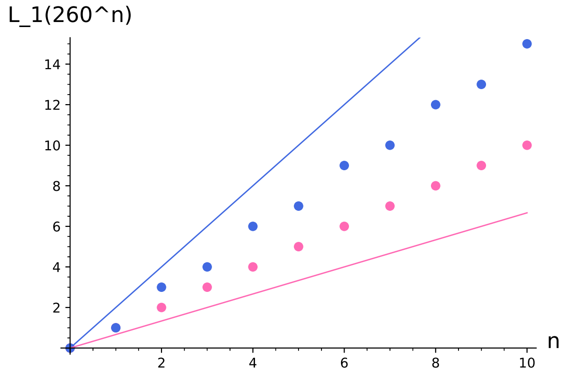

Figure 9: $L_1(260^n)$ and $l_1(260^n)$ in $M_{1,7}$

This argument is generalized below.

Theorem 5.1.3. *Given a finitely generated Abelian group G, a subset $S \subseteq G$, $x \in \mathcal{B}(G; S)$, and $n \in \mathbb{N}$, $l_1(x^n) \sim n$ and $L_1(x^n) \sim n$.*

Proof. It suffices to bound $l_1(x^n)$ and $L_1(x^n)$ above and below by functions both linear in n.

For $S = \{[s_1], \ldots, [s_j]\}$, suppose that $x = [s_1]^{p_1} \cdots [s_j]^{p_j}$. Then, let $c = \sum_{i=1}^{j} p_i$, denoting the number of characters in x. In general, x^n will have cn characters. Also, let

$$A_S = \{[s_1]^{q_{1,1}} \cdots [s_j]^{q_{1,j}}, \ldots, [s_1]^{q_{r,1}} \cdots [s_j]^{q_{r,j}}\} \text{ and } Q = \left\{ \sum_{i=1}^{j} q_{t,i} \mid 1 \leq t \leq r \right\}.$$

To bound the functions below, let $a = \max Q$. Since the shortest possible length for x^n is achieved when x^n can be written as copies of an atom with maximal character count, we have that $\frac{cn}{a} \leq l_1(x^n) \leq L_1(x^n)$.

Then, to bound the functions above, let $b = \min Q$. Since the longest possible length for x^n is achieved when x^n can be written as copies of an atom with minimal character count, we have that $l_1(x^n) \leq L_1(x^n) \leq \frac{cn}{b}$.

Thus, since $\frac{cn}{a} \leq l_1(x^n) \leq L_1(x^n) \leq \frac{cn}{b}$, we can conclude that $l_1(x^n) \sim n$ and $L_1(x^n) \sim n$. $\square$

Based on our data, it seems that the period of $L_1(x^n)$ for a block monoid element x can be arbitrarily large. The following results provide a starting point for proving such a result, by showing that at regular multiples of n, a certain block monoid element can be factored without using atoms of a specific form, and so its maximum 1-norm is greater than it is at other values of n.

Lemma 5.1.4. *Let $[a]^p[b]^q[-1]^r$ be an atom in $\mathcal{B}(\mathbb{Z}_{ab})$, where $a < b$ and $p, q, r > 0$. Then $ap + bq = ab + r$, $p < b$, and $q, r < a$.*

Proof. Since $[a]^p[b]^q[-1]^r$ is an atom, we must have $p < b$ and $q < a$. Also, if $r \geq a$, then $[a][-1]^a$ would be a factor, so we must have $r < a$. Now since $[a]^p[b]^q[-1]^r$ is a block, we must have $ap+bq-r = kab$ for some $k \in \mathbb{N}_0$. The largest value of $ap+bq$ is $a(b-1)+b(a-1) = 2ab-a-b < 2ab$, so we get $ap + bq = kab + r < 2ab \to kab < 2ab \to k < 2$. If $k = 0$, then $ap = r - bq$, and since $p \geq 1$, we get $a \leq ap = r - bq \leq r < a$. By contradiction, $k \neq 0$. Hence $k = 1$, and $ap + bq = ab + r$ as desired. □

Theorem 5.1.5. *Let $a, b \in \mathbb{Z}^+$ such that $gcd(a, b) = 1$ and $a < b$. Then the atoms in $\mathcal{B}(\mathbb{Z}_{ab}, \{a, b, -1\})$ are:*

- $[a]^b$, $[b]^a$, $[-1]^{ab}$
- $[a][-1]^a$, $[b][-1]^b$
- $a - 1$ *atoms of the form* $[a]^p[b]^q[-1]^r$

Proof. For any $x \in \mathbb{Z}_n$, we have that $[x]^{\text{ord}(x)}$ is an atom in $\mathcal{B}(\mathbb{Z}_n)$. Here $n = ab$, so the order of a is b and the order of b is a. Also, since the gcd of two consecutive integers is 1, we have that the order of -1 is ab. So we see that the blocks $[a]^b$, $[b]^a$, and $[-1]^{ab}$ are atoms, and it is simple to verify that they are the only atoms consisting of a single letter repeated some number of times.

We now characterize the two-letter atoms. We first note that the block $[a][-1]^a$ cannot be factored into atoms, since one of those atoms would contain only the letter $[-1]$ and there are fewer than ab copies of $[-1]$ in the block. So $[a][-1]^a$ is itself an atom. Now consider the block $[a]^m[-1]^n$ for some integer $m > 1$. If $m \geq b$, then the block contains $[a]^b$ as a factor, so it is not an atom. If instead $m < b$, then the sum of the $[a]$ letters in the block is $am < ab$, so at least am copies of $[-1]$ are required in order for the block to sum to 0. But then the block contains $[a][-1]^a$ as a factor, so it is not an atom. A similar argument shows that $[b][-1]^b$ is the only atom containing only $[b]$ and $[-1]$.

Suppose that there is an atom of the form $[a]^p[b]^q$ where $p, q > 0$. First, since $[a]^b$ and $[b]^a$ are atoms, we must have $p < b$ and $q < a$. Now the sum of the block is $ap + bq$, which by assumption is a multiple of ab, so bq must be a multiple of a. But b is not a multiple of a (since $\gcd(a, b) = 1$), and q is not a multiple of a, so bq cannot be a multiple of a. By contradiction, there are no atoms of the form $[a]^p[b]^q$, and so $[a][-1]^a$ and $[b][-1]^b$ are the only two-letter atoms.

We now characterize the three-letter atoms. By the lemma, all atoms of the form $[a]^p[b]^q[-1]^r$ must have $0 < p < b$, $0 < q, r < a$, and $ap + bq = ab + r$. Reducing the previous equation modulo a gives $bq \equiv r \pmod{a}$. Since $\gcd(a, b) = 1$, this equation has a unique solution for q such that $0 \leq q < a$. Then since $r \neq 0$, we also have $q \neq 0$ as required. Similarly, reducing the equation modulo b shows that there is a unique solution for p such that $0 < p < b$. Hence there is exactly one atom of the form $[a]^p[b]^q[-1]^r$ for each value of r. □

Theorem 5.1.6. *Let $x = [a]^{p_1}[b]^{q_1}[-1] \in \mathcal{B}(\mathbb{Z}_{ab})$, and suppose $a < b$. If a factorization of x^{ma} uses an atom of the form $[a]^p[b]^q[-1]^r$ where $p, q, r > 0$, then there is another factorization of x^{ma} which has a greater 1-norm.*

Proof. Consider a factorization of x^{ma} which uses at least one atom of the form $[a]^p[b]^q[-1]^r$. Then this factorization contains the block $\left([a]^p[b]^q[-1]^r\right)^h$ where $p, q, r > 0$ and $h > 1$. Note that this block contains qh copies of $[b]$. Now we can write qh as $ak - g$ where $0 \leq g < a$. Since the total number of copies of $[b]$ must be a multiple of a, we know that there must be g copies of the atom $[b][-1]^b$ in the factorization of x^{ma}. We can now construct another factorization of x^{ma} by replacing the blocks $\left([a]^p[b]^q[-1]^r\right)^h$ and $\left([b][-1^b]\right)^g$ with other blocks. First, since $qh + g = ak$, we can use all of the copies of $[b]$ from the original two blocks to construct the block $([b]^a)^k$. We are left with ph copies of $[a]$ and $rh+bg$ copies of $[-1]$.

We will now show that all the copies of $[-1]$ can be used in atoms of the form $[a][-1]^a$. We have $ap+bq=ab+r$, so $r=ap+bq-ab \to rh=aph+bqh-abh$. We also have $g=ak-qh \to bg=abk-bqh$. We then get $rh+bg=aph+bqh-abh+abk-bqh=aph+ab(k-h)$. Also, since $g<a$ and $q<a$, we have $ak=g+qh<a+ah=a(1+h)$; therefore $k<1+h \to k\leq h \to k-h\leq 0$. Together, these imply that we can construct $ph+b(k-h)$ copies of the atom $[a][-1]^a$, which together use all the copies of $[-1]$ from the original two blocks. We are left with $h-k$ copies of the atom $[a]^b$. Hence we get our new factorization by performing the following exchange of blocks:

$$([a]^p[b]^q[-1]^r)^h \left([b][-1^b]\right)^g \Leftrightarrow ([b]^a)^k \left([a]^b\right)^{h-k} ([a][-1]^a)^{ph+b(k-h)}.$$

We will now show that the length of the right-hand side is greater than the length of the left-hand side using some basic properties of cross numbers. For more information on cross numbers, see [3]. First note that the total number of copies of $([b]^a)$ and $\left([a]^b\right)$ on the right-hand side is h, the same as the number of copies of $([a]^p[b]^q[-1]^r)$ on the left-hand side. Now the cross number of any single-letter atom is 1, so the total cross number of all single-letter atoms on the right-hand side is h. But the cross number of $([a]^p[b]^q[-1]^r)$ is

$$\frac{p}{b}+\frac{q}{a}+\frac{r}{ab}=\frac{ap+bq+r}{ab}=\frac{ab+2r}{ab}=1+\frac{2r}{ab}>1.$$

Since the cross numbers of the left-hand and right-hand sides must be equal, we see that the total cross number of $\left([b][-1^b]\right)^g$ must be greater than the total cross number of $([a][-1]^a)^{ph+b(k-h)}$. The cross number of $[b][-1^b]$ is $\frac{2}{a}$, and likewise the cross number of $[a][-1]^a$ is $\frac{2}{b}$. Now $a<b$ by assumption, so we get $\frac{2}{a}>\frac{2}{b}$. Then if the number of copies of $[a][-1]^a$ was g, we would have

$$\mathbf{k}\left(([b][-1^b])^g\right)=\frac{2g}{a}>\frac{2g}{b}=\mathbf{k}\left(([a][-1^a])^g\right).$$

This would imply that the total cross number of $([a][-1]^a)^{ph+b(k-h)}$ was less than that of $\left([b][-1^b]\right)^g$, which is impossible by our earlier argument. So the number of copies of $[a][-1]^a$ is greater than g, and the length of the factorization has increased. $\square$

5.2 Characterizing $t=0$ in Regular ACMs and Block Monoids

The following results will mirror those found for numerical semigroups, in the sense that the maximal and minimal factorization lengths for powers of an element x will be bounded by some constant independent of the power n. We will now present an example that illustrates the below argument for max length being eventually constant.

As we did above, consider $x=[1][3][4]^2\in\mathcal{B}(\mathbb{Z}_6)$. Given a factorization of x^2, $([1]^3[3])([3]^2)([4]^3)^2$ (which has 0-length 3), we can construct a factorization of x^3, $([1]^3[3])([3]^2)([4]^3)^2([1][3][4]^2)$, using the factorization of x^2 and appending x (which in this case is an atom itself). In this way, we can argue that the max length function will never decrease. Further, given that we only have a finite number of atoms available to us, there is a limit to how long this factorization can get, within the 0-norm. Thus, we need only provide an example of x^n for some n that can be factored in a way that uses all available atoms to show that the max length factorization is constant as a function of n, and $x^{12}=([1]^6)([1]^3[3])([3]^2)^5([1]^2[4])([1][3][4]^2)([4]^3)^6$ is one such example.

Theorem 5.2.1. *Let $\mathcal{B}(G)$ be a block monoid and let $S\subseteq G$. Let $x\in\mathcal{B}(G,S)$, then $L_0(x^n)=|A_S|$ for sufficiently large n.*

Proof. Let $x \in \mathcal{B}(G, S)$. To prove this, it suffices to show that: (1) $L_0(x^n)$ (as a function of n) is a non-decreasing function, (2) $|A_S|$ is the largest value of $L_0(x^n)$ possible, and (3) $L_0(x^n)$ achieves a value of $|A_S|$ for n large enough.

1. Let $x = [a_1]^{n_1}[a_2]^{n_2} \cdots [a_i]^{n_i}$ be a maximum factorization of x for distinct atoms $a_i \geq 0$. This yields a maximum 0-norm length of $L_0(x) = i$. Now consider x^n, which has some maximum factorization $[b_1]^{m_1}[b_2]^{m_2} \cdots [b_k]^{m_k}$ of distinct atoms b_k where $i \leq k$ (with length $L_0(x^n) = k$). Take the element x^{n+1}, which we can express as

$$x^{n+1} = ([b_1]^{m_1}[b_2]^{m_2} \cdots [b_k]^{m_k})([a_1]^{n_1}[a_2]^{n_2} \cdots [a_i]^{n_i})$$

   and notice that the length of this factorization of x^{n+1} satisfies $L_0(x^{n+1}) \geq L_0(x^n)$.

2. For any $x \in \mathcal{B}(G, S)$, we have that the only possible atoms that x can contain are the atoms in $\mathcal{B}(G, S)$, of which only use the characters in S. Since there are only finitely many atoms in $\mathcal{B}(G, S)$ (because G is finitely generated), the largest possible value for $L_0(x^n)$ is $|A_S|$.

3. Finally, we will show that there exists a value of n such that $L_0(x^n) = |A_S|$. Suppose that $S = \{s_1, \ldots, s_j\}$, $x = [s_1]^{p_1} \cdots [s_j]^{p_j}$,

$$A_S = \{[s_1]^{q_{1,1}} \cdots [s_j]^{q_{1,j}}, \ldots, [s_1]^{q_{r,1}} \cdots [s_j]^{q_{r,j}}\},$$

   and

$$A_S^* = [s_1]^{q_{1,1}} \cdots [s_j]^{q_{1,j}}[s_1]^{q_{2,1}} \cdots [s_j]^{q_{r-1,j}}[s_1]^{q_{r,1}} \cdots [s_j]^{q_{r,j}} = [s_1]^{c_1} \cdots [s_j]^{c_j}.$$

   Now, choose $n = \sum_{i=1}^{j} \left\lceil \frac{c_i}{p_i} \right\rceil$. In this way, we guarantee that there will be enough of each element in S to create a factorization of x^n consisting of one copy of each atom in A_S, followed by some large leftover string of characters which can be factored into atoms in some way. Because we have guaranteed that x^n can be factorized in a way that includes every atom in A_S, we have that $L_0(x^n) = |A_S|$.

$\square$

Lemma 5.2.2. *Let H be a homomorphism from $M_{1,b}$ to $\mathcal{B}(\mathbb{Z}_f)$, where $f = \varphi(b)$. For $x \in M_{1,b}$, let S be the set of characters in $H(x)$ and let A_S be the set of atoms of $\mathcal{B}(\mathbb{Z}_f, S)$. Finally, let A_M be the preimage of A_S under H. Then A_M is finite.*

Proof. Let h be an isomorphism from $(\mathbb{Z}_b^*, \cdot)$ to $(\mathbb{Z}_f, +)$. Also, for $0 < i < b$, let P_i be the number of primes in the integer prime factorization of x which are congruent to $i \pmod b$. Then every $h(i)$ corresponds to a letter in $\mathcal{B}(\mathbb{Z}_f)$. Now let $a = [0]^{q_0}[1]^{q_1} \cdots [f-1]^{q_{f-1}}$ be an atom in A_S. Then for each letter $[s] \in a$, there are P_i primes in the factorization of x which map to $[s]$ under h. It follows that the number of atoms in $M_{1,b}$ constructed from prime factors of x which map to a under H is at most

$$\prod_{j=0}^{f-1} \left(P_{h^{-1}(j)}\right)^{q_j}.$$

Repeating this argument for each a in A_S shows that there are finitely many atoms in $M_{1,b}$ constructed from prime factors of x which map to an atom in A_S. Finally, by the definition of A_M, we get that $|A_M|$ is finite. $\square$

Theorem 5.2.3. *Let $M_{1,b}$ be a regular ACM and $x \in M_{1,b}$. Then $L_0(x^n)$ is constant for sufficiently large n.*

Proof. Let $x \in M_{1,b}$ and let A_M be defined as in Lemma 5.2.2. To prove this, it suffices to show that: (1) $L_0(x^n)$ (as a function of n) is a non-decreasing function, (2) $|A_M|$ is the largest value of $L_0(x^n)$ possible, and (3) $L_0(x^n)$ achieves a value of $|A_M|$ for n large enough.

1. Let $x = a_1^{n_1} a_2^{n_2} \cdots a_i^{n_i}$ be a factorization of x with maximum 0-norm, where the a_i are distinct atoms in $M_{1,b}$. Then $L_0(x) = i$. Now consider x^n, which has some maximum 0-norm factorization $b_1^{m_1} b_2^{m_2} \cdots b_k^{m_k}$, where the b_k are distinct atoms in $M_{1,b}$ and $i \leq k$. Then we have $L_0(x^n) = k$. Now take the element x^{n+1}, which we can express as

$$x^{n+1} = (b_1^{m_1} b_2^{m_2} \cdots b_k^{m_k})(a_1^{n_1} a_2^{n_2} \cdots a_i^{n_i})$$

and notice that the length of this factorization of x^{n+1} satisfies $L_0(x^{n+1}) \geq L_0(x^n)$.

2. Let H and S be defined as in Lemma 5.2.2. We have that the only possible atoms that $H(x)$ can contain are the atoms in $\mathcal{B}(\mathbb{Z}_f, S)$, which only use the characters in S. Therefore the only possible atoms that x can contain are the atoms in A_M. Since there are only finitely many atoms in A_M by Lemma 5.2.2, the largest possible value for $L_0(x^n)$ is $|A_M|$.

3. Finally, we will show that there exists a value of n such that $L_0(x^n) = |A_M|$. Let $x = s_1^{p_1} \cdots s_j^{p_j}$ be the integer prime factorization of x, and let A_M^* be the product of all atoms in A_M. Then we have

$$A_M^* = s_1^{c_1} \cdots s_j^{c_j}.$$

Now, choose $n = \sum_{i=1}^{j} \left\lceil \frac{c_i}{p_i} \right\rceil$. In this way, we guarantee that there will be enough of each s_i to create a factorization of x^n consisting of one copy of each atom in A_M, followed by some large leftover element of $M_{1,b}$ which can be factored into atoms in some way. Because we have guaranteed that x^n can be factorized in a way that includes every atom in A_M, we have that $L_0(x^n) = |A_M|$.

□

Since we have a constant bound for the max length factorization of x^n, the min length factorization function is guaranteed to be asymptotically constant as well. As it turns out, many examples turn out to be precisely constant in n. However, it is possible to construct examples that do not follow this behavior. One of these is $x = [1][2][6][12][16][17] \in \mathcal{B}(\mathbb{Z}_{18})$, which is pictured in Figure 10 (which uses $513590 \in M_{1,19}$). Despite this function not being constant, it still does appear to be periodic. The proof below demonstrates that this will hold in general.

Theorem 5.2.4. *Given a finitely generated Abelian group G, a subset $S \subseteq G$, and an element $x \in \mathcal{B}(G, S)$, $l_0(x^n)$ is periodic as a function of n.*

Proof. Consider the function

$$P(x; n; a_1, \ldots, a_k) = \begin{cases} 1 & \text{if } x^n \in \langle a_1, \ldots, a_k \rangle\} \\ 0 & \text{if } x^n \notin \langle a_1, \ldots, a_k \rangle\}, \end{cases}$$

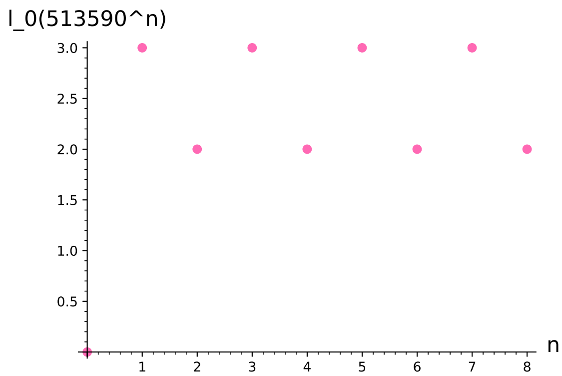

Figure 10: $l_1(513590^n)$ in $M_{1,19}$

where $a_i \in A_S$. Suppose, for fixed $A = \{a_1, \ldots, a_k\}$, that

$$P(x; p; A) = P(x; q; A) = 1,$$

so $x^p = a_1^{c_{p,1}} \cdots a_k^{c_{p,k}}$ and $x^q = a_1^{c_{q,1}} \cdots a_k^{c_{q,k}}$. Then, consider $pi + qj$ for $i, j \in \mathbb{N}_0$, not both 0. Now,

$$a_1^{c_{p,1}\cdot i + c_{q,1}\cdot j} \cdots a_k^{c_{p,k}\cdot i + c_{q,k}\cdot j} = \left(a_1^{c_{p,1}\cdot i} \cdots a_k^{c_{p,k}\cdot i}\right)\left(a_1^{c_{q,1}\cdot j} \cdots a_k^{c_{q,k}\cdot j}\right) = x^{pi}x^{qj} = x^{pi+qj},$$

so $P(x; pi + qj; A) = 1$, and so the function is closed under addition. Now, consider the semigroup

$$T = \{n \mid P(x; n; A) = 1\} \subseteq (\mathbb{N}, +).$$

Then, when $n > \bar{F}(T)$, we have that

$$P(x; n; A) = \begin{cases} 1 & \text{if } \gcd(T) \mid n \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $P(x; n; A)$ is an eventually periodic function.

Then, let

$$Q(x; n; k) = \max\{P(x; n; A) \mid A \subseteq A_S, |A| = k\}.$$

Note that since each $P(x; n; A)$ is eventually periodic, $Q(x; n; k)$ will also be periodic, having period equal to the lowest common multiple of the periods of the individual $P(x; n; A)$ functions.

Finally, observe that

$$l_0(x^n) = \min\{k \mid Q(x; n; k) = 1\},$$

which is also periodic itself.

□

5.3 Characterizing $t = \infty$ in Regular ACMs and Block Monoids

Theorem 5.3.1. *Let $x = [s][-s] \in \mathcal{B}(\mathbb{Z}_k)$ where s has order k. Then $l_\infty(x^n)$ is quasi-linear with period $k + 1$ for all $n \geq k(k + 1)$.*

Proof. Note that the only atoms in $\mathcal{B}(\mathbb{Z}_k, \{s, -s\})$ are $[s]^k$, $[-s]^k$, and $[s][-s]$. We now write n as $(k + 1)q - r$ where $0 \leq r \leq k$. Since $n \geq k(k + 1)$, we have $(k + 1)q \geq (k + 1)q - r \geq k(k + 1)$, and therefore $q \geq k$. So $\left([s]^k\right)^q \left([-s]^k\right)^q ([s][-s])^{q-r}$ is a factorization of x^n with ∞-norm q. In order to show that this factorization has the minimum ∞-norm, we will show that any change to the factorization will result in a greater ∞-norm. The only way to change the factorization is to apply the trade $([s][-s])^k \Leftrightarrow \left([s]^k\right)\left([-s]^k\right)$. Applying this trade a times for $a \in \mathbb{Z} \setminus \{0\}$ gives the factorization $\left([s]^k\right)^{q+a}\left([-s]^k\right)^{q+a}([s][-s])^{q-r-ak}$, where negative values of a correspond to applying the trade in reverse. If $a > 0$, then the ∞-norm of the factorization is $q + a > q$. If instead $a < 0$, then the ∞-norm is $q - r - ak \geq q - r + k \geq q$. In either case, the ∞-norm of the new factorization is greater than or equal to q, so the minimum ∞-norm of x^n is q. Finally, since n was arbitrary, this shows that $l_\infty(n+k+1) = l_\infty((k+1)(q+1)-r) = q+1 = l_\infty(n)+1$ for all $n \geq k(k+1)$. $\square$

Corollary 5.3.2. *For $x = [s][-s] \in \mathcal{B}(\mathbb{Z}_k)$, we have $l_\infty(x^n) = \left\lfloor \frac{n}{k+1} \right\rfloor$.*

The following result is very similar to the one given in Theorem 5.1.3.

Theorem 5.3.3. *Given a finitely generated Abelian group G, a subset $S \subseteq G$, $x \in \mathcal{B}(G;S)$, and $n \in \mathbb{N}$, $L_\infty(x^n) \sim n$.*

Proof. It suffices to bound $L_\infty(x^n)$ above and below by functions that are both linear in n.

To bound the function below, note that given a factorization $\vec{x} \in \mathsf{Z}(x)$ such that $|\vec{x}|_\infty = \lambda$, we can construct a factorization $\vec{y} \in \mathsf{Z}(x^n)$ such that $|\vec{y}|_\infty = \lambda n$, simply by using n copies of $\vec{x}$. Then, since $1 \leq \lambda$, we have that $n \leq \lambda n \leq L_\infty(x^n)$.

To bound the function above, for $S = \{[s_1], \ldots, [s_j]\}$, suppose that $x = [s_1]^{p_1} \cdots [s_j]^{p_j}$. Then, let $c = \sum_{i=1}^{j} p_i$, denoting the number of characters in x. In general, x^n will have cn characters. Then, given $A_S = \{[s_1]^{q_{1,1}} \cdots [s_j]^{q_{1,j}}, \ldots, [s_1]^{q_{r,1}} \cdots [s_j]^{q_{r,j}}\}$, let $a = \min\left\{\sum_{i=1}^{j} q_{t,i} \mid 1 \leq t \leq r\right\}$. Since the longest possible length for x^n is achieved when x^n can be written as copies of an atom with minimal character count, we have that $L_\infty(x^n) \leq \frac{cn}{a}$.

Thus, since $n \leq L_\infty(x^n) \leq \frac{cn}{a}$, we can conclude that $L_\infty(x^n) \sim n$. $\square$

Lemma 5.3.4. *Let $x \in \mathcal{B}(\mathbb{Z}_m, S)$ and $s \in S$. Suppose that the lowest power of $[s]$ among all the atoms of $\mathcal{B}(\mathbb{Z}_m, S)$ is p. Then the power of $[s]$ in x is a multiple of p.*

Proof. We will first show that all the powers of $[s]$ among the atoms of $\mathcal{B}(\mathbb{Z}_m, S)$ are multiples of p. Let A_1 be an atom containing $[s]^p$, and suppose by way of contradiction that there is an atom A_2 containing $[s]^q$, where $p \nmid q$. Since p is the lowest power of $[s]$ in an atom, we must have $p < q$. It follows that we can write q as $pk+r$, where $0 < r < p$. Now consider A_2/A_1. This is not an element of the block monoid, because A_2 is an atom. We can still represent it, however, if we allow the powers on the letters to be negative. We can then convert A_2/A_1 into a block monoid element B_1 by multiplying by sufficiently many copies of $[s_i]^m$ for each $[s_i]$ which appears to a negative power in A_2/A_1. Now since $q > p$, we see that B_1 contains $[s]^{q-p}$. This process can then be repeated a total of k times, where k was defined above. The resulting block B_k contains $[s]^{q-pk} = [s]^r$. So there is a block which contains fewer than p copies of $[s]$. Since $\mathcal{B}(\mathbb{Z}_m, S)$ is atomic, this implies that there is an atom of $\mathcal{B}(\mathbb{Z}_m, S)$ which contains fewer than p copies of $[s]$. But this contradicts the fact that p is the smallest power of $[s]$ in an atom. By contradiction, the powers of $[s]$ in all atoms must be multiples of p. Finally, since x is an element of an atomic block monoid, it must factor into atoms, all of which contain $[s]$ a multiple of p times. Hence the power of $[s]$ in x is also a multiple of p. $\square$

Theorem 5.3.5. *Let $x = [s_1]^{p_1}[s_2]^{p_2} \ldots [s_k]^{p_k} \in \mathcal{B}(\mathbb{Z}_m)$. Then for n sufficiently large, $l_\infty(x^n) \sim n$.*

Proof. We will show that $l_\infty(x^n)$ is bounded above and below by linear functions of n. First, let $\lambda = l_\infty(x)$. Then for each n, there is a factorization of x^n with ∞-norm λn, which is obtained by multiplying the power of each atom in the minimum ∞-norm factorization of x by n. Hence $l_\infty(x^n) \leq \lambda n$.

We will now show that $l_\infty(x^n)$ is bounded below by a linear function of n. Let $S = \{[s_1], [s_2], \cdots [s_k]\}$, and let a_i be the number of times that $[s_i]$ appears in the atoms of $\mathcal{B}(\mathbb{Z}_m, S)$, counted with multiplicity. We now write np_i as $a_i q_i - r_i$, where $0 \leq r_i < a_i$. Consider a block B which consists of exactly the atoms containing $[s_i]$, each raised to the power of a_i. This block contains $a_i q_i$ copies of $[s_i]$ in total. Let P_i be the lowest power of $[s_i]$ that appears in an atom of $\mathcal{B}(\mathbb{Z}_m, S)$. Then by Lemma 5.3.4, we have that np_i is a multiple of P_i, and that a_i is a multiple of P_i. From here we get that r_i must be a multiple of P_i as well. We now create a sub-block B' of B by removing $\frac{r_i}{P_i}$ copies of an atom containing $s_i^{P_i}$. To do this, we must have $q_i > a_i$, so we must have $n > a_i^2$. We note that B' contains $a_i q_i - r_i$ copies of $[s_i]$. Furthermore, there is at least one atom which appears in B' a total of q_i times. This is because if every atom appeared fewer than q_i times, the total number of copies of $[s_i]$ in B' would be at most $a_i(q_i - 1)$, and we would have $r_i \geq a_i$. It follows that the ∞-norm of B' is q_i.

Suppose there is a factorization F of x^n whose ∞-norm is less than q_i. Then the powers on all atoms in F are less than q_i, including the atoms containing $[s_i]$. But then the number of copies of $[s_i]$ in F is less than $a_i(q_i - 1) = a_i q_i - a_i < a_i q_i - r_i$. So there are fewer copies of $[s_i]$ in F than there are in x^n. This is clearly impossible, so by contradiction, no such F exists. Thus $l_\infty(x^n) \geq q_i$ as desired. Now since $[s_i]$ was arbitrary, the above argument shows that $l_\infty(x^n) \geq \max\{q_i : 0 \leq i \leq k\}$. Recall that $np_i = a_i q_i - r_i$, so $q_i = \frac{np_i + r_i}{a_i} \geq \frac{np_i}{a_i}$. Since p_i and a_i are constant with respect to n, we see that q_i grows linearly with n. Hence $l_\infty(x^n)$ is bounded below by a linear function of n. Together with the upper bound, this completes the proof. □

6 Results in Singular ACMs

6.1 Characterizing $t = 1$ in Singular ACMs

Theorem 6.1.1. *Let $M_{a,b}$ be an Arithmetic Congruence Monoid, and let $x \in M_{a,b}$. Then $L_1(x^n)$ has a linear growth rate.*

Proof. We will show that $L_1(x^n)$ is bounded below and above by functions that both scale linearly. Let $x \in M_{a,b}$, and write x as its prime factorization in $\mathbb{Z}$, $x = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ for some primes p_i and $k \in \mathbb{Z}_{\geq 0}$. If x is an atom in $M_{a,b}$, it follows that x^n has length n in $M_{a,b}$, so $n \leq L_1(x^n)$. If every prime p_i in the factorization of x is in $M_{a,b}$, then x^n has maximum length $\left(\sum_{i=1}^k m_i\right) \cdot n$, therefore $L_1(x^n) \leq \left(\sum_{i=1}^k m_i\right) \cdot n$. Since n and $\left(\sum_{i=1}^k m_i\right) \cdot n$ scale linearly, we have that $L_1(x^n)$ grows linearly as well. □

6.2 Characterizing $t = 0$ in Singular ACMs

Theorem 6.2.1. *Given an ACM $M_{a,b}$, $x \in M_{a,b}$, and $n \in \mathbb{N}_0$, $l_0(x^n) \leq l_0(x)$. In particular, $l_0(x^n)$ is bounded independent of n.*

Proof. It suffices to bound $l_0(x^n)$ above by a function that is constant in n. Suppose we have a factorization $x = a_1^{p_1} \cdots a_k^{p_k}$, with all a_i being atoms in $M_{a,b}$. Here, the length of this factorization under the 0-norm is k. Then, we can construct a factorization $x^n = a_1^{p_1 \cdot n} \cdots a_k^{p_k \cdot n}$, which also has length k. Therefore, we can choose a factorization of x which has minimal length $l_0(x)$, and conclude that $l_0(x^n) \leq l_0(x)$. $\square$

Based on our data, we suspect that $L_0(x^n)$ has a square-root growth rate in singular ACMs. Here we provide some bounds on $L_0(x^n)$ for certain classes of elements in the ACM $M_{4,6}$. We first characterize the atoms in this monoid.

Lemma 6.2.2. *All atoms in $M_{4,6}$ are of one of the following forms:*

1. 2^2,
2. $2p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$ *where the p_i are primes congruent to* $5 \pmod 6$, *the a_i are positive integers whose sum is odd, and $k \in \mathbb{Z}^+$,*
3. *any of the above times any number of primes congruent to* $1 \pmod 6$.

Proof. We first note that the only possible prime factors of an element in $M_{4,6}$ are primes congruent to 1, 2, or 5 modulo 6, and that the only prime congruent to $2 \pmod 6$ is 2.

1. Since $2^2 = 4$ is the smallest nonunit in $M_{4,6}$, it must be an atom.
2. We have that $2 \cdot 5 \equiv 4 \pmod 6$, and that $5 \cdot 5 \equiv 1 \pmod 6$. Hence every integer of the form $2 \cdot p^k$, where p is a prime congruent to $5 \pmod 6$ and k is odd, is in $M_{4,6}$. Now since all elements of $M_{4,6}$ are even, attempting to factor such an integer in $M_{4,6}$ must involve factoring it into two even integers. But all these integers only contain one factor of 2, and so cannot be factored into two even integers. So we get that all such integers are atoms in $M_{4,6}$.
3. Finally, since 1 is the multiplicative identity modulo 6, we see that any of the above atoms multiplied by a prime congruent to $1 \pmod 6$ is an element of $M_{4,6}$. In attempting to factor one of these elements, one of the factors must be the original atom. But then the remaining factor is odd, and so is not an element of $M_{4,6}$. Hence all such elements are atoms in $M_{4,6}$. This process may be repeated indefinitely to create atoms containing any number of primes congruent to $1 \pmod 6$.

We now show that the above are the only types of atoms in $M_{4,6}$. We first note that all elements of $M_{4,6}$ are divisible by 2. Those which contain exactly one or two factors of 2 were considered above. Now suppose an element x of $M_{4,6}$ is a multiple of 2^3. Then it may be divided by 4, leaving an even quotient. But since $x \equiv 4 \pmod 6$, we must have $\frac{x}{4}$ congruent to $4 \pmod 6$. Hence $\frac{x}{4} \in M_{4,6}$, and so x is not an atom in $M_{4,6}$. $\square$

Theorem 6.2.3. *Let $x \in M_{4,6}$ such that x is not a power of 2. Then for all $n \in \mathbb{Z}^+$, there is some $i \in \mathbb{Z}^+$ such that $L_0(x^{n+i}) > L_0(x^n)$.*

Proof. Let A_n be the set of atoms in the factorization of x^n which gives the maximum 0-norm. We now split the proof into two cases based on the prime factorization of x.

1. If x contains a 1-mod-6 prime, then let p be the maximum power of a 1-mod-6 prime among the elements of A_n, and choose $i = p + 1$. We can now construct a factorization of x^{n+i} by starting with the maximum 0-norm factorization of x^n and adding (among others) an atom of the form $2^2 \cdot a_1^{p+1}$, where a_1 is a 1-mod-6 prime. Note that since $p \geq 1$, we have $i \geq 2$, so there will be sufficient copies of 2 in the prime factorization of x_{n+i}. This atom was not in the maximum 0-norm factorization of x^n, so the 0-norm has strictly increased.

2. If x does not contain a 1-mod-6 prime, then it must contain a 5-mod-6 prime. Let p be the maximum power of a 5-mod-6 prime among the elements of A_n. Note that p must be odd, since it is the power of a 5-mod-6 prime in an atom of $M_{4,6}$. Now choose $i = p + 2$. Since $p + 2$ is odd, we can construct a factorization of x^{n+i} by starting with the maximum 0-norm factorization of x^n and adding an atom of the form $2 \cdot a_5^{p+2}$, where a_5 is a 5-mod-6 prime. As before, this atom was not in the maximum 0-norm factorization of x^n, so the 0-norm has strictly increased.

□

We now present examples of elements in $M_{4,6}$ for which $L_0(x^n) \sim n^{1/2}$ and $l_0(x^n) \sim n^{1/2}$.

Theorem 6.2.4. *Let $x = 28 \in M_{4,6}$, and $n \in [T_k, T_{k+1})$, where T_k denotes the k-th triangular number. Then*

$$\sqrt{T_{k-1} + T_k} + 1 \leq L_0(x^n) \leq \sqrt{T_{k-1} + T_k} + 2.$$

Proof. Let $x = 28 \in M_{4,6}$, and suppose $n \in [T_k, T_{k+1})$, where T_k denotes the k-th triangular number. We will show that $L_0(x^n)$ is bounded above and below by the functions $\sqrt{T_{k-1} + T_k} + 2$ and $\sqrt{T_{k-1} + T_k} + 1$, respectively. To begin, note that the prime factorization of 28 over $\mathbb{Z}$ is given by $28 = 2^2 \cdot 7$. In $M_{4,6}$, however, 28 is irreducible, since $28 = 2^2 \cdot 7 = 4 \cdot 7$ but $7 \notin M_{4,6}$. The fact that $7 \notin M_{4,6}$, implies that for all $n \geq 1$, $7^n \notin M_{4,6}$. As such, we see that any element of the form $4 \cdot 7^n = (2^2) \cdot 7^n$ is irreducible in $M_{4,6}$. Now, if we take $k > 2$, and $n \in [T_k, T_{k+1})$, then $n = 1 + 2 + \cdots + k + i$, for some $0 \leq i < k + 1$, so we can always write

$$x^n = (2^2 \cdot 7)^n = (2^2)^n 7^n = (2^2)(2^2 \cdot 7)(2^2 \cdot 7^2) \cdots (2^2 \cdot 7^k)(2^{2n-(2k+2)} \cdot 7^i),$$

where the last term contains powers of (2^2)'s and 7's which can be distributed amongst the first $(k+1)$ terms, (that is, by increasing the power of any of the preceding $(2^{2j} \cdot 7^r)$ terms), to balance out the factorization as needed. Since each of the first $k + 1$ factors in the above factorization is irreducible and distinct in $M_{4,6}$, we must have

$$L_0(x^n) \geq k + 1 = \sqrt{T_{k-1} + T_k} + 1.$$

Similarly, notice that each of the first k elements of $M_{4,6}$, whose prime factorization is made up exclusively of powers of (2^2) and 7, appears as one of the first $k + 1$ factors in the factorization of x^n given above, (since each of these factors is irreducible and distinct). As such, it follows that the (2^2)'s and the 7's appearing in last "factor" of the given factorization, namely, the term $((2^{2n-(2k+2)} \cdot 7^i))$, can only be distributed amongst the factors which have already been used (i.e. it cant form a new irreducible element of $M_{4,6}$ since $i < k + 1$). In particular, we see that since $i < k+1$, any factorization of x^n can't have each of the first $k+1$ factors given in the factorization above, and a factor of $(2^2 \cdot 7^j)$, where $k + 1 \leq j \leq T_k + i$. Since, together, these make up all of the factors of $x^n = (2^2)^n 7^n$, it follows that

$$L_0(x^n) \leq \|(2^2)(2^2 \cdot 7)(2^2 \cdot 7^2) \cdots (2^2 \cdot 7^k)(2^2 \cdot 7^{k+1})\|_0 = k + 2 = \sqrt{T_{k-1} + T_k} + 2.$$

This completes the proof. □

Theorem 6.2.5. *Let $x = 40 \in M_{4,6}$, and $n \in [T_{k-1} + T_k, T_k + T_{k+1}) = [k^2, (k+1)^2)$, where T_k denotes the k-th triangular number. Then*

$$\sqrt{T_{k-1} + T_k} + 1 \leq L_0(x^n) \leq \sqrt{T_{k-1} + T_k} + 2.$$

Proof. Let $x = 40 \in M_{4,6}$, and suppose $n \in [T_{k-1}+T_k, T_k+T_{k+1}) = [k^2, (k+1)^2)$, where T_k denotes the k-th triangular number. We will show that $L_0(x^n)$ is bounded above and below by the functions $\sqrt{T_{k-1}+T_k}+2$ and $\sqrt{T_{k-1}+T_k}+1$, respectively. To begin, note that the prime factorization of 40 over $\mathbb{Z}$ is given by $40 = 2^3 \cdot 5$. In $M_{4,6}$, however, 40 only factors as $40 = 4 \cdot 10 = (2^2) \cdot (2 \cdot 5)$. Moreover, we point out that while $5^2 \notin M_{4,6}$, since $5^2 \equiv 1 \bmod 6$, it follows that, for all $n \in \mathbb{Z}_{\geq 1}$, $(2 \cdot 5) \cdot 5^{2n}$ is an irreducible element in $M_{4,6}$. Thus, for any $k \geq 2$, and $n \in [T_{k-1}+T_k, T_k+T_{k+1})$, we have $n = T_{k-1}+T_k+i$, for some $0 \leq i < 2k+1$, and we can always write

$$x^n = (2^3 \cdot 5)^n = 2^{3n}5^n = (2^2)(2\cdot 5)(2\cdot 5\cdot 5^2)(2\cdot 5\cdot (5^2)^2)\cdots(2\cdot 5\cdot (5^2)^{k-1})(2^{3n-(k+2)}\cdot 5^i),$$

where the last term consists of powers of (2^2)'s and 5's which can be added to some preceding term to balance out the factorization. Now, since the first $k+1$ factors are all irreducible in $M_{4,6}$, we must have

$$L_0(x^n) \geq k+1 = \sqrt{T_{k-1}+T_k}+1,$$

for all $n \in [T_{k-1}+T_k, T_k+T_{k+1})$. Similarly, notice that each of the first k elements of $M_{4,6}$, whose prime factorization (over $M_{4,6}$) is made up exclusively of positive powers of (2^2) and 5, already appear as one of the first $k+1$ factors in the factorization of x^n given above. Moreover, since $i < 2k+1$, the remaining powers of 2's and 5's, which make up the last term, that is, the term $(2^{3n-(k+2)} \cdot 5^i)$, can only be distributed amongst the factors that have already been used. In particular, this means that no factorization of x^n can have each of the first $k+1$ factors used in the factorization above, and a factor of $(2 \cdot 5^j)$, where $2k+1 \leq j$ (as this would result in an element greater than x^n). Since, together, these encompass all of the factors of x^n, we must have

$$L_0(x^n) \leq \|(2^2)(2\cdot 5)(2\cdot 5\cdot 5^2)\cdots(2\cdot 5\cdot (5^2)^{k-1})(2\cdot 5\cdot (5^2)^k)\|_0 = k+2 = \sqrt{T_{k-1}+T_k}+2,$$

as desired. □

Theorem 6.2.6. *Let $x = 2^a5^b7^c \in M_{4,6}$. Then $L_0(x^n) = O(n^{2/3})$.*

Proof. We first note that the prime factorization of x^n contains $(b+c)n$ total copies of 5 and 7. Now consider the atoms of $M_{4,6}$ of the form $2 \cdot 5^p \cdot 7^q$, where p is odd. We wish to factor x^n in a way that contains as many distinct atoms of this form as possible. To do this, we must prioritize selecting atoms with smaller values of p and q before those with larger values. If we modify the allowed atoms by allowing p to take on even values, we could do this by first selecting the atom $2 \cdot 5$, then $2 \cdot 5^2$, then $2 \cdot 5 \cdot 7$, then $2 \cdot 5^3$, then $2 \cdot 5^2 \cdot 7$, and so on. In this way, we would select i modified atoms such that $p+q = i$ for each $i \in \mathbb{Z}^+$, up to some.integer k. The total number of such modified atoms is

$$a_1 = \sum_{i=0}^{k} i \frac{k(k+1)}{2},$$

and the total number of 5's and 7's in these modified atoms is

$$\sum_{i=0}^{k} i^2 \frac{k(k+1)(2k+1)}{6}.$$

In actuality, the total number of 5's and 7's in the real atoms will be greater, since replacing the power p of 5 with $2p-1$ will convert every modified atom into an atom while also not decreasing the total number of 5's. We now find an upper bound on k such that the total number of 5's and 7's in the modified atoms is at least $(b+c)n$.

$$\frac{k(k+1)(2k+1)}{6} \geq (b+c)n$$

$$\Rightarrow \frac{k(2k)(3k)}{6} = k^3 \geq (b+c)n$$

$$\Rightarrow k \geq \sqrt[3]{(b+c)n}$$

Hence $k = \left\lceil \sqrt[3]{(b+c)n} \right\rceil$ provides the desired upper bound. Now since $k \sim n^{1/3}$, and $a_1 \sim k^2$, we get that $a_1 \sim n^{2/3}$.

We now consider atoms of the form $2^2 \cdot 7^r$. The maximum number of such atoms which can be used in a factorization of x^n is given by $a_2 = m+1$, where m is the largest integer such that $\frac{m(m+1)}{2} \leq cn$. We see that $a_2 \sim n^{1/2}$. In actuality, the maximum number of such atoms that can be used may be less than $m+1$, because some of the atoms considered previously may also contain 7. However, the value of a_2 obtained here still suffices to show an upper bound. We now obtain an upper bound on $L_0(x^n)$ by summing a_1 and a_2. The growth rate of $a_1 + a_2$ is $\Theta(n^{2/3}) + \Theta(n^{1/2}) = \Theta(n^{2/3})$, and since this is an upper bound for $L_0(x^n)$, we see that $L_0(x^n) = O(n^{2/3})$. ☐

6.3 Characterizing $t = \infty$ in Singular ACMs

Theorem 6.3.1. *Let $M_{a,b}$ be an Arithmetic Congruence Monoid, and let $x \in M_{a,b}$. Then $L_\infty(x^n)$ has a linear growth rate.*

Proof. We will show that $L_\infty(x^n)$ is bounded below and above by functions that both scale linearly. Let $x \in M_{a,b}$, and write x as its prime factorization in $\mathbb{Z}$, $x = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ for some primes p_i and $k \in \mathbb{Z}_{\geq 0}$. If x is an atom in $M_{a,b}$, then x^n has infinity length n in $M_{a,b}$, so $n \leq L_\infty(x^n)$. If every prime p_i in the factorization of x is in $M_{a,b}$, then x^n has maximum infinity length $\max\{m_i\} \cdot n$, so $L_\infty(x^n) \leq \max\{m_i\} \cdot n$. Since n and $\max\{m_i\} \cdot n$ scale linearly, we have that $L_\infty(x^n)$ grows linearly as well. ☐

Theorem 6.3.2. *Given $x \in M_{4,6}$ and $n \in \mathbb{N}$, $l_\infty(x^n)$ has a linear lower bound.*

Proof. Let $x = 2^a 5^b 7^c$. Then, define the set of good atoms $\mathbf{G} = \{2^p 5^q 7^r \in A \mid a(q+r) \leq 3p(b+c)\}$ and the set of evil atoms $\mathbf{E} = \{2^s 5^t 7^u \in A \mid a(t+u) > 3s(b+c)\}$, where A refers to the atoms of $M_{4,6}$ containing just copies of 2, 5, and 7. Suppose for the sake of contradiction that we can factor x^n for some n using more evil atoms than good atoms. Thus, for $g_i \in \mathbf{G}$ and $e_i \in \mathbf{E}$,

$$\begin{aligned} x^n &= g_1 \cdots g_l e_1 \cdots e_m \\ &= 2^{p_1} 5^{q_1} 7^{r_1} \cdots 2^{p_l} 5^{q_l} 7^{r_l} \cdot 2^{s_1} 5^{t_1} 7^{u_1} \cdots 2^{s_m} 5^{t_m} 7^{u_m} \\ &= 2^{p_1 + \cdots + p_l + s_1 + \cdots + s_m} 5^{q_1 + \cdots + q_l + t_1 + \cdots + t_m} 7^{r_1 + \cdots + r_l + u_1 + \cdots + u_m}, \end{aligned}$$

where, by our supposition, $l < m$. Note that, by Lemma 6.2.2, each p_i and each s_i will have value either 1 or 2, so $p_1 + \cdots + p_l \leq 2 \cdot l$ and $s_1 + \cdots + s_m \geq 1 \cdot m$ (so $2 \cdot m \leq 2(s_1 + \cdots + s_m)$). Thus,

$$p_1 + \cdots + p_l \leq 2 \cdot l \leq 2l < 2m \leq 2(s_1 + \cdots + s_m).$$

Then, since the ratios between the number of copies of 2, 5, and 7 in x^n will be the same as in x, we have that

$$a(q_1 + \cdots + q_l + t_1 + \cdots + t_m + r_1 + \cdots + r_l + u_1 + \cdots + u_m) = (b+c)(p_1 + \cdots + p_l + s_1 + \cdots + s_m).$$

Observe first that

$$(b+c)(p_1+\cdots+p_l+s_1+\cdots+s_m) < (b+c)\cdot 3(s_1+\cdots+s_m).$$

Then, by the characterization of evil atoms (and because $q_1+\cdots+q_l+r_1+\cdots+r_l > 0$), note that

$$\begin{aligned} &a(q_1+\cdots+q_l+t_1+\cdots+t_m+r_1+\cdots+r_l+u_1+\cdots+u_m) \\ &> (b+c)\cdot 3(s_1+\cdots+s_m)+a(q_1+\cdots+q_l+r_1+\cdots+r_l) \\ &> (b+c)\cdot 3(s_1+\cdots+s_m). \end{aligned}$$

However, this implies that

$$(b+c)\cdot 3(s_1+\cdots+s_m) > (b+c)\cdot 3(s_1+\cdots+s_m),$$

which is a contradiction. Therefore, for an arbitrary factorization of x^n, written as

$$x^n = g_1\cdots g_L e_1\cdots e_M,$$

we know that $L \geq M$.

We will now briefly argue that there are finite atoms in **G**. To do so, notice that **G** is constructed by bounding the total copies of 2, 5, and 7, and so the total number of atoms in the set can be at most the product of these bounds.

Now, for $x = 2^a 5^b 7^c$, consider $x^{\frac{4(k\cdot|G|+1)}{a}}$. The factorization of x^n with minimal length will have $4(k\cdot|G|+1)$ copies of 2, and so will have between $2(k\cdot|G|+1)$ and $4(k\cdot|G|+1)$ atoms. Therefore, we know that there will be at least $k\cdot|G|+1$ good atoms in the factorization. Now, by the Pigeonhole Principle, we know that there will be at least one good atom that is used at least $k+1$ times, and so $l_\infty(x^{\frac{4(k\cdot|G|+1)}{a}}) > k$. Thus, we have a linear lower bound on $l_\infty(x^n)$. □

7 Future Work

The following conjectures are constructed purely on empirical data.

Conjecture 7.0.1. *Let $S = \langle g_1,\ldots,g_k\rangle$. For all $t \in \mathbb{Z}$ with $t \geq 2$,*

$$L'_t(x) = g_1^{-t}(x-A_x)^t + L'_t(A_x),$$

where $A_x \in Ap(S;g_1)$ such that $x \equiv A_x \pmod{g_1}$.

The following is hinted at by Theorems 6.2.4 and 6.2.5.

Conjecture 7.0.2. *For any singular ACM $M_{a,b}$ and $x \in M_{a,b}$, $L_0(x^n) \sim n^{1/2}$.*

The following is suggested by Theorem 6.3.2.

Conjecture 7.0.3. *For any singular ACM $M_{a,b}$ and $x \in M_{a,b}$, $l_1(x^n) \sim n$ and $l_\infty(x^n) \sim n$.*

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