Characteristics of glued numerical semigroups and the Kunz cone

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Abstract

Recent research has focused on the relationship between numerical semigroups and the group cone. Gluing is a process of building new numerical semigroups from old numerical semigroups. This write up seeks to examine where in the group cone we find glued numerical semigroups. We show a membership criterion for the Apéry sets of glued semigroups as well as a description of their poset relations. We then connect these results to the group cone using an injective map. We end by considering future directions to extend our work.

Contents

1 Introduction

A numerical semigroup is a subset $N \subseteq \mathbb{Z}_{\geq 0}$ which is closed under addition and has finite complement. We can specify numerical semigroups using a list of generators $\{g_1, g_2, \ldots, g_k\}$ such that for a numerical semigroup S we have

$$S = \{x \mid x = a_1 g_1 + \cdots + a_k g_k, a_i \in \mathbb{Z}_{\geq 0}\}.$$

We then write $S = \langle g_1, \ldots, g_k \rangle$. For each element μ in a numerical semigroup S there exists a subset of S called the Apéry set of S with respect to μ. We write

$$\mathrm{Ap}(S; \mu) = \{n \in S : n - \mu \notin S\}.$$

Apéry sets give rise to a poset $(\mathrm{Ap}(S; \mu), \preceq)$ where for $c, d \in \mathrm{Ap}(S; \mu)$ we say that $c \preceq d$ if and only if $d - c \in S$ (or equivalently $d - c \in \mathrm{Ap}(S; \mu)$ by Lemma 3.1.4). Posets can be described visually using a directed graph with edges representing covering relations.

One process of constructing new numerical semigroups from others is a process called gluing which works by "scaling" two semigroups and combining them. The goal of this paper is to provide a description of the Apéry posets of glued semigroups. We can then connect this description to a geometric object called the group cone.

The group cone is a pointed rational cone described in [3]. Apéry sets of numerical semigroups can be associated to integer points on the group cone. A previous paper [1] characterized where monoscopic (and Arithmetical) numerical semigroups lie on the group cone. Following from that, we seek to characterize where glued semigroups lie in the group cone, and explore what other semigroups lie in faces with glued semigroups.

We begin this write up by introducing necessary background on numerical semigroups, Apéry sets, posets, and gluings. This provides us with the background necessary to then introduce a membership criterion for glued semigroups. From there we will explore two special cases of the Apéry posets for general gluings: when we take the Apéry poset with respect to a generator, and when we restrict one of the glued semigroups to be all of the non-negative integers (monoscopic gluings). In both of these cases we provide a characterization of their Apéry poset relations as well as their covering relations. Following this description of Apéry posets, we move on to providing a more thorough background on the group cone, which is a larger family of cones to which the Kunz cone belongs. We then move into an attempt to generalize section 6 of [1] from monoscopic gluings to any gluing. We finally end by discussing remaining questions and what we would like to continue to look at in the future.

2 Background: numerical semigroups

We will now provide additional background on numerical semigroups, Apéry sets, and gluings which will be useful when we present our results in later sections.

2.1 Fundamental properties of numerical semigroups

Having introduced numerical semigroups in the introduction, we will now begin this section with a few examples of numerical semigroups.

Example 2.1.1. Consider

$$\langle 5,7\rangle = \{0,5,7,10,12,14,15,17,19,20,21,22,24,25,26,27,28\ldots\}$$

Notice that $1,2,3,4,6,8,9,11,13,16,18,23$ are the only non-negative integers not in the semigroup. Additionally, since we know that 5 is in the semigroup, because we have $24,25,26,27,28$ in the numerical semigroup, we can obtain any integer greater than 28 by adding copies of 5 to one of these integers. So we have a finite complement.

We can also have numerical semigroups with more than 2 generators, consider

$$\begin{aligned}\langle 6,9,20\rangle =&\{0,6,9,12,15,18,20,21,24,26,27,29,30,32,33,\\ &35,36,38,39,40,41,42,44,45,46,47,48,49\ldots\}\end{aligned}$$

Notice that 43 is the largest integer not in the numerical semigroup. Since we know that 6 is in the semigroup, and $44,45,46,47,48,49$ are in the semigroup, we can obtain every integer greater than 49 by adding copies of 6 to one of these integers. So we have finite complement here too.

We will now introduce an important result about numerical semigroups which is essential to understanding gluings:

Theorem 2.1.2 (section 1.2 of [5])**.** *Given a numerical semigroup S, there is a unique minimal generating set $\mathcal{A}(S)$ for S.*

Definition 2.1.3. We call elements of $\mathcal{A}(S)$ the *minimal generators* or *atoms* of S.

Example 2.1.4. Consider the numerical semigroup

$$\langle 5,7,10\rangle = \{0,5,7,10,12,14,15,17,19,20,21,22,24,\ldots\}$$

This is a valid numerical semigroup; however, notice that $10 = 2(5)$, so

$$\langle 5,7,10\rangle = \{0,5,7,10,12,14,15,17,19,20,21,22,24,\ldots\} = \langle 5,7\rangle$$

So $\{5,7\}$ is the minimal generating set for this numerical semigroup.

For the purposes of this manuscript, when we refer to "a generator" of a numerical semigroup S, we implicitly mean a minimal generator unless otherwise stated. We primarily use "atom" when referring to posets, where generators always appear directly above 0.

We now make note of one final result about generating sets for numerical semigroups which is useful for determining if a set is a generating set of a numerical semigroup:

Theorem 2.1.5 (section 1.2 of [5]). *A set* $\mathcal{A}(S) = \{g_1, g_2, \ldots, g_k\} \subseteq \mathbb{Z}_{\geq 0}$ *is the generating set of a numerical semigroup if and only if* $\gcd(g_1, g_2, \ldots, g_k) = 1$.

2.2 Gluing

One process of constructing new numerical semigroups from others is a process called *gluing*. Suppose $S_1 = \langle a_1, \ldots, a_k \rangle$ and $S_2 = \langle b_1, \ldots, b_\ell \rangle$ are numerical semigroups, and α, β are nonnegative integers. We may then define the set

$$T = \alpha S_1 + \beta S_2$$

where the usual definitions of adding sets and multiplying by a scalar are used. In particular, this means that

$$T = \langle \alpha a_1, \ldots, \alpha a_k, \beta b_1, \ldots, \beta b_\ell \rangle.$$

We next describe under what circumstances T is a numerical semigroup with the given generating set:

Theorem 2.2.1 (section 8.3 of [5]). *The set T is a numerical semigroup with the above generating set minimal if and only if*

1. $\gcd(\alpha, \beta) = 1$,
2. $\alpha \in S_2 \setminus \{b_1, \ldots, b_\ell\}$, *and*
3. $\beta \in S_1 \setminus \{a_1, \ldots, a_k\}$.

Definition 2.2.2. If all three requirements above are met, we call $T = \alpha S_1 + \beta S_2$ a *gluing.*

Example 2.2.3. Consider $\langle 68, 75, 85, 105 \rangle$, notice $\gcd(68, 85) = 17$ and $\gcd(75, 105) = 15$. Additionally notice that $\gcd(15, 17) = 1$. Thus we have a candidate for a gluing, which would be:

$$\langle 68, 75, 85, 105 \rangle = 17\langle 4, 5 \rangle + 15\langle 5, 7 \rangle.$$

The final condition that we need to check is that $17 \in \langle 5, 7 \rangle$ and $15 \in \langle 4, 5 \rangle$. Observe that $17 = 2(5) + 7 \in \langle 5, 7 \rangle$. Also, we can see that $15 = 3(5) = \langle 4, 5 \rangle$. So we have

$$\langle 68, 75, 85, 105 \rangle = 17\langle 4, 5 \rangle + 15\langle 5, 7 \rangle$$

is a gluing.

2.3 Apéry sets and posets

In the introduction we defined an Apéry set for an element of a numerical semigroup S. We will repeat the definition again here:

Definition 2.3.1. The *Apéry set* of a numerical semigroup S with respect to an element $\mu \in S$ is the set

$$\mathrm{Ap}(S;\mu) = \{n \in S : n - \mu \notin S\}.$$

Example 2.3.2. Consider the numerical semigroup $S = \langle 6, 9, 20\rangle$. We can then build the Apéry set $\mathrm{Ap}(S;6)$ through the following method. The first element we can consider is 6. Since $6 - 6 = 0 \in S$ we know that $6 \notin \mathrm{Ap}(S;6)$, but $0 - 6 = -6 \notin S$, so $0 \in \mathrm{Ap}(S;6)$. Next notice that $9 - 6 = 3 \notin S$, so $9 \in \mathrm{Ap}(6;S)$. Furthermore $20 - 6 = 14 \notin S$, so $20 \in \mathrm{Ap}(S;6)$. Continuing in this way we notice:

$$29 - 6 = 23 \notin S \qquad 40 - 6 = 34 \notin S \qquad 49 - 6 = 43 \notin S$$

So we have $\mathrm{Ap}(S;6) = \{0, 9, 20, 29, 40, 49\}$.

One important result about Apéry sets is that an Apéry set contains precisely one element in each mod class for μ:

Theorem 2.3.3 (section 1.2 of [5])**.** *The Apéry set of S with respect to $\mu \in S$ consists of exactly μ elements, with each modulus class mod μ being represented exactly once. In fact, we can give the equivalent characterization*

$$\mathrm{Ap}(S;\mu) = \{\min(S \cap [i]_\mu) : 0 \leq i \leq \mu - 1\}$$

or in other words, it consists of the minimal elements of each mod class mod μ which lie in S.

The fact that each mod class modulo μ is represented exactly once in $\mathrm{Ap}(S;\mu)$ means that we can write the Apéry set in order of mod class, i.e.

$$\mathrm{Ap}(S;\mu) = \{x_0, x_1, \ldots, x_{\mu-1}\}$$

where $x_i \equiv i \bmod \mu$. We traditionally write Apéry sets in this order whenever possible.

Example 2.3.4. Following from Example 2.3.2, $\mathrm{Ap}(S;6) = \{0, 9, 20, 29, 40, 49\}$ can be reordered as $\mathrm{Ap}(S;6) = \{0, 49, 20, 9, 40, 29\}$ since

$$\begin{array}{lll} 0 \equiv 0 \bmod 6 & 49 \equiv 1 \bmod 6 & 20 \equiv 2 \bmod 6 \\ 9 \equiv 3 \bmod 6 & 40 \equiv 4 \bmod 6 & 29 \equiv 5 \bmod 6 \end{array}$$

We now return to the concept of an Apéry poset. We assume the definition of a poset and poset relation is known.

Definition 2.3.5. For a numerical semigroup S and $\mu \in S$, then the *Apéry poset* of $\mathrm{Ap}(S;\mu)$, is defined such that for $c, d \in \mathrm{Ap}(S;\mu)$ we have $c \preceq d$ if and only if $d - c \in S$.

Additionally, we can define a specific type of precedence relation, the covering relation:

Definition 2.3.6. In a poset $(P, \preceq)$, we say an element $b \in P$ *covers* an element $a \in P$ if $a \prec b$ and there does not exist $c \in P$ such that $a \prec c \prec b$.

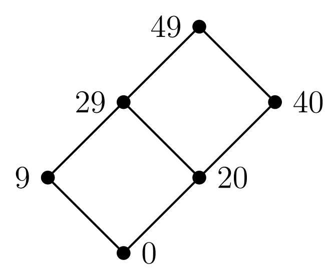
Theorem 2.3.7 (Prop. 3.10(a) in [3])**.** *In the Apéry poset* $(\mathrm{Ap}(S;\mu), \preceq_S)$*, b covers a if and only if $b - a \in \mathcal{A}(S)$. Moreover, 0 is the unique minimal element of the poset, and it is covered by the elements of $\mathcal{A}(S) \setminus \{\mu\}$.*

We call the elements which cover 0 the atoms of an Apéry poset. These correspond to the atoms (generators) of an Apéry set which are not μ. We now consider the graphical representation of a (finite) poset i.e its Hasse diagram.

Example 2.3.8. Returning to $S = \langle 6, 9, 20\rangle$, we know from Example 2.3.2 that

$$\mathrm{Ap}(S;6) = \{0, 49, 20, 9, 40, 29\}.$$

Notice that $9 - 0 = 9 \in \mathcal{A}(S)$ so 9 covers 0. Notice also $29 - 9 = 20 \in \mathcal{A}(S)$, so 29 covers 9. If we try $49 - 20 = 29 \in S$, so $20 \preceq 49$, but 49 does not cover 20 because they do not differ by a generator. If we continue to consider all combinations of elements of $\mathrm{Ap}(S;6)$, we can then build the following poset graph:



3 Apéry posets of glued semigroups

In this first set of results, we characterize the Apéry sets of arbitrary gluings with respect to arbitrary elements. We then characterize the Apéry posets of monoscopic gluings with respect to arbitrary elements, and the Apéry posets of arbitrary gluings with respect to a generator.

Our key tool in this endeavor is the idea of a *canonical factorization* for integers based on a glued semigroup, which will lead to a convenient membership criterion for both the glued semigroup and, by extension, its Apéry set, as well as allowing us to describe precisely when elements precede each other in the Apéry poset.

3.1 Canonical factorization and Apéry sets of general gluings

We first define the canonical factorization of an integer with respect to a glued semigroup and prove its fundamental properties (existence, uniqueness, and membership criterion). We then use the glued semigroup membership criterion to create an Apéry set membership criterion.

To motivate canonical factorization, consider trying to determine whether an integer n lies in the glued numerical semigroup $T = \alpha S_1 + \beta S_2$. Unfortunately, there are infinitely many ways to write n as a linear combination $\alpha c_1 + \beta c_2$ of α and β, and only one of them needs to satisfy $c_1 \in S_1, c_2 \in S_2$ for n to be in T.

To get insight into this, we can consider an element we know to be in T, namely $\alpha\beta$. There are two ways to factor this in T, either as $\alpha(\beta)+\beta(0)$ or as $\alpha(0)+\beta(\alpha)$. More generally, we can observe that for any element of T with multiple factorizations, lowering the second coefficient by α will raise the first coefficient by β, and vice versa. We might hope that by requiring us to minimize the second coefficient, we can pick out a unique factorization.

It turns out that we can always make the second coefficient c_2 small enough that it lies in $\text{Ap}(S_2; \alpha)$, and no smaller. Taking the equality $n = \alpha c_1 + \beta c_2 \bmod \alpha$ and then dividing by β, we see that $c_2 \equiv n\beta^{-1} \bmod \alpha$, which fixes the mod class of c_2 and guarantees we cannot decrease c_2 any further once $c_2 \in \text{Ap}(S_2; \alpha)$. This allows us to establish a canonical factorization for elements in T: Every element of T has a unique factorization of the form $\alpha c_1 + \beta c_2$ where $c_2 \in \text{Ap}(S_2; \alpha)$.

This then allows us to answer our question of whether an arbitrary integer n lies in T, since the canonical factorization above is still valid for any integer n: Write $n = \alpha c_1 + \beta c_2$, where c_2 is the unique element of $\text{Ap}(S_2; \alpha)$ that is equivalent mod α to $n\beta^{-1}$. If $c_1 \in S_1$, then $n \in T$ by definition, whereas if $c_1 \notin S_1$, then n can't possibly be an element of T, since it would have at least one factorization with the β coefficient c_2.

We now prove this rigorously.

Theorem 3.1.1. *Suppose $T = \alpha S_1 + \beta S_2$ is a gluing, and consider $n \in \mathbb{Z}$. Then there exists a unique pair of coefficients c_1, c_2 such that*

$$n = \alpha c_1 + \beta c_2$$

where $c_2 \in \text{Ap}(S_2; \alpha)$. Moreover, $n \in T$ if and only if $c_1 \in S_1$.

Proof. Define c_2 to be the unique integer solution in $\text{Ap}(S_2; \alpha)$ to the equation $c_2 \equiv n \bmod \alpha$. Then there is a unique integer solution c_1 to the equation

$$n = \alpha c_1 + \beta c_2.$$

This shows existence. To show uniqueness, notice that if

$$n = \alpha c_1 + \beta c_2 = \alpha c_1' + \alpha c_2'$$

where $c_2, c_2' \in \mathrm{Ap}(S_2;\alpha)$, then taking the difference gives

$$\alpha(c_1' - c_1) + \beta(c_2' - c_2) = 0$$

Taking this mod α gives $c_2' - c_2 \equiv 0 \bmod \alpha$, forcing $c_2' = c_2$, from which $c_1' = c_1$ follows.

Suppose now that n has been written in canonical factorization $n = \alpha c_1 + \beta c_2$. If $c_1 \in S_1$, then obviously $n \in T$. If $c_1 \notin S_1$, then suppose that $n \in S$ regardless, with factorization $n = \alpha c_1' + \beta c_2'$, where $c_1' \in S_1$ and $c_2' \in S_2$. Then we have

$$n - n = \alpha(c_1' - c_1) + \beta(c_2' - c_2) = 0$$

Taking this mod α, we see that $c_2' = c_2 + m\alpha$ for some $m > 0$. But then $c_1' - c_1 = -m\beta$, or equivalently $c_1 = c_1' + n\beta \in S_1$, a contradiction. $\square$

Definition 3.1.2. Suppose $T = \alpha S_1 + \beta S_2$ is a gluing, and $n \in \mathbb{Z}$. We call the unique factorization $n = \alpha c_1 + \beta c_2$ with $c_2 \in \mathrm{Ap}(S_2;\alpha)$ given above the *canonical factorization* of n.

Example 3.1.3. Suppose $T = \langle 4, 5\rangle = 4\langle 1\rangle + 5\langle 1\rangle$. Then

$$\mathrm{Ap}(S_2;\alpha) = \mathrm{Ap}(\langle 1\rangle; 4) = \{0, 1, 2, 3\}.$$

1. The element 13 has only a single factorization $4(2) + 5(1)$ in T, which is its canonical factorization.

2. The element 25 has two factorizations, $4(5) + 5(1)$, and $4(0) + 5(5)$. Notice that the two second coefficients, 1 and 5, are both 1 mod 4, but only 1 lies in $\mathrm{Ap}(S_2; 4)$, so the first factorization is the canonical one.

3. Suppose we wanted to know whether the integer 7 was an element of T. We notice that $7 \cdot 5^{-1} \equiv 3 \cdot 1^{-1} \equiv 3 \bmod 4$, so we pick out the element of $\mathrm{Ap}(\langle 1\rangle, 4)$ equivalent to 3 mod 4, namely 3, and write
$$7 = 4c_1 + 5(3).$$
Solving, we find $c_1 = -2$. Since $-2 \notin \langle 1\rangle$, we conclude that $7 \notin \langle 4, 5\rangle$. We can see this intuitively by writing
$$7 = 4(-2) + 5(3)$$
and trying to increase the coefficient -2. The only way to do so is to increase the -2 by 5 and decrease the other coefficient 3 by 4, giving
$$7 = 4(3) + 5(-1)$$
which now suffers from the problem that $-1 \notin S_2$.

Canonical factorization is convenient because it gives us an inclusion criterion for a glued semigroup $T = \alpha S_1 + \beta S_2$. Since computing the elements of $\mathrm{Ap}(T;\mu)$ essentially reduces to determining whether $c - \mu \in T$ for each element c of T, canonical factorization allows us to define an inclusion criterion for $\mathrm{Ap}(T;\mu)$.

The strategy is to break an arbitrary element $c \in T$ into canonical factorization $\alpha c_1 + \beta c_2$, break the modulus $\mu \in T$ into canonical factorization $\alpha\mu_1 + \beta\mu_2$, and then adjust the coefficients of the difference $c - \mu = \alpha(c_1 - \mu_1) + \beta(c_2 - \mu_2)$ by adding and subtracting copies of $\alpha\beta$ to put it in canonical factorization.

We include the complete result Theorem 3.1.6 below, but unfortunately it is not nearly as nice as might be hoped. Taking the difference of two elements in canonical factorization does not always give the canonical factorization of an element, and so far there seems no intuitive way to tell how many copies of $\alpha\beta$ have to be traded between the terms in order to get back a canonical factorization other than directly taking differences.

Thankfully, things become much simpler in two special cases, which we will explore in more depth later in subsection 3.2 and subsection 3.3, and turn out to be incredibly useful on their own. These are $\mu \in \alpha\mathcal{A}(S_1)$, where no trades are needed, and when $S_2 = \langle 1 \rangle$, where the number of trades n is at most 1.

Lemma 3.1.4. *Suppose $a_i, a_j \in \mathrm{Ap}(S;\mu)$. Then either $a_i - a_j \notin S$ or $a_i - a_j \in \mathrm{Ap}(S;\mu)$.*

Proof. Suppose neither is true. Then $a_i - a_j - \mu \in S$. But this means that

$$a_i - \mu = (a_i - a_j - \mu) + a_j \in S,$$

contradicting $a_i \in \mathrm{Ap}(S;\mu)$. □

Corollary 3.1.5. *For two elements $a_i, a_j \in \mathrm{Ap}(S;\mu)$, there exists a unique integer $n \geq 0$ such that*

$$a_i - a_j + n\mu = a_{i-j} \in \mathrm{Ap}(S;\mu).$$

Theorem 3.1.6. *With $T = \alpha S_1 + \beta S_2$, let*

$$\mu = \alpha\mu_1 + \beta\mu_2 \in T$$

have canonical factorization (i.e. $\mu_1 \in S_1$ and $\mu_2 \in \mathrm{Ap}(S_2;\alpha)$). Let

$$c = \alpha c_1 + \beta c_2 \in T$$

have canonical factorization. Then $c \in \mathrm{Ap}(T;\mu)$ if and only if $c_1 \in \mathrm{Ap}(S_1;\mu_1 + n\beta)$ with n the unique nonnegative integer such that $c_2 - \mu_2 + n\alpha \in \mathrm{Ap}(S_2;\alpha)$.

Proof. We begin by noting that n is unique by Corollary 3.1.5. We then move on to showing that if these conditions are met, then $c \in \mathrm{Ap}(T;\mu)$. We have

$$c - \mu = \alpha(c_1 - \mu_1) + \beta(c_2 - \mu_2).$$

So we have

$$c - \mu = \alpha(c_1 - \mu_1 - n\beta) + \beta(c_2 - \mu_2 + n\alpha).$$

Since we assume that $c_1 \in \mathrm{Ap}(S_1; \mu_1 + n\beta)$ we know that $c_1 - \mu_1 - n\beta \notin S_1$. Furthermore, since we assume $c_2 - \mu_2 + n\alpha \in \mathrm{Ap}(S_2; \alpha)$, by the uniqueness of the canonical factorization we know that $c - \mu \notin T$. So $c \in \mathrm{Ap}(T; \mu)$.

On the other hand, suppose $c - \mu \notin T$. So we have

$$c - \mu = \alpha(c_1 - \mu_1) + \beta(c_2 - \mu_2) \notin T.$$

Define n to be the unique nonnegative integer such that $c_2 - \mu_2 + n\alpha \in \mathrm{Ap}(S_2; \alpha)$ (which exists by Corollary 3.1.5). Then the canonical factorization of $c - \mu$ is

$$c - \mu = \alpha(c_1 - \mu_1 - n\beta) + \beta(c_2 - \mu_2 + n\alpha)$$

and so $c - \mu \notin T$ if and only if $c_1 - \mu_1 - n\beta \notin S_1$ that is,

$$c_1 \in \mathrm{Ap}(S_1; \mu_1 + n\beta).$$

So $c \in \mathrm{Ap}(T; \mu)$ if and only if $c_1 \in \mathrm{Ap}(S_1; \mu_1 + n\beta)$ with n the nonnegative integer such that $c_2 - \mu_2 + n\alpha \in \mathrm{Ap}(S_2; \alpha)$. □

Question 3.1.7. This theorem requires computing a family of Apéry sets $\mathrm{Ap}(S_1; \mu_1 + n\beta)$ for n ranging from 0 to some maximal value. What is the maximal value of n required? Is there an efficient way to generate this family of Apéry sets from only one of them?

Question 3.1.8. Generalize the Apéry poset descriptions of the two special cases found later to give a description of the poset associated to the above Apéry set.

While the above result is a useful membership criterion for the Apéry sets of glued semigroups with respect to arbitrary moduli, it is also useful to understand the poset and covering relations corresponding to glued semigroups. We will now present two special cases of glued semigroups with their poset and covering relations:

3.2 Apéry posets of general gluings with respect to generators

Firstly, we will describe the poset and covering relations for glued semigroups with respect to generators. Apéry sets are often considered with respect to generators so this case is especially useful.

Corollary 3.2.1 (of Theorem 3.1.6)**.** *For a gluing $T = \alpha S_1 + \beta S_2$, with $S_1 = \langle a_1, \ldots, a_k \rangle$ and $S_2 = \langle b_1, \ldots, b_\ell \rangle$, then the canonically factored*

$$c = \alpha c_1 + \beta c_2 \in \mathrm{Ap}(T; \alpha a_i)$$

if and only if $c_2 \in \mathrm{Ap}(S_2; \alpha)$, and $c_1 \in \mathrm{Ap}(S_1; a_i)$. A similar result follows for generators resulting from S_2.

Proof. Observe that $\alpha a_i = \alpha(a_i) + \beta(0)$ in canonical factorization. So by Theorem 3.1.6 we know that for all $c = c_1\alpha + c_2\beta \in \mathrm{Ap}(T; \alpha a_i)$, we need $c_2 - 0 + n\alpha \in \mathrm{Ap}(S_2; \alpha)$. This is only true when $n = 0$. Thus we have that $c_1 \in \mathrm{Ap}(S_1; a_i)$. □

Example 3.2.2. Consider the gluing $T = \langle 68, 85, 100, 140\rangle = 17\langle 4, 5\rangle + 20\langle 5, 7\rangle$. We can determine $\mathrm{Ap}(T; 17 \cdot 4)$ by writing

$$\mathrm{Ap}(T; \alpha a_1) = \alpha\,\mathrm{Ap}(S_1; a_1) + \beta\,\mathrm{Ap}(S_2; \alpha)$$

$$\mathrm{Ap}(T; 17 \cdot 4) = 17\,\mathrm{Ap}(\langle 4, 5\rangle; 4) + 20\,\mathrm{Ap}(\langle 5, 7\rangle; 17).$$

Iterating this process, we can quickly compute

$$\mathrm{Ap}(\langle 4, 5\rangle; 4) = \{0, 5, 10, 15\}$$

and

$$\mathrm{Ap}(\langle 5, 7\rangle; 17) = \{0, 35, 19, 20, 21, 5, 40, 7, 25, 26, 10, 28, 12, 30, 14, 15, 33\}$$

so by picking elements e_1, e_2 of these sets, computing the result of the expression $17e_1 + 20e_2$, and then sorting the result by mod class, we obtain the Apéry set

$$\mathrm{Ap}(T; 68) = \begin{array}{cccc}
\{\boxed{0}, & 885, & 410, & 955, \\
140, & 685, & 550, & 755, \\
280, & 485, & 690, & 555, \\
420, & 285, & 830, & 355, \\
560, & \boxed{85}, & 970, & 495, \\
700, & 225, & 770, & 635, \\
500, & 365, & 570, & 775, \\
300, & 505, & 370, & 915, \\
100, & 645, & \boxed{170}, & 1055, \\
240, & 785, & 310, & 855, \\
380, & 585, & 450, & 655, \\
520, & 385, & 590, & 455, \\
660, & 185, & 730, & \boxed{255}, \\
800, & 325, & 870, & 395, \\
600, & 465, & 670, & 535, \\
400, & 605, & 470, & 675, \\
200, & 745, & 270, & 815\}.
\end{array}$$

where the boxed elements are those that come from choosing $e_2 = 0$, and those in the first column come from choosing $e_1 = 0$. We could also write this set as a 17×4 array, in which case the boxed elements would be the first elements of the four rows and the current first elements of each column would be spaced out every four numbers throughout the table.

We now characterize the precedence and covering relations of the Apéry poset. The lemma below is a useful counterpart to Lemma 3.1.4, which will also be used.

Lemma 3.2.3. *Suppose $a \in \mathrm{Ap}(S;\mu)$ and $a - c \in \mathrm{Ap}(S;\mu)$. Then $c \in \mathrm{Ap}(S;\mu)$.*

Proof. Let c' be the element of $\mathrm{Ap}(S;\mu)$ such that $c' \equiv c \bmod \mu$. Then observe

$$a - c \equiv a - c' \bmod \mu$$

and that $a - c' \in \mathrm{Ap}(S;\mu)$ by Lemma 3.1.4. This implies $a - c = a - c'$, or $c = c'$, and therefore $c \in \mathrm{Ap}(S;\mu)$ as desired. □

Theorem 3.2.4. *For a gluing $T = \alpha S_1 + \beta S_2$, fix $c, d \in \mathrm{Ap}(T;\alpha a_1)$ with canonical factorizations $c = \alpha c_1 + \beta c_2$ and $d = \alpha d_1 + \beta d_2$ (i.e. $c_1, d_1 \in \mathrm{Ap}(S_1;a_1)$ and $c_2, d_2 \in \mathrm{Ap}(S_2;\alpha)$).*

1. *If $\beta \notin \mathrm{Ap}(S_1, a_1)$ then $c \preceq_T d$ if and only if $c_1 \preceq_{S_1} d_1$ and $c_2 \preceq_{S_2} d_2$.*
2. *If $\beta \in \mathrm{Ap}(S_1, a_1)$ then $c \preceq_T d$ if and only if $c_1 \preceq_{S_1} d_1$ and $c_2 \preceq_{S_2} d_2$, or $c_1 + n\beta \preceq_{S_1} d_1$ and $c_2 \preceq_{S_2} d_2 + n\alpha$ for some $n \geq 1$.*

Moreover, the additional allowed condition in the second case always result in additional relations.

Proof. If $c_1 \preceq_{S_1} d_1$ and $c_2 \preceq_{S_2} d_2$, then $d - c = \alpha(d_1 - c_1) + \beta(d_2 - c_2)$ with $d_1 - c_1 \in S_1$ and $d_2 - c_2 \in S_2$, so $c \preceq_T d$.

If $c_1 + n\beta \preceq_{S_1} d_2$ and $c_2 \preceq_{S_2} d_2 + n\alpha$ for $n \geq 1$, then

$$d - c = \alpha(d_1 - c_1) + \beta(d_2 - c_2) = \alpha(d_1 - c_1 + n\beta) + \beta(d_2 - c_2 - n\alpha)$$

where the first coefficient is in S_1 and the second is in S_2.

Now assume that $d - c \in T$. We then know that

$$d - c = \alpha(d_1 - c_1) + \beta(d_2 - c_2) \in T.$$

If $c_2 \preceq_{S_2} d_2$ then $d_2 - c_2 \in S_2$, and specifically in $\mathrm{Ap}(S_2;\alpha)$ by Lemma 3.1.4. So then $d_1 - c_1 \in S_1$, i.e. $c_1 \preceq_{S_1} d_1$.

We know $c_2 \not\preceq_{S_2} d_2$ if and only if $d_2 - c_2 \notin \mathrm{Ap}(S_2;\alpha)$. By Corollary 3.1.5 we know that there exists a unique $n \in \mathbb{Z}_{\geq 0}$ such that $d_2 - c_2 + n\alpha \in \mathrm{Ap}(S_2;\alpha)$. If $n = 0$ we get the case in the previous paragraph, so assume $n \geq 1$. Observe

$$d - c = \alpha(d_1 - c_1) + \beta(d_2 - c_2) = \alpha(d_1 - c_1 - n\beta) + \beta(d_2 - c_2 + n\alpha).$$

Since $d_2 - c_2 + n\alpha \in \mathrm{Ap}(S_2;\alpha)$ we know that $c_2 \preceq_{S_2} d_2 + n\alpha$. So $\alpha(d_1 - c_1 - n\beta) + \beta(d_2 - c_2 + n\alpha)$ is in canonical factorization, so $d - c \in T$ if and only if $d_1 - c_1 - n\beta \in S_1$, i.e. $c_1 + n\beta \preceq d_1$. On the other hand, the fact that $c, d \in \mathrm{Ap}(T;\alpha a_1)$ also gives that $d - c \in T$ if and only if $d - c \in \mathrm{Ap}(T;\alpha a_1)$, if and only if $d_1 - c_1 - n\beta \in \mathrm{Ap}(S_1;a_1)$. By Lemma 3.2.3, $n\beta \in \mathrm{Ap}(S_1;a_1)$.

Since we know that $\beta \in S_1$, therefore if we assume $\beta - a_1 \in S_1$ (that is $\beta \notin \mathrm{Ap}(S_1; a_1)$), then, since we are assuming $n \geq 1$, we may write

$$n\beta - a_1 = (\beta - a_1) + (n-1)\beta \in S_1.$$

This would imply $\beta n \notin \mathrm{Ap}(S_1; a_1)$, so by contradiction we must have $\beta \in \mathrm{Ap}(S_1; a_1)$ if we have an $n \geq 1$ case.

Finally, observe that if $\beta \in \mathrm{Ap}(S_1; a_1)$, then the canonical factorization of $\alpha\beta$ is $\alpha(\beta) + \beta(0)$, while the canonical factorization of $\beta(\alpha - g)$, for a generator g of S_2 such that $g \preceq_{S_2} \alpha$, is $\alpha(0) + \beta(\alpha - g)$ (since $\alpha - g \in \mathrm{Ap}(S_2; \alpha)$), and that we must have $\alpha\beta - \beta(\alpha - g) = \beta(g) \in T$, so

$$\alpha(\beta) + \beta(0) \preceq_T \alpha(0) + \beta(\alpha - g)$$

but this precedence relation cannot be given only by the $n = 0$ case. □

Corollary 3.2.5. *For a glued semigroup $T = \alpha S_1 + \beta S_2$ where we have $S_1 = \langle a_1, \ldots, a_k \rangle$, and $S_2 = \langle b_1, \ldots, b_\ell \rangle$ fix $c, d \in \mathrm{Ap}(T; \alpha a_i)$ with $c = \alpha c_1 + \beta c_2$ and $d = \alpha d_1 + \beta d_2$ in canonical factorization. Then d covers c if and only if:*

1. *$c_1 = d_1$ and d_2 covers c_2 in S_2, or*
2. *d_1 covers c_1 in S_1 and $c_2 = d_2$, or*
3. *$c_1 + n\beta = d_1$ and $d_2 + n\alpha$ covers c_2 in S_2, or*
4. *d_1 covers $c_1 + n\beta$ in S_1 and $c_2 = d_2 + n\alpha$.*

Moreover, the third and fourth cases occur (for $n \geq 1$) if and only if $\beta \in \mathrm{Ap}(S_1; a_1)$.

Proof. First suppose one of the conditions holds. For condition (1), if $c_1 = d_1$ and d_2 covers c_2 in S_2 then

$$d - c = \alpha(d_1 - c_1) + \beta(d_2 - c_2) = \alpha(0) + \beta(b_i)$$

for some $i \in \{1, \ldots, \ell\}$. Showing that condition (2) corresponds to a covering relation follows similarly.

If condition (3) holds then we have

$$d - c = \alpha(d_1 - c_1) + \beta(d_2 - c_2) = \alpha(d_1 - c_1 - n\beta) + \beta(d_1 + n\alpha - c_1) = \alpha(0) + \beta(b_i)$$

for some $i \in \{1, \ldots, \ell\}$. Showing that condition (4) corresponds to a covering relation follows similarly.

Now suppose that d covers c. Then we have two cases:

Case 1. Suppose $d - c = \alpha a_i$ for some $i \in \{1, \ldots, k\}$. Choose the corresponding i. So we have

$$\alpha a_i = d - c = \alpha(d_1 - c_1) + \beta(d_2 - c_2).$$

Taking this equation mod α and dividing by β we know that $d_2 \equiv c_2 \bmod \alpha$. Choose $n \in \mathbb{Z}$ such that $d_2 + n\alpha = c_2$. If $n = 0$ we have $d_2 = c_2$, then clearly d_1 covers c_1 in S_1 and condition (2) holds. If $n > 0$ then we know that

$$d - c = \alpha(d_1 - c_1 - n\beta) + \beta(d_2 - c_2 + n\alpha).$$

So d_1 covers $c_1 + n\beta$ which is condition (4). Since covering implies precedence by Theorem 3.2.4 we know that $n \geq 0$.

Case 2. Suppose $d - c = \beta b_i$ for some $i \in \{1, \ldots, \ell\}$. Choose the corresponding i. So we have

$$\beta b_i = d - c = \alpha(d_1 - c_1) + \beta(d_2 - c_2).$$

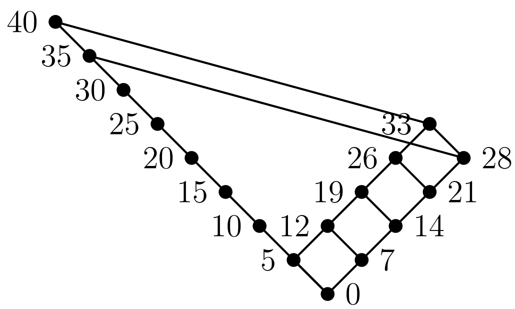
Taking this equation mod β and dividing by α we know that $d_1 \equiv c_1 \bmod \beta$. Choose $n \in \mathbb{Z}$ such that $d_1 + n\beta = c_1$. If $n = 0$ we have $d_1 = c_1$, then clearly d_2 covers c_2 in S_2 and condition (1) holds. If $n > 0$ then we know that

$$d - c = \alpha(d_1 - c_1 - n\beta) + \beta(d_2 - c_2 + n\alpha).$$

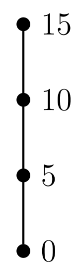
So $d_2 + n\alpha$ covers c_2 which is condition (3). Since covering implies precedence by Theorem 3.2.4 we know that $n \geq 0$. □

Example 3.2.6. We will now consider two examples, in order to demonstrate the precedence and covering relations when we have the extra relations and when we do not have the extra relations.

Before we look more closely at comparing two glued semigroups, we will begin by showing the posets for $\mathrm{Ap}(\langle 5, 7\rangle; 17)$ and $\mathrm{Ap}(\langle 4, 5\rangle; 4)$ in Figures 1a and 1b, respectively.



(a) $\mathrm{Ap}(\langle 5, 7\rangle; 17)$



(b) $\mathrm{Ap}(\langle 4, 5\rangle; 4)$

Figure 1: The posets for $\mathrm{Ap}(\langle 5, 7\rangle; 17)$ and $\mathrm{Ap}(\langle 4, 5\rangle; 4)$

This Apéry poset is useful for seeing the structure within the Apéry posets in Figures 2 and 3.

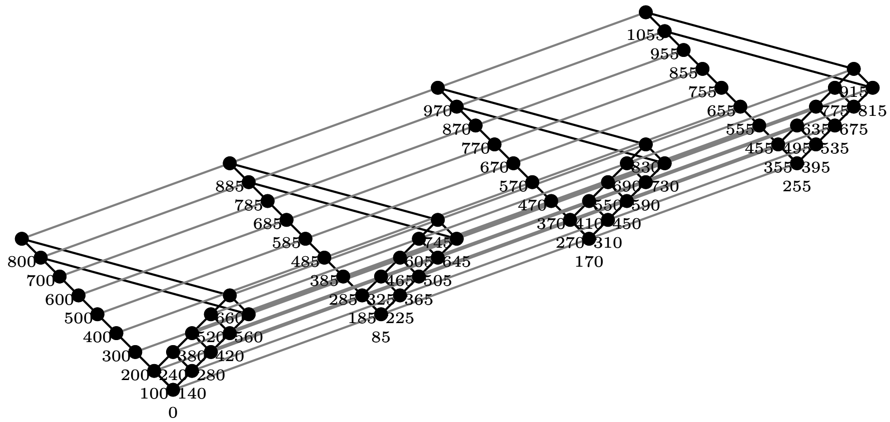


Figure 2: $\mathrm{Ap}(T;68)$

Now, consider the numerical semigroup $T = \langle 68, 85, 100, 140\rangle$. Notice that T can also be written as the gluing $T = 17\langle 4,5\rangle + 20\langle 5,7\rangle$. We have then that $\mathrm{Ap}(T;68)$ is represented by the poset in Figure 2.

Notice that this poset resembles the poset for $\mathrm{Ap}(\langle 5,7\rangle;17)$ copied out 4 times. Thus, we can view this Apéry set as a sort of Cartesian product between $\mathrm{Ap}(\langle 4,5\rangle;4)$ and $\mathrm{Ap}(\langle 5,7\rangle;17)$. The covering relations are highlighted by the different colored lines in this image, for two elements $c = \alpha c_1 + \beta c_2$ and $d = \alpha d_1 + \beta d_2$ located within the poset the black lines correspond to when the associated covering relation is that $c_1 = d_1$ and d_2 covers c_2 in $\langle 5,7\rangle$. Furthermore, the gray lines correspond to the covering relation arising when d_1 covers c_1 in $\langle 4,5\rangle$ and $c_2 = d_2$. Note that in this example $20 \notin \mathrm{Ap}(\langle 4,5\rangle;4)$ so we expect not to get any extra relations.

Now consider the numerical semigroup $T' = \langle 68, 75, 85, 105\rangle$. Notice that T' can also be written as the gluing $T' = 17\langle 4,5\rangle + 15\langle 5,7\rangle$. Note that $15 \in \mathrm{Ap}(\langle 4,5\rangle;4)$, so we expect to find extra relations. We have then that $\mathrm{Ap}(T';68)$ is represented by the poset:

For two elements $c = \alpha c_1 + \beta c_2$ and $d = \alpha d_1 + \beta d_2$ located within the poset the black and gray lines correspond to cases (1) and (2) of Corollary 3.2.5 respectively. The red lines then correspond to the extra relations that occur because $15 \in \mathrm{Ap}(\langle 4,5\rangle;4)$, these are cases (3) and (4) of Corollary 3.2.5.

In general, the red relations connect the bottom copy of the poset $(\mathrm{Ap}(S_2;\alpha), \preceq_{S_2})$ to the copy at the position in $(\mathrm{Ap}(S_1;a_1), \preceq_{S_1})$ corresponding to the position of β in the poset $(\mathrm{Ap}(S_1;a_1), \preceq_{S_1})$. Moreover, *any* two poset copies whose overall positions differ by β in the poset $(\mathrm{Ap}(S_1;a_1), \preceq_{S_1})$ will be connected by these extra relations, and similarly for any poset copies whose overall positions differ by $n\beta$ for $n \geq 2$.

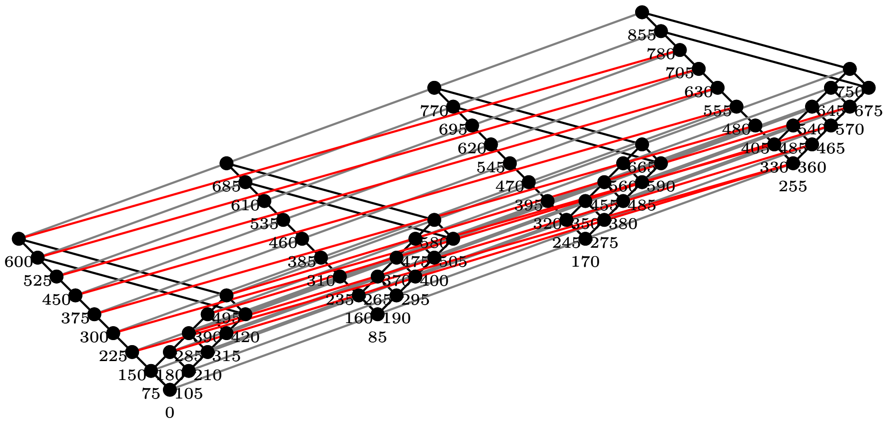


Figure 3: $\mathrm{Ap}(T'; 68)$

3.3 Apéry posets of monoscopic gluings with respect to arbitrary moduli

We take the definition of monoscopic gluing from [1]:

Definition 3.3.1. Fix a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle$, an integer $\alpha \in \mathbb{Z}_{\geq 2}$, and an element $\beta \in S \setminus \{n_1, \ldots, n_k\}$ such that $\gcd(\alpha, \beta) = 1$.The numerical semigroup

$$T = \alpha S + \beta \langle 1 \rangle = \langle \alpha n_1, \ldots, \alpha n_k, \beta \rangle$$

is called a *monoscopic gluing* of S.

It is previously known [5] that the above T is a numerical semigroup and that the generating set is minimal.

Example 3.3.2. The McNugget semigroup given by $\langle 6, 9, 20 \rangle$ is one example of a monoscopic gluing. Notice that $\gcd(6, 9) = 3$, so 3 is a candidate for α. Next notice that $\gcd(3, 20) = 1$, so we can write

$$\langle 6, 9, 20 \rangle = 3\langle 2, 3 \rangle + 20\langle 1 \rangle$$

We now will give a membership criterion for the Apéry sets of monoscopic gluings with respect to arbitrary moduli. This is a corollary of our membership criterion for general gluings with respect to arbitrary moduli (Theorem 3.1.6) but we share here before sharing the poset precedence and covering relations of this case.

Corollary 3.3.3 (of Theorem 3.1.6)**.** *Let $T = \alpha S + \beta \langle 1 \rangle$ be a monoscopic gluing. Let $\mu \in T$ have canonical decomposition $\mu = \alpha \mu_1 + \beta \mu_2$. Then $c = \alpha c_1 + \beta c_2$ (in canonical factorization) is in $\mathrm{Ap}(T; \mu)$ if and only if either*

1. *$c_1 \in \mathrm{Ap}(S; \mu_1 + \beta)$ and $0 \leq c_2 \leq \mu_2 - 1$, or*

2. $c_1 \in \mathrm{Ap}(S; \mu_1)$ *and* $\mu_2 \leq c_2 \leq \alpha - 1$

Proof. First note that in either case we have $c_2 \in \mathrm{Ap}(\langle 1 \rangle; \alpha)$. Since $\mathrm{Ap}(\langle 1 \rangle; \alpha) = [0, \alpha - 1]$, we know that $0 \leq c_2, \mu_2 \leq \alpha - 1$. So $1 - \alpha \leq c_2 - \mu_2 \leq \alpha - 1$. If $c_2 < \mu_2$ we have $1 - \alpha \leq c_2 - \mu_2 \leq -1$. So $c_2 - \mu_2 + \alpha \in \mathrm{Ap}(\langle 1 \rangle; \alpha)$. So by Theorem 3.1.6 we know that $c \in \mathrm{Ap}(T; \mu)$ if and only if $c_1 \in \mathrm{Ap}(S; \mu_1 + \beta)$. If $c_2 \geq \mu_2$ we have $0 \leq c_2 - \mu_2 \leq \alpha - 1$, so $c_2 - \mu_2 \in \mathrm{Ap}(\langle 1 \rangle; \alpha)$. So by Theorem 3.1.6 we know that $c \in \mathrm{Ap}(T; \mu)$ if and only if $c_1 \in \mathrm{Ap}(S; \mu_1)$. □

A useful exercise is to verify that letting μ be a generator (either α times a generator or β) in Corollary 3.3.3 gives the same result as letting $S_2 = \langle 1 \rangle$ in Corollary 3.2.1.

We now list the precedence and covering relations for the poset.

Theorem 3.3.4. *Let* $T = \alpha S + \beta \langle 1 \rangle$ *be a monoscopic gluing. Let* $\mu \in T$. *Let* $c, d \in \mathrm{Ap}(T; \mu)$ *have canonical decompositions:*

$$c = \alpha c_1 + \beta c_2 \quad \text{and} \quad d = \alpha d_1 + \beta d_2$$

Then $c \preceq_T d$ *if and only if either:*

1. $c_2 \leq d_2$ *and* $c_1 \preceq_S d_1$, *or*

2. $c_1 \preceq_S d_1 - \beta$.

Proof. To begin with, we will show that if $c \preceq_T d$ then either $c_2 \leq d_2$ and $c_1 \preceq_S d_1$, or $c_1 \preceq_S d_1 - \beta$. Since $c \preceq_T d$, we know that $d - c \in T$. We can expand $d - c$ to find

$$d - c = \alpha(d_1 - c_1) + \beta(d_2 - c_2).$$

We now have two cases: $d_2 - c_2 \geq 0$ and $d_2 - c_2 < 0$.

Case 1. Suppose $d_2 - c_2 \geq 0$. Then we know that $d - c$ is in the canonical decomposition. Since the canonical decomposition of an element of T is unique, we know that $d_1 - c_1 \in S$ and hence $c_1 \preceq_S d_1$.

Case 2. Suppose $d_2 - c_2 < 0$. Then, since $0 \leq d_2$, and $c_2 \leq \alpha - 1$ we know that

$$1 - \alpha \leq d_2 - c_2 \leq -1.$$

So we have

$$1 \leq d_2 - c_2 + \alpha \leq \alpha - 1.$$

Observe

$$\begin{aligned} d - c &= \alpha(d_1 - c_1) + \beta(d_2 - c_2) \\ &= \alpha(d_1 - \beta - c_1) + \beta(d_2 - c_2 + \alpha) \end{aligned}$$

Since $1 \leq d_2 - c_2 + \alpha \leq \alpha - 1$, we know that $d - c$ is in canonical decomposition. By the uniqueness of the canonical decomposition, since $d - c \in T$ it follows that $d_1 - \beta - c_1 \in S$. Therefore $c_1 \preceq_S d_1 - \beta$.

Now we will show that if one of the two conditions is met, then $c \preceq_T d$.

Case 1: $c_2 \leq d_2$ and $c_1 \preceq_S d_1$. In this case,

$$d - c = \alpha(d_1 - c_1) + \beta(d_2 - c_2).$$

Since $d_1 - c_1 \in S$ and $d_2 - c_2 \geq 0$, this is manifestly in T.

Case 2: $c_1 \preceq_S d_1 - \beta$. We write

$$d - c = \alpha(d_1 - c_1) + \beta(d_2 - c_2)$$

If $d_2 - c_2 \geq 0$ this is the canonical decomposition of $d - c$ and we have $d_1 - c_1 = (d_1 - \beta - c_1) + \beta$, which is the sum of the elements $d_1 - \beta - c_1$ and β in S. Otherwise, $d_2 - c_2 \leq 0$. Since $0 \leq c_2, d_2 \leq \alpha - 1$, $1 - \alpha \leq d_2 - c_2 \leq -1$, implying $1 \leq d_2 - c_2 + \alpha \leq \alpha - 1$ so that

$$d - c = \alpha(d_1 - \beta - c_1) + \beta(d_2 - c_2 + \alpha)$$

Since $d_2 - c_2 + \alpha$ is nonnegative and $d_1 - \beta - c_1 \in S$ from $c_1 \preceq_S d_1 - \beta$, this is manifestly in T. □

Theorem 3.3.5. *With notation as in the previous theorem, specifically with $S = \langle n_1, n_2, \ldots, n_k \rangle$ then d covers c in T if and only if one of the following criteria is met:*

1. *$c_1 = d_1$ and $d_2 - c_2 = 1$, or*
2. *d_1 covers c_1 in S and $d_2 = c_2$, or*
3. *$d_2 = 0$, $c_2 = \alpha - 1$, and $d_1 = c_1 + \beta$.*

Proof. For the forwards direction, we know that d covers c if and only if

$$d - c \in \mathcal{A}(T) = \{\beta, \alpha n_1, \alpha n_2, \ldots, \alpha n_k\}.$$

Case 1. Suppose $d - c = \beta$. Then we know that $\beta = \alpha(d_1 - c_1) + \beta(d_2 - c_2)$. We can take both sides of the equation mod α, to find $1 \equiv d_2 - c_2 \bmod \alpha$. Since c and d are in canonical decomposition, we know that $0 \leq c_2, d_2 \leq \alpha - 1$. Thus $1 - \alpha \leq d_2 - c_2 \leq \alpha - 1$. Within that range, we know that 1 and $1 - \alpha$ are the only two values which are equivalent to 1 mod α. Suppose $d_2 - c_2 = 1$. Then we have $\beta = \alpha(d_1 - c_1) + \beta$. Thus we can solve that $d_1 = c_1$. This satisfies criteria 1. Suppose $d_2 - c_2 = 1 - \alpha$. Thus $d_2 = c_2 + 1 - \alpha$. Since $0 \leq d_2, c_2 \leq \alpha - 1$, we know that

$$1 - \alpha \leq d_2 = c_2 + 1 - \alpha \leq 0.$$

So $d_2 = 0$. Thus $c_2 = \alpha - 1$. Furthermore, since $\beta = \alpha(d_1 - c_1) + \beta(d_2 - c_2)$ we can solve that $d_1 = c_1 + \beta$. This is criterion 3.

Case 2. Suppose $d - c = \alpha n_i$ for some $n_i \in \mathcal{A}(S) = \{n_1, n_2, \ldots, n_k\}$. Then we have $\alpha n_i = \alpha(d_1 - c_1) + \beta(d_2 - c_2)$. We can take both sides of the equation mod α, to find $d_2 \equiv c_2 \bmod \alpha$, and given that $0 \leq d_2, c_2 \leq \alpha - 1$ it follows that $d_2 = c_2$. Thus we have

$$\alpha n_i = \alpha(d_1 - c_1) + \beta(d_2 - c_2) = \alpha(d_1 - c_1).$$

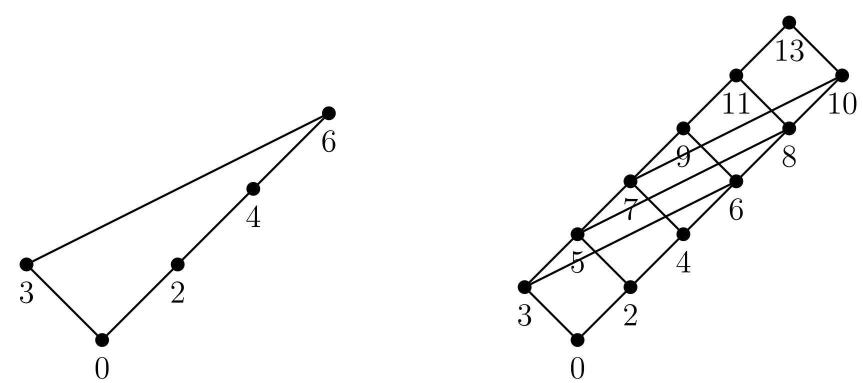
So $n_i = d_1 - c_1$. We know that $d_1 - c_1 = n_i \in \mathcal{A}(S)$ if and only if d_1 covers c_1 in S. This is criterion 2.

For the reverse direction, observe that if condition 1 is true, then

$$d - c = \alpha(0) + \beta(1) = \beta \in \mathcal{A}(T).$$

If condition 2 is true, then $d - c = \alpha(n_i) + \beta(0) = \alpha n_i$, where n_i is a generator of S, and thus αn_i is a generator of T. If condition 3 is true, then $d - c = \alpha(\beta) + \beta(-(\alpha - 1)) = \alpha$. □

Example 3.3.6. In order to demonstrate the covering relations and precedence relations for monoscopic gluings with respect to arbitrary elements we will demonstrate using the numerical semigroup $T = \langle 4, 6, 7 \rangle = 2\langle 2, 3 \rangle + 7\langle 1 \rangle$. In this example we will be exploring $\mathrm{Ap}(T; 17)$. Before we show this poset however, we will begin by showing two preliminary posets. Recall from Corollary 3.3.3, that for a gluing $T = \alpha S + \beta\langle 1 \rangle$ that for an element $c \in \mathrm{Ap}(T; \mu)$ we are concerned with when $c_1 \in \mathrm{Ap}(S; \mu_1)$ and $c_1 \in \mathrm{Ap}(S; \mu_1 + \beta)$. So, we will now look at the posets for $\mathrm{Ap}(\langle 2, 3 \rangle; 5)$ and $\mathrm{Ap}(\langle 2, 3 \rangle; 12)$, shown in Figures 4a and 4b, respectively.



(a) $\mathrm{Ap}(\langle 2, 3 \rangle; 5)$ (b) $\mathrm{Ap}(\langle 2, 3 \rangle; 12)$

Figure 4: The Apéry posets for $\mathrm{Ap}(\langle 2, 3 \rangle; 5)$ and $\mathrm{Ap}(\langle 2, 3 \rangle; 12)$

Now we will show that Apéry poset for $\mathrm{Ap}(T; 17)$ in Figure 5.

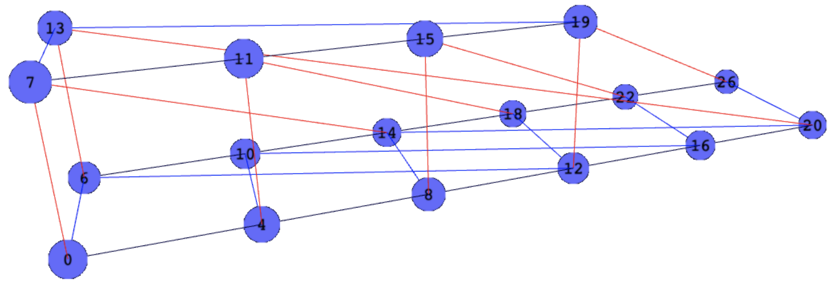


Figure 5: The Apéry poset for $\text{Ap}(T;17)$.

The black lines in the image correspond to a difference of 4, the blue to a difference of 6, and red to a difference of 7. Furthermore, notice the two different layers of this poset, the bottom layer correspond to $\text{Ap}(\langle 2,3\rangle;12)$ and the top to $\text{Ap}(\langle 2,3\rangle;5)$. This matches Corollary 3.3.3 describing two different cases one $c_1 \in \text{Ap}(S;\mu_1)$ and the other $c_1 \in \text{Ap}(S;\mu_1+\beta)$.

Then we can see the covering relations from Theorem 3.3.5, that case (1) corresponds to the vertical relations, then case (2) are the relations within a slice and (3) are the diagonal lines connecting slices.

4 Background: Kunz posets and the group cone

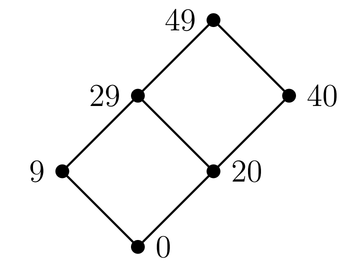
An interesting question when considering numerical semigroups is how two numerical semigroups relate to each other. One way we can consider this is when their Apéry posets look "similar". In the next subsection we will introduce a method of viewing "similarity" called Kunz posets, and then in the following subsection we will introduce the group cone which will help us to geometrically represent relations between numerical semigroups.

4.1 Kunz posets

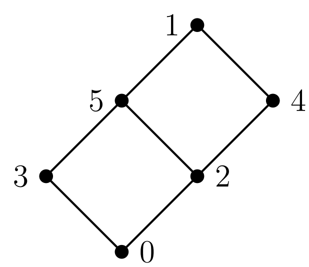
While in the next subsection we will introduce a more general definition of a Kunz poset, we will begin with the numerical semigroup-specific definition which will help to motivate the importance of looking at the group cone later on. The following definition was taken from [2]

Definition 4.1.1. Given a numerical semigroup S and an element μ, the Kunz poset of S with respect to μ denoted $\text{Ku}(S;\mu)$ is the partially ordered set with groundset $\mathbb{Z}_\mu$ where we replace each element of $\text{Ap}(S;\mu)$ with its equivalence class in $\mathbb{Z}_\mu$

Example 4.1.2. Consider the numerical semigroup $S = \langle 6, 9, 20 \rangle$. We can take $\mathrm{Ap}(S; 6)$. The poset corresponding to $\mathrm{Ap}(S; 6)$ is given by:

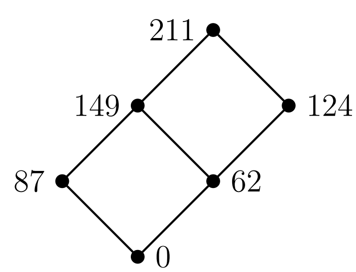


To find the Kunz poset we can relabel every node with its equivalence class modulo 6. For example we know that $0 \equiv 0 \bmod 6$. Additionally, notice that $49 = 8(6) + 1$, so $49 \equiv 1 \bmod 6$. By taking the equivalence class modulo 6 for every element of $\mathrm{Ap}(S; 6)$ we have:

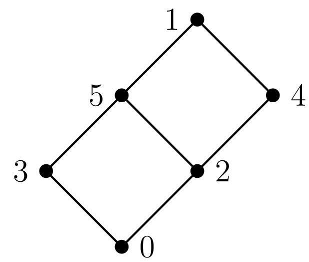


Notice that Kunz posets are not unique to one semigroup, more than one semigroup may give rise to the same Kunz poset.

Consider the numerical semigroup $S' = \langle 6, 62, 187 \rangle$. Firstly, we can find $\mathrm{Ap}(S'; 6)$, which is:



Using this Apéry poset we can find the Kunz poset $\mathrm{Ku}(S'; 6)$



So $\mathrm{Ku}(S;6) = \mathrm{Ku}(S';6)$.

A natural question to ask next is when do two numerical semigroups have the same Kunz poset? One important tool for answering this question is the group cone, which provides a geometric perspective to the question. We will introduce the group cone in the next subsection.

4.2 The group cone

We will now introduce relevant background about the group cone, a geometric object useful for studying numerical semigroups. The group cone is based on ideas introduced in [4]. In this section we are following closely to the introduction to the group cone given in [1]. The following definitions and theorem come from [3]:

Definition 4.2.1. Fix $m \in \mathbb{Z}_{\geq 2}$ and a numerical semigroup S containing m. Write

$$\mathrm{Ap}(S;m) = \{0, a_1, \ldots, a_{m-1}\}$$

where $a_i \equiv i \bmod m$ for each $i \in \{1, \ldots, m-1\}$. We call $(a_1, \ldots, a_{m-1})$ as the *Apéry tuple* or *coordinate* of S.

Definition 4.2.2. Fix a finite Abelian group $(G, \oplus)$ with identity 0_G, and let $m = |G|$. The *group cone* $\mathcal{C}(G) \subset \mathbb{R}^{m-1}$ is the pointed cone with *facet inequalities*

$$x_i + x_j \geq x_{i \oplus j} \qquad \text{for } i, j \in G \setminus \{0_G\} \text{ with } i \oplus j \neq 0_G$$

Theorem 4.2.3. *Fix an integer $m \geq 2$, then the set of all Apéry tuples of numerical semigroups containing m coincides with the set of integer points $(a_1, \ldots, a_{m-1})$ in $\mathcal{C}(\mathbb{Z}_m)$ with $a_i \equiv i \bmod m$ for every i.*

Drawing from polyhedral geometry we will note a further definition:

Definition 4.2.4. A *face* is a subset of the group cone that satisfies some subset of the facet inequalities with strict equality.

We then say that an Apéry coordinate lies "in" a face when said coordinate lies in a given face but does not lie in any proper subface. We will now include a final theorem from [3]:

Theorem 4.2.5 (Theorem 3.4 in [3]). *Fix a finite Abelian group G and a face $F \subset \mathcal{C}(G)$.*

(a) The set $H = \{h \in G : x_h = 0 \text{ forall } x \in F\}$ is a subgroup of G which we call the Kunz subgroup *of G. Furthermore, the relation $P = (G/H, \preceq)$ with unique minimal element $\overline{0}$ and $\overline{a} \preceq \overline{b}$ whenever $x_a + x_{b-a} = x_b$ for distinct $a, b \in G$ is a well-defined partial order (called the* Kunz poset *of F).*

(b) If $G = \mathbb{Z}_m$ with $m \geq 2$ and F contains a numerical semigroup S, then the Kunz subgroup of F is trivial and the Kunz poset of F equals the Kunz poset of S.

5 Glued semigroups and the group cone

A recent paper [1] presents a method for describing where monoscopic gluings lie in the group cone when the multiplicity is an element of the glued semigroup. Here we seek to generalize these results to more general gluings.

In section 3 we gave a characterization of the Apéry posets of glued semigroups with respect to generators. A next step is to consider where in the group cone we find glued semigroups. A first key step to this is understanding the Kunz posets of glued semigroups. In our characterization of the Apéry set elements for a general gluing we relate the Apéry sets of glued semigroups to the Apéry set of the overall gluing.

In this section we define a gluing extension that essentially translates the relations between Apéry poset relations into Kunz poset relations. We also define a map which injects the Apéry coordinate of a semigroup into a group cone of higher dimension, and specifically to a point corresponding to an Apéry set of a gluing of the original semigroup, given a few additional inputs. This map allows us to move from the Kunz poset for one semigroup into a Kunz poset of a gluing involving that semigroup, which we show corresponds to the gluing extension. These results work for numerical semigroups, but the map also works for points that do not correspond to numerical semigroups, the map will "work" for any input which is in the group cone (assuming the other required inputs are correct).

Notation 5.0.1. In order to simplify numerous expressions in this section, we adopt the convention of prepending a "0" entry to each point in $\mathcal{C}(G)$, indexed by the identity element of G. More precisely, we write each $(x_1, x_2, ...) \in \mathcal{C}(G)$ in the form $(x_0, x_1, x_2, ...)$ with $x_0 = 0$, replacing $\mathcal{C}(G)$ with $\{0\} \times \mathcal{C}(G)$, this is in agreement with section 6 of [1], which this section seeks to generalize.

5.1 The face injection Φ_{ρ,S_2}

We will now introduce a family of combinatorial embeddings which are very similar to those in section 6 of [1], with the change that we require k to be in an Apéry set rather than a closed interval (which in fact is an Apéry set for $\langle 1 \rangle$).

Definition 5.1.1. Fix a finite abelian group G, a subgroup $H \subset G$ with G/H cyclic, fix $\rho \in G$ whose image in G/H is a generator. Let $\alpha = |G/H|$. Fix $S_2 = \langle g_1, \ldots, g_k \rangle$ a numerical semigroup with $\alpha \in S_2 \setminus \{g_1, \ldots, g_k\}$. We define

$$\Phi_{\rho,S_2} : \mathcal{C}(H) \to \mathcal{C}(G)$$

$$w \mapsto x$$

where $x_{h+k\rho} = \alpha w_h + k w_{\alpha\rho}$ for each $h \in H$ and $k \in \mathrm{Ap}(S_2; \alpha)$.

Lemma 5.1.2. *Φ_{ρ,S_2} is well-defined and injective.*

Proof. Every element of G can be written uniquely as $h + k\rho$ for $h \in H$ and $k \in \mathrm{Ap}(S_2; \alpha)$. If $w \in \mathcal{C}(H)$ and $x = \Phi_{\rho,S_2}(w)$ then $x_0 = \alpha w_0 + 0 w_{\alpha\rho} = 0$. For any $h_1, h_2 \in H$, and $k_1, k_2 \in \mathrm{Ap}(S_2; \alpha)$ we have:

$$\begin{aligned} x_{h_1+k_1\rho} + x_{(h_2-h_1)+(k_2-k_1)\rho} &= \alpha w_{h_1} + k_1 w_{\alpha\rho} + \alpha w_{h_2-h_1} + (k_2 - k_1) w_{\alpha\rho} \\ &= \alpha(w_{h_1} + w_{h_2-h_1}) + k_2 w_{\alpha\rho} \\ &\geq \alpha w_{h_2} + k_2 w_{\alpha\rho} \\ &= x_{h_2+k_2\rho} \end{aligned}$$

So $\mathrm{Im}\,(\Phi_{\rho,S_2}) \in \mathcal{C}(G)$. Furthermore, Φ_{ρ,S_2} is clearly linear. By applying to $\mathrm{Im}\,(\Phi_{\rho,S_2})$ the projection that keeps only the $|H|$ components of the form $x_{h+0\rho}$ we get an injective map from $\mathbb{R}^{|H|}$ to $\mathbb{R}^{|H|}$ that multiplies by α, so Φ_{ρ,S_2} must be injective. □

In application, the group G is $\mathbb{Z}_{\alpha a_1}$, the subgroup H is $\alpha\mathbb{Z}_{a_1}$, and $\rho = [\beta]_{\alpha a_1} \in \mathbb{Z}_{\alpha a_1}$ must be relatively prime to α. One important side effect of this notation is that the Φ map, as written, is actually a map with domain $\mathcal{C}(\alpha\mathbb{Z}_{a_1})$ rather than domain $\mathcal{C}(\mathbb{Z}_{a_1})$.

Example 5.1.3. Consider the group $G = \mathbb{Z}_6$ and its subgroup $2\mathbb{Z}_3$. For the purposes of applying the map Φ_{ρ,S_2}, we require $\alpha = |G/H| = |\mathbb{Z}_6/2\mathbb{Z}_3| = 2$, and we require that $\rho \in \mathbb{Z}_6$ satisfy $\gcd(\rho, 2) = 1$, so ρ can be either 1, 3, or 5 (mod 6). Evaluating the map for each of these ρ, we will observe three different types of behavior and correlate these to later results.

Choose $G = \mathbb{Z}_6$, $H = 2\mathbb{Z}_3$, and $S_2 = \langle 1 \rangle$. We will apply the map Φ_{ρ,S_2} to the point

$$w = (\underset{0}{0}, \underset{2}{4}, \underset{4}{5}) \in \mathcal{C}(2\mathbb{Z}_3)$$

where the blue numbers are the indices (each an element of $2\mathbb{Z}_3$) of the corresponding coordinate. That is, we write $w_0 = w_{[0]_6} = 0$, $w_2 = w_{[2]_6} = 4$, and $w_4 = w_{[4]_6} = 5$. (As a sidenote, we may immediately observe that $\{0, 4, 5\}$ is $\mathrm{Ap}(\langle 3, 4, 5 \rangle, 3)$.)

The image of $\Phi_{\rho,S_2} = \Phi_{\rho,\langle 1 \rangle}$ is a point

$$x = (x_0, x_1, x_2, x_3, x_4, x_5) \in C(\mathbb{Z}_6)$$

where each coordinate is defined by

$$x_{h+k\rho} = \alpha w_h + k w_{\alpha\rho} = 2w_h + k w_{[2\rho]_6}$$

This depends on the fact that each element of $\mathbb{Z}_6$ has a unique decomposition as an element h of $2\mathbb{Z}_3$ plus an element $k\rho$ of $\mathbb{Z}_6$, where ρ is fixed and k varies over the set

$$\mathrm{Ap}(S_2; \alpha) = \mathrm{Ap}(\langle 1 \rangle; 2) = \{0, 1\}.$$

To evaluate this map, we may begin by ignoring k and ρ by setting $k = 0$. This gives that $x_h = 2w_h$ for $h \in 2\mathbb{Z}_3$, or in particular,

$$x = (\overset{2w_0}{\underset{0}{0}}, x_1, \overset{2w_2}{\underset{2}{8}}, x_3, \overset{2w_4}{\underset{4}{10}}, x_5) \in C(\mathbb{Z}_6).$$

Now suppose $\rho = 1$. Then in particular, we decompose

$$1 \equiv 0 + 1\rho, \quad 3 \equiv 2 + 1\rho, \quad 5 \equiv 4 + 1\rho$$

so

$$x_1 = 2w_0 + 1w_2 = 0 + 1(4) = 4$$

$$x_3 = 2w_2 + 1w_2 = 8 + 1(4) = 12$$

$$x_5 = 2w_4 + 1w_2 = 10 + 1(4) = 14$$

so all together,

$$\Phi_{[1]_6,\langle 1 \rangle}(w) = x = (\underset{0}{0}, \underset{1}{4}, \underset{2}{8}, \underset{3}{12}, \underset{4}{10}, \underset{5}{14}) \in C(\mathbb{Z}_6).$$

Performing similar calculations with $\rho = [3]_6$ gives

$$x_1 = x_{4+1\rho} = 2w_4 + 1w_0 = 10$$

$$x_3 = x_{0+1\rho} = 2w_0 + 1w_0 = 0$$

$$x_5 = x_{2+1\rho} = 2w_2 + 1w_0 = 8$$

$$\Phi_{3,\langle 1 \rangle}(w) = (\underset{0}{0}, \underset{1}{10}, \underset{2}{8}, \underset{3}{0}, \underset{4}{10}, \underset{5}{8})$$

and with $\rho = 5$, we get

$$x_1 = x_{2+1\rho} = 2w_2 + 1w_4 = 13$$

$$x_3 = x_{4+1\rho} = 2w_4 + 1w_4 = 15$$

$$x_5 = x_{0+1\rho} = 2w_0 + 1w_4 = 5$$

$$\Phi_{5,\langle 1 \rangle}(w) = (\underset{0}{0}, \underset{1}{13}, \underset{2}{8}, \underset{3}{15}, \underset{4}{10}, \underset{5}{5}).$$

Let's now compare the resulting points:

$$p_1 = \Phi_{1,\langle 1\rangle}(w) = (\underset{0}{0}, \underset{1}{4}, \underset{2}{8}, \underset{3}{12}, \underset{4}{10}, \underset{5}{14}).$$

$$p_3 = \Phi_{3,\langle 1\rangle}(w) = (\underset{0}{0}, \underset{1}{10}, \underset{2}{8}, \underset{3}{0}, \underset{4}{10}, \underset{5}{8})$$

$$p_5 = \Phi_{5,\langle 1\rangle}(w) = (\underset{0}{0}, \underset{1}{13}, \underset{2}{8}, \underset{3}{15}, \underset{4}{10}, \underset{5}{5}).$$

Of these, we see that only p_5 looks like an Apéry set, with each component having the corresponding mod class mod 6. This is $\mathrm{Ap}(\langle 5, 6, 8\rangle; 6)$. It is no coincidence that this is the same as the not-quite-valid "gluing" $2\langle 3, 4, 5\rangle + 5\langle 1\rangle$. (The dimensions in this example were too small to properly illustrate a gluing.) The reason that p_5 turns out to correspond to the Apéry set of an actual semigroup turns out to be the fact that there is an element of $\mathrm{Ap}(\langle 3, 4, 5\rangle; 3) = \{0, 4, 5\}$ that is equivalent to $\rho = 5$ mod 6. The same is not true of the other two choices for ρ, 1 and 3.

Looking at p_1, we observe that it does not correspond to a valid Apéry set. In fact, not all of the mod classes mod 6 are represented. That being said, we can still construct a Kunz poset from this point, with the corresponding equalities $x_0 = 0$, $x_2 = 2x_1$, $x_3 = 3x_1$, and $x_5 = x_1 + x_4$, and there are numerical semigroups with this Kunz poset in $\mathcal{C}(\mathbb{Z}_6)$, such as $\langle 6, 7, 22\rangle$.

Finally, looking at p_3, we may notice that it is periodic. This is a side-effect of the fact that $\alpha\rho = 2[3]_6 = [0]_6$, so that $w_{\alpha\rho} = 0$ in the map and the value of $x_{h+k\rho}$ is dependent only on h. Because two coordinates of p_3 are equal, it does not have a conventional Kunz poset: The poset contains only 3 points, each labeled with two numbers. There are no numerical semigroups in the associated face of $\mathcal{C}(\mathbb{Z}_6)$, but by viewing the poset as a Kunz poset for a face of $\mathcal{C}(\mathbb{Z}_3)$, we get get a face that *does* have numerical semigroups in it.

Definition 5.1.4. Fix a poset $P = (H/H', \preceq_P)$ where $H' \leq H$ is a subgroup and suppose $h_1, h_2 \in H/H'$ and $k_1, k_2 \in \mathrm{Ap}(S_2; \alpha)$.

(a) The *gluing extension* of P by S_2 along ρ is the poset $Q = (G/H', \preceq_Q)$ which satisfies $h_1 + k_1\rho \preceq_Q h_2 + k_2\rho$ if and only if $h_1 \preceq_P h_2$ and $k_1 \preceq_{S_2} k_2$.

(b) The *augmented gluing extension* of P by S_2 along ρ is the poset Q defined as follows

  (i) If $\alpha\rho \neq 0$, then $Q = (G/H', \preceq_Q)$ is the poset satisfying satisfying $h_1 + k_1\rho \preceq_Q h_2 + k_2\rho$ if and only if $h_1 \preceq_P h_2$ and, for some $n \geq 0$, $n\alpha\rho \preceq_P h_2 - h_1$ and $k_1 - k_2 \preceq_{S_2} n\alpha$.

  (ii) If $\alpha\rho = 0$, then Q is the poset on $G/(H' + \langle\rho\rangle)$ identical to P under the natural group isomorphism $G/(H' + \langle\rho\rangle) \cong H/H'$.

Remark 5.1.5. The non-augmented extension of P is isomorphic to the Cartesian product of P with $\mathrm{Ap}(S_2; \alpha)$. Furthermore, notice that the augmented gluing extension is a refinement of a gluing extension. If $\alpha\rho \neq 0$, then new relations are added between the elements, (in particular, the Kunz subgroup H' stays the same) while if $\alpha\rho = 0$, all of the elements that would come from $\mathrm{Ap}(S_2;\alpha)$ are collapsed onto a single node, making the Kunz subgroup H' into the larger $H' + \langle\rho\rangle$. In particular, this means that the element $h + k\rho \in G$ now lies in the same equivalence class as $h \in G$, and satisfies same poset relations satisfied by the elements $h \in H$ in the poset P. The natural isomorphism $G/(H' + \langle\rho\rangle) \cong H/H'$ is thus expressed by

$$G/(H' + \langle\rho\rangle) \ni [h + k\rho]_{H'+\langle\rho\rangle} = [h]_{H'+\langle\rho\rangle} \mapsto [h]_{H'} \in H/H'.$$

Corollary 5.1.6. *If $S_1 = \langle a_1, \ldots, a_k\rangle$ and $T = \alpha S_1 + \beta S_2$ is a gluing then the Kunz poset of T with respect to αa_i is the gluing extension of the Kunz poset of S_1 with respect to a_i by S_2 along $\rho = [\beta]_{\alpha a_i}$ which is augmented if and only if $\beta \in \mathrm{Ap}(S_1; a_i)$.*

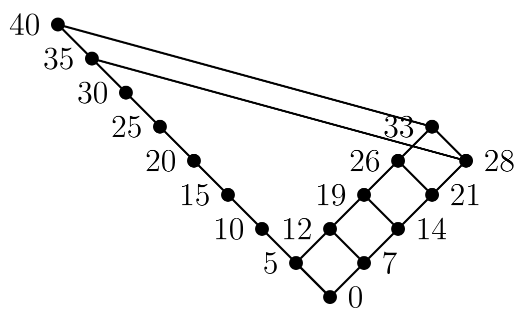
Proof. If $\alpha\rho = 0$ then $\alpha a_i \mid \alpha\beta$ so $a_i \mid \beta$ so $\beta \in \mathrm{Ap}(S_1; a_i)$ is impossible. By Corollary 3.2.1 we know that $\mathrm{Ap}(T; \alpha a_i)$ consists of all $c = \alpha c_1 + \beta c_2$ such that both $c_1 \in \mathrm{Ap}(S_1, a_i)$ and $c_2 \in \mathrm{Ap}(S_2; \alpha)$ with relations corresponding to Theorem 3.2.4. From Theorem 3.2.4 we know that $c \preceq_T d$ if and only if $c_1 \preceq_{S_1} d_1$ and $c_2 \preceq_{S_2} d_2$ or $c_1 + n\beta \preceq_{S_1} d_1$ and $c_2 \preceq_{S_2} d_2 + n\alpha$ with $n \geq 0$ and the latter case occurring if and only if $\beta \in \mathrm{Ap}(S_1; a_1)$. Set $h_1 = [\alpha c_1]_{\alpha a_i}$, and $h_2 = [\alpha d_1]_{\alpha a_i}$, set $c_2 = k_1$ and $d_2 = k_2$. The conditions $c_1 \preceq_{S_1} d_1$ and $h_1 \preceq_P h_2$ with $H/H' = \alpha\mathbb{Z}_{a_i}$ become identical. Furthermore $c_1 + n\beta \preceq_{S_1} d_1$ implies $c_1 \preceq_{S_1} d_1$ and $n\beta \preceq_{S_1} d_1 - c_1$. This corresponds to $h_1 \preceq_P h_2$ and $n\alpha\rho \preceq_P h_2 - h_1$. Finally $c_1 \preceq_{S_2} d_2 + n\alpha$ implies $c_1 - d_1 \preceq_{S_2} n\alpha$ which corresponds to $k_1 - k_2 \preceq_{S_2} n\alpha$. $\square$

This result is probably best understood visually, that is by showing how the Kunz posets relate to the Apéry posets:

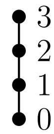
Example 5.1.7. Consider the gluing $T = 17\langle 4, 5\rangle + 20\langle 5, 7\rangle = \langle 68, 85, 100, 140\rangle$. Here we have $S_1 = \langle 4, 5\rangle$, $S_2 = \langle 5, 7\rangle$, $\alpha = 17$, and $\rho = 20$. From Example 3.2.6 we know that

$$\mathrm{Ap}(\langle 5, 7\rangle; 17) = \{0, 35, 19, 20, 21, 5, 40, 7, 25, 26, 10, 28, 12, 30, 14, 15, 33\}$$

with poset:

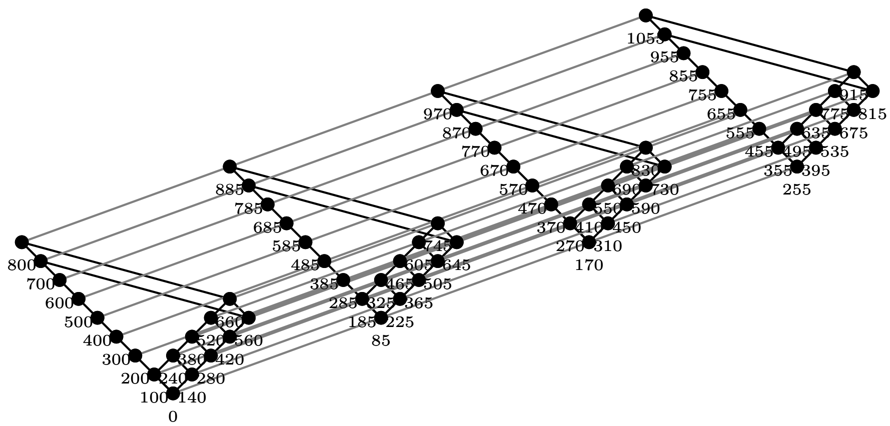


We also know that the Kunz poset for $\mathrm{Ku}(\langle 4, 5\rangle; 4)$ is



And we know the Apéry coordinate is $(0, 5, 10, 15)$.

From Example 3.2.6 we know $\mathrm{Ap}(T; 68)$ is



In order to check that $\mathrm{Ku}(T; 68)$ corresponds to the gluing extension of the $\mathrm{Ku}(\langle 4, 5\rangle; 4)$ by $\langle 5, 7\rangle$ along 20 we can check a few components to confirm that relationships are the same. Since we are dealing with a coordinate in $\mathbb{R}^{68}$, we will not be exhaustive, we will just choose an example to demonstrate the idea.

Since $G = \mathbb{Z}_{68}$ and $H = 17\mathbb{Z}_4$ we know that the coordinate corresponding to $\mathrm{Ap}(\langle 4, 5\rangle; 4)$ is indexed using $\{0, 17, 34, 51\}$. Thus we have

$$w = (\underset{0}{0}, \underset{17}{5}, \underset{34}{10}, \underset{51}{15}) \in \mathcal{C}(17\mathbb{Z}_4).$$

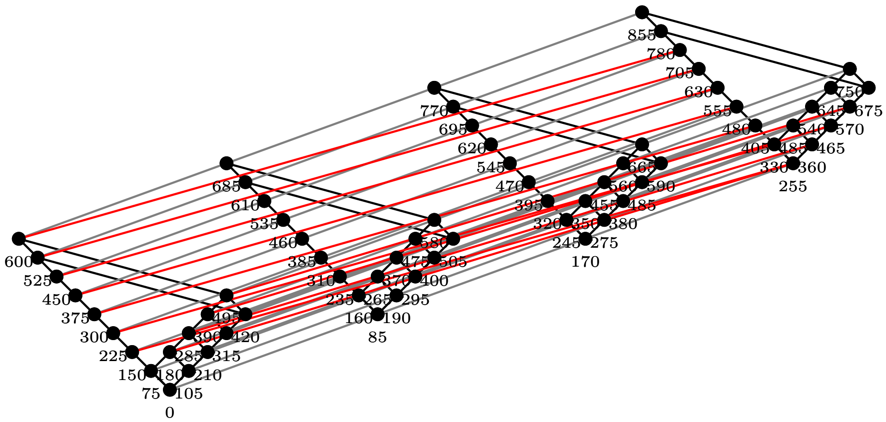
Since $5 \preceq_P 15$, we can notice that $w_{17} \preceq_P w_{51}$ in $\mathrm{Ap}(\langle 4, 5\rangle; 4)$. Furthermore, from $\mathrm{Ap}(\langle 5, 7\rangle; 17)$ we can observe that $7 \preceq_{S_2} 26$. So in the gluing extension we have $17 + 7(20) \preceq_Q 51 + 26(20)$, rewritten mod 68 we have $21 \preceq_Q 27$.

The element of $\mathrm{Ap}(T; 68)$ corresponding to 21 is 225 and the element corresponding to 27 is 775. Looking at the diagram for $\mathrm{Ap}(T; 68)$ we can see that $225 \preceq_T 775$.

Example 5.1.8. If we want to check a point with augmented relations, we can use

$$T' = 17\langle 4, 5\rangle + 15\langle 5, 7\rangle = \langle 68, 75, 85, 105\rangle.$$

Once again we have $S_1 = \langle 4, 5\rangle$, $S_2 = \langle 5, 7\rangle$, and $\alpha = 17$. What differs from Example 5.1.7 is that $\beta = 15 = \rho$. Since $15 \in \mathrm{Ap}(\langle 4, 5\rangle; 4)$ we can expect augmented relations. From Example 3.2.6 we know that the poset for $\mathrm{Ap}(T'; 68)$ is



Since we have the same α and S_1, the point corresponding to $\mathrm{Ap}(\langle 4,5\rangle;4)$ is still

$$w = (\underset{0}{0}, \underset{17}{5}, \underset{34}{10}, \underset{51}{15}) \in \mathcal{C}(17\mathbb{Z}_4).$$

In order to check that the augmented relations work as expected, we will want to pick values that are suitably "far apart" for these relations to take effect. Notice that $w_0 \preceq w_{51-17(15)}$ since $17(15) \equiv 51 \bmod 68$. Also notice that $5 - 5 \preceq_{S_2} 17$. So in the augmented gluing extension we have $0 + 12(15) \preceq 51 + 5(15)$, rewritten mod 68 we have $44 \preceq_Q 58$.

The element of $\mathrm{Ap}(T';68)$ corresponding to 44 is 180 and the element corresponding to 58 is 330. Looking at the diagram for $\mathrm{Ap}(T';68)$ we can see that $180 \preceq_T 330$. The fact that the augmentation relations are necessary can be seen in the diagram: in order to trace a path from 330 down to 180 we need to use one of the red lines.

Theorem 5.1.9. *For a face $F \subset \mathcal{C}(H)$, the image of $\Phi_{\rho,S_2}(F)$ lies within a face of $\mathcal{C}(G)$ whose Kunz poset is the augmented gluing extension of P along ρ by S_2.*

Proof. First, fix $w \in F \subset \mathcal{C}(H)$, let $x = \Phi_{\rho,S_2}(w)$, let F' denote the face containing x and let Q denote the corresponding Kunz poset of F'. If $\alpha\rho = 0$ we know that $kw_{\alpha\rho} = 0$, so $x_{h+k\rho} = 0$ whenever $w_h = 0$. If $\alpha\rho \neq 0$, then this occurs when $w_h = 0$ and $k = 0$. Notice that $w_h = 0$ when $h \in H'$. In either case Q has the claimed ground set.

Now suppose $h_1, h_2 \in H$ and $k_1, k_2 \in \mathrm{Ap}(S_2;\alpha)$. If $\alpha\rho = 0$, then

$$\begin{aligned} x_{h_1+k_1\rho} + x_{h_2-h_1+(k_2-k_1)\rho} &= \alpha w_{h_1} + k_1 w_{\alpha\rho} + \alpha w_{h_2-h_1} + (k_2-k_1)w_{\alpha\rho} \\ &= \alpha(w_{h_1} + w_{h_2-h_1}) \geq \alpha w_{h_2} \\ &= x_{h_2+k_2\rho} \end{aligned}$$

with equality if and only if $h_1 \preceq_P h_2$. If $\alpha\rho \neq 0$ then there are two possibilities:

If $k_1 \preceq_{S_2} k_2$ then we have:

$$\begin{aligned} x_{h_1+k_1\rho} + x_{h_2-h_1+(k_2-k_1)\rho} &= (\alpha w_{h_1} + k_1 w_{\alpha\rho}) + (\alpha w_{h_2-h_1} + (k_2-k_1)w_{\alpha\rho}) \\ &= \alpha(w_{h_1} + w_{h_2-h_1}) + k_2 w_{\alpha\rho} \\ &\geq \alpha w_{h_2} + k_2 w_{\alpha\rho} \\ &= x_{h_2+k_2\rho} \end{aligned}$$

with equality if and only if $h_1 \preceq_P h_2$.

If $k_1 \not\preceq_{S_2} k_2$ then

$$(h_2 + k_2\rho) - (h_1 + k_1\rho) = (h_2 - h_1 - n\alpha\rho) + (k_2 - k_1 + n\alpha)\rho$$

with $k_2 - k_1 + n\alpha \in \mathrm{Ap}(S_2; \alpha)$. Thus we have:

$$\begin{aligned}
x_{h_1+k_1\rho} + x_{h_2-h_1+(k_2-k_1)\rho} &= x_{h_1+k_1\rho} + x_{h_2-h_1-n\alpha\rho+(k_2-k_1+n\alpha)\rho} \\
&= \alpha w_{h_1} + k_1 w_{\alpha\rho} + \alpha w_{h_2-h_1-n\alpha\rho} + (k_2 - k_1 + n\alpha) w_{\alpha\rho} \\
&= \alpha(w_{h_1} + w_{h_2-h_1 n\alpha\rho} + n w_{\alpha\rho}) + k_2 w_{\alpha\rho} \\
&\geq \alpha(w_{h_1} + w_{h_2-h_1}) + k_2 w_{\alpha\rho} \\
&\geq \alpha w_{h_1} + k_2 w_{\alpha\rho} \\
&= x_{h_2+k_2\rho}
\end{aligned}$$

with equality if and only if $n\alpha\rho \preceq_P h_2 - h_1$ and $h_1 \preceq_P h_2$. In either case $h_1 + k_1\rho \preceq_Q h_2 + k_2\rho$ in the cases required by Definition 5.1.4. Thus $\Phi_{\rho,S_2}(w)$ lies in the interior of the face. □

Example 5.1.10. Consider the gluing

$$T = \langle 68, 75, 85, 105\rangle = 17\langle 4, 5\rangle + 15\langle 5, 7\rangle$$

From Example 3.2.6 we know what $\mathrm{Ap}(T; 68)$ looks like. We now want to demonstrate that the Apéry coordinate corresponding to $\mathrm{Ap}(T; 68)$ matches the output of $\Phi_{15,\langle 5,7\rangle}(\mathrm{Ap}(\langle 4, 5\rangle; 4))$. Rather than being exhaustive, we will check for a few components of the Apéry coordinate corresponding to $\mathrm{Ap}(T; 68)$. If we let $\Phi_{15,\langle 5,7\rangle}(\mathrm{Ap}(\langle 4, 5\rangle; 4)) = x$, then $x_3 = x_{34+7(15)}$. So

$$x_3 = x_{34+7(15)} = 17w_{34} + 7w_{17(15)} = 17w_{34} + 7w_{51}.$$

From Example 5.1.8 we know that $w_{34} = 10$ and $w_{51} = 15$. So

$$x_3 = 17(10) + 7(15) = 275.$$

Then from Example 3.2.6 we know that $275 \in \mathrm{Ap}(T; 68)$ and $275 \equiv 3 \bmod 68$. Thus the map works for x_3.

We can now try with a different component. Consider

$$x_2 = x_{51+33(15)} = 17w_{51} + 33w_{17(15)} = 17w_{51} + 33w_{51} = (17 + 33)15 = 750$$

Then from Example 3.2.6 we know that $750 \in \mathrm{Ap}(T; 68)$ and $750 \equiv 2 \bmod 68$. Thus the map works for x_2.

Definition 5.1.11. The *gluing ray* $\vec{s}$ of a gluing embedding Φ_{ρ,S_2} is defined

$$s_{h+k\rho} = k$$

for each $h \in H$ and $k \in \mathrm{Ap}(S_2; \alpha)$. Notice that $s_h = 0$ precisely when $h \in H$, so $\vec{s}$ must lie in a face whose Kunz subgroup is H.

Lemma 5.1.12. *The gluing ray* $\vec{s}$

(a) lies in a face of $\mathcal{C}(G)$ *whose corresponding Kunz subgroup is* H*, and*

(b) is linearly independent to each vector in the image of Φ_{ρ,S_2}

Proof. As noted above, $s_h = 0$ precisely when $h \in H$, so $\vec{s}$ must lie in a face whose Kunz subgroup is H. For the second part, since Φ_{ρ,S_2} is linear and $\mathcal{C}(H)$ is full-dimensional, it suffices to show that $\vec{s}$ lies outside the image of Φ_{ρ,S_2}. Projecting the image of Φ_{ρ,S_2} onto the coordinates indexed by H is injective by the proof of Lemma 5.1.2, while applying the same projection to $\vec{s}$ gives us 0. □

Theorem 5.1.13. *For any face* $F \subset \mathcal{C}(H)$*, the set* $\mathbb{R}_{\geq 0}\vec{s} + \Phi_{S_2,\rho}(F)$ *is in a face of* $\mathcal{C}(G)$ *whose Kunz poset is the gluing extension of the Kunz poset of* F*.*

Proof. To begin with we will demonstrate that the Kunz poset of the set $\mathbb{R}_{\geq 0}\vec{s} + \Phi_{S_2,\rho}(F)$ is the gluing extension of the Kunz poset of F. Let $P = (H/H', \preceq_P)$ be the Kunz poset of F. Let F' denote the smallest face containing the set $\mathbb{R}_{\geq 0}\vec{s} + \Phi_{S_2,\rho}(F)$, and let $Q = (G/H'', \preceq_Q)$ be the Kunz poset of F'. We want to show that Q is the gluing extension of P. To begin with, we need to show that the gluing extension of P, and the poset Q have the same ground sets. Since $\Phi_{\rho,S_2}(F) \subset F'$, by Theorem 5.1.9 we know that $H'' \subset H'$. Furthermore, the coordinates in which $\vec{s}$ is 0 are those indexed by H, by Lemma 5.1.12, which implies $H'' = H'$. Thus we know that the grounds sets are equal.

Now we need to show that the same poset relations hold. Fix $x \in \mathbb{R}_{\geq 0}\vec{s} + \Phi_{S_2,\rho}(F)$, and write $x = y + cs$ for $y \in \Phi_{\rho,S_2}(F)$ and $c \geq 0$. If $h_1 + k_1\rho, h_2 + k_2\rho \in G$, then by Theorem 5.1.9 we know that

$$x_{h_1+k_1\rho} + x_{h_2-h_1+(k_2-k_1)\rho)} = y_{h_1+k_1\rho} + ck_1 + y_{h_2-h_1+(k_2-k_1)\rho)} + c(k_2-k_1) = y_{h_2+k_2rho} + ck_2 = x_{h_2+k_2\rho}.$$

Thus we know that F' satisfies all of the Kunz equalities of the gluing extension. We know that these are the only equalities satisfied by F' because we know that the Kunz poset of Φ_{ρ,S_2} is a refinement of the Kunz poset of F', and $\vec{s}$ does not satisfy the additional relations $h_1 + n\alpha\rho \preceq_P h_2$ and $k_1 \preceq_{S_2} k_2 + n\alpha$. Thus we know that Q is equal to the monoscopic extension of P. □

Example 5.1.14. Consider the gluing

$$T = \langle 68, 85, 100, 140\rangle = 17\langle 4, 5\rangle + 20\langle 5, 7\rangle$$

From Example 3.2.6 we know what $\text{Ap}(T; 68)$ looks like. We now want to demonstrate that the Apéry coordinate corresponding to $\text{Ap}(T; 68)$ is in the span of $\Phi_{20,\langle 5,7\rangle}(\text{Ap}(\langle 4, 5\rangle; 4))$ and $\vec{s}$. We want to show that the coordinate for $\text{Ap}(T; 68)$ is equal to $\Phi_{20,\langle 5,7\rangle}(\text{Ap}(\langle 4, 5\rangle; 4)) + c\vec{s}$ Rather than being exhaustive, we will check for a few components of the Apéry coordinate corresponding to $\text{Ap}(T; 68)$. If we let $\Phi_{20,\langle 5,7\rangle}(\text{Ap}(\langle 4, 5\rangle; 4)) = x$, then $x_2 = x_{34+12(20)}$. So

$$x_2 = x_{34+12(20)} = 17w_{34} + 12w_{17(20)} = 17w_{34} + 12w_0.$$

From Example 5.1.8 we know that $w_{34} = 10$ and $w_0 = 0$. So

$$x_2 = 17(10) + 12(0) = 170.$$

The value of $\mathrm{Ap}(T; 68)$ corresponding to mod class 2 is 410. So then $410 - 170 = 240$. Then $\vec{s}_2 = 12$. So if $y = \mathrm{Ap}(T; 68)$ we have $y_2 = x_2 + 20s_2$. We can now try with a different component. Consider

$$x_9 = x_{17+20(20)} = 17w_{17} + 20w_{17(20)} = 17w_{17} + 20w_0 = 17(5) = 85$$

The value of $\mathrm{Ap}(T; 68)$ corresponding to mod class 9 is 485. So then $485 - 85 = 400$. Then $\vec{s}_9 = 20$. So if $y = \mathrm{Ap}(T; 68)$ we have $y_9 = x_9 + 20s_2$. Notice that we have the same coefficient on $\vec{s}$ in both components. There are 68 components, so this is not proof, but it is a demonstration of how Theorem 5.1.13 works.

5.2 Face-filling

Section 5 of this write up is an attempt to generalize section 6 of [1]. While many of the results generalize quite nicely, one notable exception is Theorem 5.1.9 which is a generalization of:

Theorem 5.2.1 (Theorem 6.7 in [1])**.** *The image of $\Phi_{\rho,\langle 1\rangle}$ is a face of $\mathcal{C}(G)$. More precisely, given any face $F \subset \mathcal{C}(H)$ with Kunz poset $P = (H/H', \preceq_P)$, the image $\Phi_{\rho,\langle 1\rangle}(F)$ is a face of $\mathcal{C}(G)$ whose Kunz poset is the augment monoscopic extension Q of P along ρ.*

In [1] the image of the Φ_ρ map is guaranteed to be an entire face of $\mathcal{C}(G)$, however in Theorem 5.1.9 the image of Φ_{ρ,S_2} is not always an entire face of $\mathcal{C}(G)$. A similar issue arises for:

Theorem 5.2.2 (Theorem 6.10 in [1])**.** *For any face $F \subset \mathcal{C}(H)$, the set $\mathbb{R}_{\geq 0}\vec{s} + \Phi_{\rho,\langle 1\rangle}(F)$ is a face of $\mathcal{C}(G)$ whose Kunz poset is the monoscopic extension of the Kunz poset of F.*

Here we have that $\mathbb{R}_{\geq 0}\vec{s} + \Phi_\rho(F)$ is a face of $\mathcal{C}(G)$ but in Theorem 5.1.13 we only have that$\mathbb{R}_{\geq 0}\vec{s} + \Phi_{\rho,S_2}(F)$ is contained within a certain face of $\mathcal{C}(G)$. We thus want to establish when the image of Φ_{ρ,S_2} is a face of $\mathcal{C}(G)$.

This proved to be rather difficult for us. In [1], Theorem 6.11 says that for a face with a monoscopic gluing every point that is an Apéry coordinate in that face corresponds to a monoscopic gluing. We cannot guarantee for a glued numerical semigroup that all numerical semigroups with Apéry coordinates in the same face are also gluings.

Example 5.2.3. Consider the glued numerical semigroup

$$T_1 = \langle 198, 220, 242, 1111, 1212, 1313\rangle = 22\langle 9, 10, 11\rangle + 101\langle 11, 12, 13\rangle.$$

We can also consider the numerical semigroup

$$T_2 = \langle 198, 220, 242, 1511, 2400, 3289\rangle$$

which is not a gluing. One can use Sage (or GAP) to check that these two numerical semigroups have the same Kunz poset, but the poset is to large to include here.

The following definitions and theorems will prove that for a face F of $\mathcal{C}(H)$, that $\Phi_{\rho,S_2}(F)$ is a face (F') of $\mathcal{C}(G)$ if and only if $\dim(F) = \dim(F')$. This moves us closer to a condition on when a face of the group cone contains only gluings, because this amounts to showing that the two faces are of equal dimension.

Definition 5.2.4. Suppose G, H are finite abelian groups with $H \leq G$, G/H cyclic. Let $\alpha = |G/H|$. Then define the map

$$\pi : \mathcal{C}(G) \to \mathcal{C}(H)$$

by $\pi(w) = x$ with

$$w_h = \frac{1}{\alpha} x_h.$$

We call this the *projection map* from $\mathcal{C}(G)$ (down) to $\mathcal{C}(H)$.

Theorem 5.2.5. *For any ρ, S_2, the projection map π is a left inverse on F for Φ_{ρ,S_2}. That is,*

$$\pi \circ \Phi_{\rho,S_2} = \mathrm{Id}_F.$$

Proof. Suppose $w \in F$, and define $x = \Phi_{\rho,S_2}$ and $z = \pi(x)$. Then for $h \in H$,

$$z_h = \frac{1}{\alpha} w_h = \frac{1}{\alpha}(w_{h+0\rho}) = \frac{1}{\alpha}(\alpha w_h + 0 w_{\alpha\rho}) = w_h$$

as desired. □

We know that Φ_{ρ,S_2}, applied to a face with poset P, gives back the face whose poset Q is essentially a cartesian product of P with $\mathrm{Ap}(S_2; \alpha)$, with some extra relations. As expected, π takes such a poset and returns only the information corresponding to the starting poset P.

Theorem 5.2.6. *Suppose F is a face of $\mathcal{C}(H)$, F' is the face of $\mathcal{C}(G)$ whose Kunz poset Q is either the augmented or non-augmented gluing extension of the Kunz poset P of F, and x is a point in F'. Then $\pi(x) \in F$.*

Proof. If Q is the non-augmented gluing extension of P, then, limiting ourselves only to the coordinates of $\mathcal{C}(G)$ indexed by elements of H, we get the relations $h_1 + 0\rho \preceq_Q h_2 + 0\rho$ if and only if $h_1 \preceq_P h_2$, if and only if $x_{h_1} + x_{h_2 - h_1} = x_{h_2}$, if and only if

$$\frac{1}{\alpha} x_{h_1} + \frac{1}{\alpha} x_{h_2 - h_1} = \frac{1}{\alpha} x_{h_2}$$

if and only if $w_{h_1} + w_{h_2 - h_1} = w_{h_2}$. But this is the description of the exact facet equalities encoded in the Kunz poset P for the face F, so the point w must lie in F.

Suppose now that the Kunz poset Q of F' is the *augmented* Kunz poset of P. If $\alpha\rho \neq 0$ (as in Definition 5.1.4), then we have $h_1 + 0\rho \preceq_Q h_2 + 0\rho$ if and only if (combining the cases)

$$h_1 \preceq_P h_2 \quad \text{and} \quad n\alpha\rho \preceq_P h_2 - h_1 \quad \text{and} \quad 0 - 0 \preceq_{S_2} n\alpha$$

for some $n \geq 0$. The third condition is vacuously true since $\alpha \in S_2$, and we can then always choose $n = 0$ to satisfy the second condition. This then gives the same list of conditions as above, and the logic follows identically.

Finally, if Q is the augmented Kunz poset of P and $\alpha\rho = 0$, then we have the isomorphism $G/(H' + \langle\rho\rangle) \cong H/H'$ defined by

$$[h + k\rho]_{H'+\langle\rho\rangle} \mapsto [h]_{H'}.$$

In particular, we see that $h_1 + 0\rho \preceq_Q h_2 + 0\rho$ if and only if $[h_1]_{H'} \preceq_P [h_2]_{H'}$, if and only if (stripping away the equivalence classes) $h_1 \preceq_P h_2$. As before, this gives us the same set of conditions, and translates to the same result. $\square$

Theorem 5.2.7. *Suppose F is a face of $C(H)$ and F' is the face of $C(G)$ in which $\Phi_{\rho,S_2}(F)$ lies. Then $\Phi_{\rho,S_2}(F) = F'$ if and only if $\dim F' = \dim F$.*

Proof. If $\Phi_{\rho,S_2}(F) = F'$, then in particular $\dim F = \dim \Phi_{\rho,S_2}(F) = \dim F'$.

Now assume $\dim F' = \dim F$. Then since π restricted to F' is a surjective linear map (Theorem 5.2.5) from F' to F (Theorem 5.2.6) and both F' and F have the same dimension, π is in fact a bijection from F' to F, and is thus the *two-sided* inverse of $\Phi_{\rho,S_2} : F \to F'$, showing that $\Phi_{\rho,S_2}(F) = F'$. $\square$

5.3 A special case: $n \times 2$ gluings

While we do not have a condition for when general gluings "fill a face" of $\mathcal{C}(G)$, when we restrict $S_2 = \langle b_1, b_2\rangle$, we have made more progress. We then have two cases: when $b_1b_2 \in \mathrm{Ap}(S_2;\alpha)$ and when $b_1b_2 \notin \mathrm{Ap}(S_2;\alpha)$.

When $b_1b_2 \in \mathrm{Ap}(S_2;\alpha)$ we have the following result:

Theorem 5.3.1. *If $S_2 = \langle b_1, b_2\rangle$ and $b_1b_2 \in \mathrm{Ap}(S_2;\alpha)$ for a face $F \subset \mathcal{C}(H)$, the image of $\Phi_{\rho,S_2}(F)$ is a face of $\mathcal{C}(G)$ whose Kunz poset is the augmented gluing extension of P along ρ by S_2.*

Proof. By Theorem 5.1.9 we know that $\Phi_{\rho,S_2}(F)$ is in a face of $\mathcal{C}(G)$ whose Kunz poset is the augmented gluing extension of P along ρ by S_2.

Let $F' \subset \mathcal{C}(G)$ denote the face whose Kunz poset is the augmented gluing extension Q or P (which must exist by above), and fix $x \in F'$. Define $w_h = \frac{1}{\alpha}x_h$ for $h \in H$. Let $k = c_1b_1 + c_2b_2$ and $\alpha = \alpha_1b_1 + \alpha_2b_2$. We can see

$$
\begin{aligned}
x_{h+k\rho} &= x_h + x_{k\rho} && \text{because } \mathrm{h} \preceq_{\mathrm{Q}} \mathrm{h}+\mathrm{k}\rho \\
&= x_h + \frac{1}{\alpha}(\alpha x_{k\rho}) \\
&= x_h + \frac{1}{\alpha}(\alpha(x_{(c_1b_1+c_2b_2)\rho})) \\
&= x_h + \frac{1}{\alpha}(\alpha(x_{c_1b_1\rho} + x_{c_2b_2\rho})) && \text{because } \ c_1b_1 \preceq_{S_2} k \\
&= x_h + \frac{1}{\alpha}((\alpha_1b_1+\alpha_2b_2)(x_{c_1b_1\rho} + x_{c_2b_2\rho})) \\
&= x_h + \frac{1}{\alpha}(\alpha_1b_1(x_{c_1b_1\rho}) + \alpha_1b_1(x_{c_2b_2\rho}) + \alpha_2b_2(x_{c_1b_1\rho}) + \alpha_2b_2(x_{c_2b_2\rho})) \\
&= x_h + \frac{1}{\alpha}(\alpha_1b_1c_1(x_{b_1\rho}) + \alpha_1b_1c_2(x_{b_2\rho}) + \alpha_2b_2c_1(x_{b_1\rho}) + \alpha_2b_2c_2(x_{b_2\rho})) \\
&= x_h + \frac{1}{\alpha}(\alpha_1b_1c_1(x_{b_1\rho}) + \alpha_1b_2c_2(x_{b_1\rho}) + \alpha_2b_1c_1(x_{b_2\rho}) + \alpha_2b_2c_2(x_{b_2\rho})) && \text{because } \ b_1x_{b_2} = b_2x_{b_1} \\
&= x_h + \frac{1}{\alpha}(\alpha_1c_1b_1x_{b_1\rho} + \alpha_1c_2b_2x_{b_1\rho} + \alpha_2c_1b_1x_{b_2\rho} + \alpha_2c_2b_2x_{b_2\rho}) \\
&= x_h + \frac{1}{\alpha}(k(\alpha_2x_{b_2\rho} + \alpha_1x_{b_1\rho})) \\
&\vdots \\
&= x_h + \frac{1}{\alpha}(k(x_{\alpha_2b_2\rho} + \alpha_1x_{b_1\rho})) && \text{because } \ b_2 \preceq_{S_2} \alpha b_2 \\
&\vdots \\
&= x_h + \frac{1}{\alpha}(k(x_{\alpha-b_1)\rho} + x_{b_1\rho})) \\
&= x_h + \frac{1}{\alpha}(k_{\alpha\rho}) && \text{by} \quad \text{augmentation} \\
&= \alpha w_h + kw_{\alpha\rho} = \Phi_{\rho,S_2}(w)
\end{aligned}
$$

This proves set equality for $\Phi_{\rho,S_2}(F) = F'$ □

In the case where $b_1b_2 \notin \mathrm{Ap}(S_2;\alpha)$, we then have the following result:

Theorem 5.3.2. *For a gluing $T = \alpha S_1 + \beta S_2$, if $S_2 = \langle b_1, b_2\rangle$, and $b_1b_2 \notin \mathrm{Ap}(S_2;\alpha)$ then there exists a monoscopic gluing $T = \alpha' S' + \beta'\langle 1\rangle$ where the multiplicity of T is not β'.*

Proof. Fix a $k \times 2$ gluing $T = \alpha S_2 + \beta S_2$. Since we assume that $b_1b_2 \notin \mathrm{Ap}(S_2;\alpha)$ we know that without loss of generality $\alpha = xb_1$ with $0 \leq x \leq b_2 - 1$. So $b_1 \mid \alpha$. Thus we can rewrite

$$T = b_1\langle \frac{\alpha}{b_1}a_1, \dots \frac{\alpha}{b_1}a_k, \beta\rangle + \langle \beta b_2\rangle.$$

We will now show that this is a valid gluing. First note that $\beta \mid \beta b_2$ and $b_2 \neq 0$ so βb_2 is a non-generator element of the opposite semigroup. Next note that $\gcd(b_1, b_2) = 1$. So since $\gcd(\alpha, \beta) = 1$ and $b_1 \mid \alpha$, we know $\gcd(b_1, \beta b_2) = 1$.

Now we just need to show that

$$\left\langle \frac{\alpha}{b_1} a_1, \ldots \frac{\alpha}{b_1} a_k, \beta \right\rangle$$

is a valid semigroup. Since T is minimally generated, it suffices to show that $\gcd(\frac{\alpha}{b_1}, \beta) = 1$. This follows from the fact that $\gcd(\alpha, \beta) = 1$ □

Remark 5.3.3. This means that we can apply results from [1] when $b_1 b_2 \notin \mathrm{Ap}(S_2; \alpha)$. When $b_1 b_2 \notin \mathrm{Ap}(S_2; \alpha)$ we sometimes have face-filling and we sometimes do not. However, we are able to avoid describing these cases by defaulting to [1] instead.

6 Further questions

In subsection 3.1, we introduced the idea of a canonical factorization for elements of an semigroup T when T is a gluing. This allowed us to deduce a membership criterion for T, and thence a membership criterion for the Apéry sets of T. While the idea of a canonical factorization does not seem to work nearly as well for numerical semigroups in general, such a concept may well prove useful for investigating the Ap ery sets and posets of other families of semigroups.

While we fully described the Apéry set of a gluing T with respect to an arbitrary element μ (Theorem 3.1.6), the description was rather cumbersome, and we did not give a completely general description of the associated Apéry posets.

Question 6.0.1. Generalize the Apéry poset descriptions of the two special cases Corollary 3.2.1 and Corollary 3.3.3 found later to give a description of the poset associated to the Apéry set given in Theorem 3.1.6.

Question 6.0.2. This theorem requires computing a family of Apéry sets $\mathrm{Ap}(S_1; \mu_1 + n\beta)$ for n ranging from 0 to some maximal value. What is the maximal value of n required? Is there an efficient way to generate this family of Apéry sets from only one of them?

One of the consequences of having a description of the Apéry posets of general gluings, at least with respect to generators, is in particular, the ability to determine the number of maximal elements of an Apéry set, one way of looking at the faces of the group cone. A cursory glance seems to indicate that the number of maximal elements of the Apéry set $(\mathrm{Ap}(T; \alpha a_1), \preceq_T)$ should be the product of the number of maximal elements of $(\mathrm{Ap}(S_1; a_1), \preceq_{S_1})$ and of $(\mathrm{Ap}(S_2; \alpha), \preceq_{S_2})$.

Question 6.0.3. Confirm or disprove the above assertion, and if false, find the correct answer.

In [1], it was found that when $\langle S_2 \rangle = \langle 1 \rangle$, that $\Phi_{\rho,\langle 1 \rangle}(F)$ for F a face of $\mathcal{C}(H)$ is always a face of $\mathcal{C}(G)$ and that every numerical semigroup in that face had a gluing of the same type, with S_1 coming from the same face F and with $S_2 = \langle 1 \rangle$. Experiments seem to indicate that, whenever $\Phi_{\rho,S_2}(F)$ is a face F' of $\mathcal{C}(G)$, every numerical semigroup in F' is a gluing of the same type, and the same also seems to hold true for the non-augmented face obtained by spanning with the gluing ray $\vec{s}_{\rho,S_2}$.

Question 6.0.4. Is the above conjecture true?

If every numerical semigroup in a face can be written as a gluing (it is possible to have a mix of gluings and non-gluings in the same face, as in Example 5.2.3), it seems like at least one gluing type is shared across the entire face.

Question 6.0.5. Show whether this is always the case, and whether, by extension, any face containing a gluing T can be expressed as the image of a face injection Φ_{ρ,S_2}, where S_2 is derived from some gluing of T.

In Theorem 5.2.7 we were able to show that for a face $F \subset \mathcal{C}(H)$, and $F' \subset \mathcal{C}(G)$ the face which contains $\Phi_{\rho,S_2}(F)$, that $\Phi_{\rho,S_2}(F) = F'$ if and only if $\dim F' = \dim F$. This gives us a starting point to find a condition for when $\Phi_{\rho,S_2}(F) = F'$. A recent paper [2] created a number of useful tools for computing face dimension from Kunz poset, hopefully we can use these tools to create a condition for when $\Phi_{\rho,S_2}(F) = F'$.

Question 6.0.6. Find a condition dependent on the Kunz poset of F, $\mathrm{Ap}(S_2; \alpha)$ and ρ, for when $\dim F' = \dim F$, and thus by Theorem 5.2.7 $\Phi_{\rho,S_2}(F) = F'$.

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