On the Preimage of Associated Semigroups

Deepriliam

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1 Introduction

Let $T \subseteq \mathbb{N}_0$. T is a *numerical set* if it includes 0 and is cofinite, and a *numerical semigroup* if it is also closed under addition. In this paper, S will be reserved for numerical semigroups, and T for numerical sets.
For every T, the set $A(T) = \{t \in T \mid t + T \subseteq T\}$ is a numerical semigroup contained in T; this is referred to as the *associated numerical semigroup.* Our paper concerns $P(S) = \#\{T \mid A(T) = S\}$; in other words, we wish to count the numerical sets which associate to a given S, which shall henceforth be referred to as "good" numerical sets. In particular, we wish to classify $P^{-1}(n)$ for $n > 1$, with special attention paid to small n.

We introduce the notion of the void which is defined as $B(S) = \{a|a \notin S, F - a \notin S\}$ and give it a poset structure in a natural way. This ensures that if $A(T) = S$ then $T \setminus S$ must be an order ideal of the Void poset. We further introduce to notion of red triangles which are particular triples of the void in terms of which we give necessary and sufficient conditions for an order ideal to be a good numerical Set.

We apply this machinery to prove several results relating the pseudofrobenius numbers and type of a semigroup to the good numerical sets it has. We also give an algorithm for computing $P(S)$ that works significantly faster than a brute force search specially for semigroups of small type.

In later sections we consider several families of numerical semigroups for e.g. the staircase family $St(m, n) = \{m, 2m, \ldots, mn, \rightarrow\}$ and several others and obtain polynomial growth of $P(S)$ in each case.

We finally consider the kunz polyhedron that has all numerical semigroups of a fixed multiplicity and investigate the density of different P values on the polyhedron. The geometry of the polyhedron seem to play a key role in determining $P(S)$ with certain hyperplanes separating different values of $P(S)$, the behaviour on the hyperplanes being more complicated. We prove that for multiplicity 3, $P(S) = 2$ has density 1; for $m = 4$, $P(S) = 2, 4$ have positive density with the density of $4 \approx 0.62$, density of $2 \approx 0.38$ and for multiplicity 5, $P(S) = 4, 8$ are precisely the values of P that have positive densities which are ≈ 0.29, ≈ 0.71 respectively. Finally we make a conjecture for multiplicity m in general based on collected data.

1.1 Basic Definitions

For every numerical semigroup, there exists a unique minimal set $\mathcal{A}(S)$ which generates S under addition; this set is known as the *atoms* of S. The minimum of $\mathcal{A}(S)$ is called the *multiplicity* and is denoted $m(S)$; note that it is also the minimal nonzero element of S.
For this paper, the minimal elements of S in each residue class modulo $m(S)$ are of particular note; this set is called the *Apery Set* $Ap(S)$, and its elements will be denoted $\mathcal{A}_i$, where $\bar{i}$ is the residue class containing it.
It is often useful to endow $Ap(S)$ with the poset structure $\mathcal{A}_i \preccurlyeq \mathcal{A}_j \iff \mathcal{A}_j - \mathcal{A}_i \in S$. From this, we derive the *Pseudo-Frobenius Numbers* $PF(S) =$

$\{\mathcal{A} - m \mid \mathcal{A} \in Ap(S) \text{ maximal}\} = \{P \in \mathbb{N}_0 \setminus S \mid P + S \in \{P\} \cup S\}$. The largest Pseudo-Frobenius numbers is called the *Frobenius Number* $F(S)$ (simply F when the choice of S is clear); this is equivalent to the standard definition $F(S) = \max \mathbb{N}_0 \setminus S$. All other Pseudo-Frobenius numbers will be labelled P_i when the residue class $\bar{i}$ modulo m is known. In general, P, Q, R will be reserved for labelling Pseudo-Frobenius numbers. Finally, the *type* $t(S) = |PF(S)|$.

1.2 Prior Results

A semigroup S is *symmetric* if $a \in S \iff F - a \notin S$. It is known that S is symmetric $\iff t(S) = 1 \iff P(S) = 1$.
Similarly, a semigroup S is *pseudo-symmetric* if $2 \mid F(S)$ and $a \neq F/2 \in S \iff F - a \notin S$ (note that $F/2 \in S$ would violate additive closure). It is also known that S is pseudo-symmetric $\Rightarrow t(S) = 2, P(S) = 2$.

2 The Void

Definition 2.1 (The Void)**.** *B, the* Void *of a Numerical Semigroup is defined as $B := \{a | a \notin S, F(S) - a \notin S\}$. The elements of B are known as the* paired gaps *of S.*

Note that the paired gaps are particularly useful elements. For instance, since $a \in S$ implies $F - a \notin PF(S)$, $PF(S) \setminus \{F(S)\} \subseteq B$. For the purposes of this paper, they are relevant because of their connection to good numerical sets, as shown below:

Theorem 2.2 (TBUS Theorem)**.** *For a numerical semigroup S, the set $T = B \cup S$ must satisfy $A(T) = S$. Furthermore, $A(T) = S$ implies $T \subseteq B \cup S$.*

The proof of the TBUS Theorem, as well as several proofs to follow, relies on the following lemma:

Lemma 2.3. *$B \subseteq B + S \subseteq B \cup S$*

Proof of Lemma 2.3: The left inequality is trivial, as $0 \in S$. For the sake of contradiction, suppose there exist $b \in B, s \in S$ such that $b + s \notin B \cup S$. Then we must have $F - b - s \in S$, but that implies $(F - b - s) + s = F - b \in S$, which is impossible. $\square$

Proof of Theorem 2.2: Let $T = B \cup S$. Firstly note that $F(S) \notin T$, as $F(S) \notin S, F(S) - F(S) = 0 \in S$. Now if $a \in B$, then $F(S) - a \in B$ so $a \notin A(T)$; thus $A(T) \subseteq T \setminus B = S$. By Lemma 2.3, $B \cup S \subseteq (B + S) \cup S = (B + S) \cup (S + S) = (B \cup S) + S$, implying $S \subseteq A(T)$.

Now suppose $A(T) = S$. Then firstly $F(S) \notin T$ as otherwise $F(S) \in A(T)$. Next if $a \in T \setminus S$ and $a \notin B$ then $F - a \in S$. And $F - a \in A(T)$ implies $F - a + T \subseteq T$; thus $F = F - a + a \in T$, contradiction. Therefore $T \setminus S \subseteq B$ and $T \subseteq B \cup S$. $\square$

With this established, we can now offer more concise proofs for the previously known results on $P(S)$:

Proposition 2.4. *The numerical semigroups with $P(S) = 1$ are precisely the symmetric semigroups.*

Proof: If S is symmetric, then B is empty. Therefore by Theorem 2.2 if $A(T) = S$ then $T \subseteq B \cup S = S$, i.e. $T = S$. In the other direction, if S is not symmetric, then B is non empty and $B \cup S \neq S$ and hence $P(S) \geq 2$ $\square$

Proposition 2.5. *Pseudosymmetric semigroups have $P(S) = 2$.*

Proof: For Pseudosymmetric semigroups, $B = \{\frac{F}{2}\}$. Since $T \subseteq B \cup S$, either $T = S$ or $T = S \cup \{\frac{F}{2}\} = B \cup S$. Therefore $P(S) = 2$. Note that the converse is not true. $\square$

2.1 Determining Semigroups with a Given Void

With the void established, the natural following step is to determine its pre-image.

Lemma 2.6. *A void with Frobenius number F has an even number of elements if F is odd, and odd if F is even. If the Frobenius number is even, $\frac{F}{2}$ is always in the void.*

For finite $B \subseteq \mathbb{N}$, we say B is a *self-dual set* if there exists $N \in \mathbb{N}$ such that $b \in B \iff N - b \in B$.

Lemma 2.7. *Every self-dual set is the void of some numerical semigroup.*

Proof: Represent the complement of the void as $\{a_1, a_2, \dots a_n, N - a_n, \dots, N - a_1\}$. Let S be $\{0, N - a_n, N - a_{n-1}, \dots, N - a_1, N + 1 \rightarrow\}$. This is a semigroup closed under addition whose void is precisely the elements not in $\{a_1, a_2, \dots a_n, N - a_n, \dots, N - a_1\}$ (note that $F(S) = N$). $\square$

Definition 2.8. *For a self-dual set B, the* Diov *$V(B)$ is the set of semigroups which have B as their void; i.e. $V(B) = \{S \mid B(S) = B\}$*

The following examples serve to illustrate the properties of $V(B)$ (note that by Lemma 2.6, $2 \nmid F - |B|$):

Example 2.9. *If $|B| = F - 1$, every number less than the Frobenius number is in the void, so clearly the only possible semigroup is $\{0, F+1 \rightarrow\}$, so $|V(B)| = 1$.*

Example 2.10. *If $|B| = F - 3$, the complement of the void is simply $(a, F - a)$, so the only possible semigroup is the one described in lemma 2.7, $\{0, F - a, F + 1 \rightarrow\}$, so $|V(B)| = 1$.*

Example 2.11. *If $|B| = F - 5$, $|V(B)| = 1$ or $|V(B)| = 2$.*

Proof: By default, the semigroup described in lemma 2.7 has void B. Denoting the complement of the void as $\{a, b, F-b, F-a\}$, there is also an additional semigroup if some combination of $(a, F-b)$, $(b, F-a)$, and (a, b) is in S. Note $a \notin S$, because if $a, b \in S$, then $a+b < F$ so $a+b \in S$ which is impossible, and if $a, F-b \in S$, since $a < b$, $a+F-b \in S$ which is also a contradiction. So the only additional possibility is $b, F-a \in S$. Then, $2b = F-a$, and in this case, $V(B) = 2$.

Example 2.12. *If $|B| = F-7$, $|V(B)| = 1, 2, 3$.*

Proof: If the complement of the void is $\{a, b, c, F-c, F-b, F-a\}$, the nontrivial semigroups with void B must contain $\{c, F-b, F-a\}$ or $\{b, F-c, F-a\}$. From the same argument as the previous example, a cannot be in S, so $F-a \in S$. Then, b and c cannot simultaneously be in S because $2b < 2c < F$, so these are the only two possibilities.

For $\{c, F-b, F-a\}$, $2c = F-a$ or $2c = F-b$. For $\{b, F-c, F-a\}$, $2b = F-c$ and $3b = F-a$. If both $2c = F-a$ and $2b = F-c$ and $3b = F-a$, then $2c = 3b$, so if the complement of the void has form $\{n, 2n, 3n, 4n, 5n, 6n\}$, it is the void of three different semigroups. Otherwise, it has $V(B) = 2$ or $V(B) = 1$.

Theorem 2.13. *For a given F and for each possible length $|B| \neq 1, 3$ there is at least one B with $N = F, V(B) = 1$.*

Proof: If F is odd, we must have $|B| = 2k$, so let $B = \{1, 2, \ldots, k, F-k, F-k+1, \ldots F-1\}$. We claim $|V(B)| = 1$.

If S is a semigroup with void B, then it cannot contain any elements less than $F/2$. Suppose this was not the case; i.e., let $m(S) < F/2$. Then, $F-m$ must not be in S. Since $F-1 \notin S$, $F-m-1$ also cannot be in S, but since $F-m-1 \notin B$, $m+1 \in S$. Continuing this process, we eventually find that $\lfloor \frac{F}{2} \rfloor \in S$. But then, $2\lfloor \frac{F}{2} \rfloor = F-1 \in S$, which is a contradiction. So $|V(B)| = 1$.

Similarly, if F is even we must have $|B| = 2k+1$ and $k > 1$, so let $B = \{1, 2, \ldots, k, \frac{F}{2}, F-k, F-k+1, \ldots F-1\}$. We again claim $|V(B)| = 1$.

Again, suppose $m(S) < \frac{F}{2}$. Since $F, F-1 \notin S$, $F-m \notin S$ and $F-m-1 \notin S$, but then $m+1 \in S$. Continuing, we get $\frac{F}{2} - 1 \in S$. But then, $2(\frac{F}{2} - 1) = F-2 \in S$, which is a contradiction as $F-2 \in B$. Thus, $|V(B)| = 1$. $\square$

2.2 The Void Poset

Definition 2.14 (Void Poset)**.** *For a Numerical semigroup S, consider the poset on $B(S)$ with $a, b \in B$, $a \preccurlyeq b$ iff $b-a \in S$. This poset shall henceforth be referred to as the* Void Poset*.*

Example 2.15. *The B poset of $S = \{0, 4, 8, 10 \rightarrow\}$, $B = \{2, 3, 6, 7\}$ is*

$$\begin{array}{cc} 6 & 7 \\ | & | \\ 2 & 3 \end{array}$$

And the B poset of $S = \langle 6, 25, 29\rangle$ is

$$
\begin{tikzcd}
 & 52 & \\
23 \arrow[ur, no head] & & 46 \arrow[ul, no head] \\
 & 17 \arrow[ul, no head] \arrow[ur, no head] &
\end{tikzcd}
$$

The void poset has many useful structural properties, as outlined below:

Recall a poset is *self-dual* if there exists an isomorphism $\phi : P \to P$ such that $a \preccurlyeq b \iff \phi(b) \preccurlyeq \phi(a)$

Proposition 2.16. *The B poset is self-dual.*

Proof: If $a \preccurlyeq b$, then $F - b \preccurlyeq F - a$, as $b - a = (F - a) - (F - b)$ $\square$

This transformation will serve several purposes in the future, so we shall name it:

Definition 2.17 (Conjugation)**.** *For $a \in B$, we define $\overline{a} = F(S) - a$ as the* conjugate *of a.*

Corollary 2.17.1. *P is maximal $\iff$ $\bar{P}$ is minimal*

Theorem 2.18. *The set of maximal elements of B poset is precisely $PF(S) \setminus \{F(S)\}$*

Proof: Let a be a maximal element of $B(S)$. Then, $\nexists x \in B$ such that for some $s \in S$, $a + s = x$. So $\forall s \in S$, either $a + s \in S$, or $a + s \in \mathit{Gaps} \setminus B$. In the latter case, then $F - a - s \in S$, but then $F - a - s + s = F - a \in S$ which is a contradiction, so $a + s \in S$. By definition, then $a \in PF(S) \setminus \{F(S)\}$.

In the other direction, since we know $PF(S) \setminus \{F(S)\} \subseteq B(S)$, we only need to show these elements are also maximal. Let $a \in PF(S) \setminus \{F(S)\}$. For the sake of contradiction, assume there exists some $x \in B$ such that $\exists s \in S$ with $a + s = x$. But $a \in PF(S)$ implies $a + s \in S$, which is a contradiction, so a must be maximal. $\square$

Proposition 2.19. *If y covers x, then $y - x \in \mathcal{A}(S)$*

Proof: If $x, y \in B$ and $y - x = s_1 + s_2$ where $s_1, s_2 \in S \setminus \{0\}$, then let $z = x + s_1$. $z \in (B + S) \subseteq (B \cup S)$.

If $z \in S$ then $y = z + s_2 \in S$ which is impossible. So $z \in B$, $x \preccurlyeq z \preccurlyeq y$ and $z \neq x$, $z \neq y$. Therefore y does not cover x, contradiction.

Corollary 2.19.1. *If S has r atoms less than F, then each point of the B-Poset can have at most r direct edges above it, one for each atom.*

Proposition 2.20. *If $a \in \mathcal{A}(S)$, $x + a \notin B$, and $x \preccurlyeq y$, then $y + a \notin B$.*

Proof: We are given $x, y \in B$ and $x + a, y - x \in S$. It follows that $y + a = (x + a) + (y - x) \in S$.

Corollary 2.20.1. *If $x, y \in B$, $x \preccurlyeq y$ then number of edges directly above y is at most the number of edges above x*

Proposition 2.21. *Suppose $a \preccurlyeq x \preccurlyeq b$; then $y = a + b - x \in B$ with $a \preccurlyeq y \preccurlyeq b$*

Proof: If $y \in S$ then $b = y + (x - a) \in S$ which is impossible.

If $F - y \in S$ then $F - y = F + x - a - b$ so $F - a = (F - y) + (b - x) \in S$ which is again impossible.

So $y \in B$ and $b - y = x - a \in S$, $y - a = b - x \in S$ so $a \preccurlyeq y \preccurlyeq b$

As it turns out, the Void Poset can be obtained from the Apery Set by first constructing the *Gap*-Poset, which is the set of Gaps with $x \preccurlyeq y$ iff $y - x \in S$, and then deleting everything below the Frobenius Number.

3 The Void Poset and Good Numerical Sets

Recall that an *Order Ideal* of a poset is a subposet I where $x \in I, x \preccurlyeq y$ implies $y \in I$

Proposition 3.1. *Let $I \subseteq B$, then $S \subseteq A(I \cup S)$ iff I is an order ideal of the Void Poset*

Proof: First, assuming I is an order ideal, if $s \in S$ we want to show $s + I \subseteq S \cup I$. Pick $a \in I$

- Case 1: if $s + a \in S$, this works.
- Case 2: if $s + a \in Gap \setminus B$, $F - s - a \in S$, so $F - a = F - s - a + s \in S$, so $a \notin B$, which is a contradiction, so this case is not possible.
- Case 3: if $s + a \in B$, $a \preccurlyeq s + a$ in B, so $s + a \in I$, so this works.

Thus, $S \subseteq A(I \cup S)$.

Conversely if $S \subseteq A(I \cup S)$ then given $s \in S$ and $a, a + s \in B$, If $a \in I$ then $s + I \in S \cup I$, here $s + a \in B$ so $s + a \in I$. Thus I is an order ideal. $\square$

With this refinement of Theorem 2.2 in hand, we now have enough theory in place to tackle the following theorem:

Theorem 3.2. *For a semigroup S, $t(S) = 2$ implies that $P(S) = 2$.*

Lemma 3.3. *If $P = \max(B)$ and $A(T) = S$ then $P \in T$ implies $\bar{P} \in T$*

Proof: Since $P \in T \setminus A(T)$, we need $x \in T$ such that $P + x \notin T$. Since $P = \max(B)$, $P + x \notin B$, so $P + x = F$. $\square$

Proof of Theorem 3.2: If $t(S) = 2$, $|PF(S) \setminus \{F(S)\}| = 1$, and so B has a unique maximal element. Thus, B must have a unique minimal element by 2.17.1.

In the B poset, by Proposition 3.1, if an element x is in a numerical set T, then every element above x in the poset must also be in the numerical set. Thus if any element of B is in T, we must have $P \in T$; furthermore, by Lemma 3.3 $\bar{P} \in T$; since this the unique minimal element, all of B lies above it and hence $T = B \cup S$. Thus, either $T = S$, or $T = B \cup S$, so $P(S) = 2$. □

3.1 Self-Dual Order Ideals

We've seen the importance of order ideals and the self-duality of the Void Poset previously; combining these properties yields even more powerful results.
Note: When a self-dual order ideal I is referred to in this paper, it will be assumed that the isomorphism under which I is self-dual is the same as the original poset.

Proposition 3.4. *If I is a self-dual order ideal of the Void Poset, then $A(I \cup S) = S$*

Proof: Given a self dual order ideal I, we know by Proposition 3.1 that $S \subseteq A(I \cup S)$. Given $a \in I$, by definition $F - a \in I$ and $a + F - a \notin I \cup S$. So $a + (I \cup S) \nsubseteq (I \cup S)$ and $a \notin A(I \cup S)$. Hence $A(I \cup S) = S$.

Proposition 3.5. *If I is a self-dual order ideal, then $a \in I, b \preccurlyeq a \Rightarrow b \in I$*

Proof: $a \in I \Rightarrow \bar{a} \in I \Rightarrow \bar{b} \in I \Rightarrow b \in I$ □

Proposition 3.6. *A self dual order ideal is determined by which Pseudo-Frobenius numbers are contained in it.*

Proof: If $I_1 \cap PF(S) = I_2 \cap PF(S)$ then given $x \in I_1$ pick a maximal element above it $x \preccurlyeq a$. Now $a \in I_1 \cap PF(S)$ so $a \in I_2$ and by lemma 3.5 $x \in I_2$. So $I_1 \subseteq I_2$ and by symmetry $I_1 = I_2$ □

Definition 3.7. *The* Pseudo-Frobenius Graph $GPF(S)$ *is the graph with vertices $PF(S) \setminus \{F\}$ and edges $PQ \iff P + Q - F \in S$ (Note that this happens iff $\bar{P} \preccurlyeq Q \iff \bar{Q} \preccurlyeq P$)*

Theorem 3.8. *If I is a self dual order ideal then $I \cap PF(S)$ forms a union of connected components of $GPF(S)$*

Conversely if we take a union of connected components of $GPF(S)$ and then the order ideal generated by the conjugates of the chosen Pseudo-Frobenius numbers is a self dual order ideal.

Proof: Say the connected components of the graph are $C_1 \sqcup C_2 \sqcup \cdots \sqcup C_k$, and a subset of $\{1, \ldots, k\}$ as J.

First assuming I is an self dual order ideal. If $a \in I \cap PF(S)$, and $a, b \in C_i$ for some component of the graph, $\overline{a} \in I$, and $\overline{a} \preccurlyeq b$, so $b \in I$ so $C_i \subseteq I \cap PF(S)$.

Conversely, let I be the order ideal generated by the conjugates of $\bigcup_{i \in J}$ for some J. If $a \in I$, then $\exists b \in C_i$ such that $\overline{b} \preccurlyeq a$, and $\exists$ maximal c such that $a \preccurlyeq c$. Then $\overline{b} \preccurlyeq c$ so b and c are connected. Then, $c \in C_i$ so $\overline{c} \in I$, so since $\overline{c} \preccurlyeq \overline{a}$, $\overline{a} \in I$. Thus, I is self dual.

Corollary 3.8.1. *$P(S) \geq 2^{\kappa}$, where κ is the number of connected components of $GPF(S)$.*

3.2 General Order Ideals

Definition 3.9 (Red Triangles)**.** *Unordered triple $(a,b,c)^r$ where $a,b,c \in B$ is called a* Red triangle *if $a+b+c=F$.*

Lemma 3.10. *$(a,b,c)^r$ is a red triangle iff $a+b=\overline{c}$ iff $b+c=\overline{a}$ iff $a+c=\overline{b}$*

Theorem 3.11. *Let $I \subseteq B$, then $S = A(I \cup S)$ iff*

- *I is an order ideal of the B poset*
- *$\forall a \in I$ either $F-a \in I$ or there is a red triangle $(a,b,c)^r$ for which $b \in I$ and $F-c \notin I$*

Proof: Let $a \in I$; we need to ensure $a \notin A(I \cup S)$, which happens iff $a + (I \cup S) \nsubseteq I \cup S$. Because I is an order ideal, $a + S \subseteq I \cup S$, so we need to ensure $a + I \nsubseteq I \cup S$, which happens iff $\exists b \in I$ such that $a + b \notin I \cup S$

Case 1: $a+b \in Gap \setminus B$, so $F-a-b = s \in S$ i.e. $b \preccurlyeq F-a$ and hence $F-a \in I$.

Case 2: $a+b \in B \setminus I$, let $c = F-(a+b) \in B$ then $(a,b,c)^r$ is a red triangle and $b \in I$, $F-c = a+b \notin I$

The converse is trivial. □

Corollary 3.11.1. *If $|B(S_1)| = |B(S_2)|$ and the B Poset of S_2 is a refinement of the B poset of S_1 and the set of red triangles of S_2 is a subset of red triangles of S_1. Then $P(S_2) \leq P(S_1)$*

(Both the properties are checked under a common identification between the two Posets)

Definition 3.12. *We say that $a \in T$* satisfies *a triangle $(a,b,c)^r$ if $b \in T$, $\overline{c} \notin T$*

We can refine the previous theorem in the following manner:

Theorem 3.13. *Let $I \subseteq B$; then $S = A(I \cup S)$ iff*

i) I is an order ideal of the B poset

ii) $\forall P \in I \cap PF(S)$ either $F-P \in I$ or there is a red triangle $(P,b,c)^r$ which P satisfies

Proof: Say $A(I \cup S) = T \neq S$; then T is a numerical semigroup, with $S \subset T$. It follows that $T \setminus S$ is an order ideal of the Void Poset, so it must contain a maximal element P. However, $P \in I \cap PF(S)$ implies either $F-P \in I$, in which case $P + I \nsubseteq I$, or P satisfies some $(P,b,c)^r$, i.e. $P+b = F-c \notin I$, so again $P + I \nsubseteq I$. Thus $P \notin A(I \cup S)$ and we have a contradiction. □

This Theorem allows for the current algorithm we use to determine $P(S)$ (see Appendix A). It also allows us to henceforth ignore red triangles which do not include Pseudo-Frobenius numbers.

3.3 Structure among Red Triangles

Proposition 3.14. *If $Q \in PF(S) \setminus \{F\}$ and $(Q,a,b)^r$ is a red triangle then $x \prec a$ implies $x \prec F-b$*

Proof: We know that $Q+a = F-b$. Say $x = a-s$, $s \in S \setminus \{0\}$; then $(F-b)-x = Q+s \in S$ because Q is a Pseudo-Frobenius number and $s \neq 0$

Corollary 3.14.1. *If $P, Q \in PF(S) \setminus \{F\}$ and $P-Q \in B$ then $x \prec P-Q$ implies $x \preccurlyeq P$*

Proof: $(Q, P-Q, F-P)$ is a red triangle

Corollary 3.14.2. *If $Q \in PF(S) \setminus \{F\}$ and $(Q,a,b)^r$ is a red triangle then $\overline{b} \prec x$ implies $a \prec x$*

Proof: $\overline{b} \prec x \implies \overline{x} \prec b \implies \overline{x} \prec \overline{a} \implies a \prec x$

If $(Q,a,b)^r$ is satisfied then $Q, a \in T$, $F-b \notin T$ and so $x \prec a$ implies $x \notin T$, so a is a minimal element of T. Furthermore, $\overline{b} \prec y \implies y \in T$, so $\overline{b}$ is a maximal element of $B \setminus T$.

Corollary 3.14.3. *If $(a,b,c)^r$ is a red triangle with $b \preccurlyeq c$ and we pick an intermediate element $b \preccurlyeq x \preccurlyeq c$, then if $y = b+c-x$, $(a,x,y)^r$ is another red triangle.*

Lemma 3.15. *If $(a,b,c)^r$ is a red triangle, $x \preccurlyeq a$ and $x \not\preccurlyeq \bar{c}$, then $y = a+b-x \in B$, $b \preccurlyeq y$ and $(x,y,c)^r$ is another red triangle.*

Proof: If $y \in S$, $y = a+b-x = F-c-x = (F-x)-c$ which contradicts $c \not\preccurlyeq F-x$

If $F-y \in S$, $F-y = F-a-b+x$ so $F-b = (F-y)+(a-x) \in S$ which is a contradiction.

Therefore $y \in B$, $y-b = a-x \in S$ and $(x,y,c)^r$ is a red triangle. □

Notice that this theorem does not rely on the order of the triple, and thus is true for any permutation of $(a,b,c)^r$.

Corollary 3.15.1. *If If $(P,a,b)^r$ is a red triangle, $x \preccurlyeq a$ then $y = a+b-x \in B$, $b \preccurlyeq y$ and $(P,x,y)^r$ is another red triangle.*

The next corollary is incredibly powerful, and will motivate the rest of the section:

Corollary 3.15.2. *If P is a Pseudo-Frobenius number, it has a triangle $(P,a,b)^r$. Then if $F-Q \preccurlyeq b$ for some Pseudo-Frobenius number Q then $Q-P \in B$ and $a \preccurlyeq Q-P$*

Proof: $a+b-(F-Q) = (F-P)-(F-Q) = Q-P$

Definition 3.16. *Given a Pseudo-Frobenius number P, $Tri(P) = \{a \in B | \exists b \in B \text{ s.t } (P,a,b)^r \text{ is a red triangle}\}$*

Lemma 3.17. *If $P \in PF(S) \setminus \{F\}$. Then $Tri(P) = \{a | \exists Q \in PF(S)\{F\}$ s.t. $Q - P \in B$ and $a \preccurlyeq Q - P\}$*

Proof: Corollary 3.15.2 tells us $Tri(P) \subseteq \{a | \exists Q \in PF(S)\{F\}$ s.t. $Q - P \in B$ and $a \preccurlyeq Q - P\}$

If $Q - P \in B$ then $(P, Q - P, F - Q)^r$ is a red triangle and by corollary 3.15.1 $a \preccurlyeq Q - P \implies a \in Tri(P)$

Definition 3.18. *If T is an order ideal of B, define $Tri(T) = \{(a, b) \in B^2 \mid \exists P \in T \cap PF(S), P$ satisfies $(P, a, b)^r\}$, $X_1(T) = \{a \in B \mid \exists b \in B, (a, b) \in Tri(T)\}$, $X_2(T) = \{\bar{b} \in B \mid \exists a \in B, (a, b) \in Tri(T)\}$, and $Mi(T) = \{\bar{P} \mid P \in T \cap PF(S)\}$*

Lemma 3.19. *If $(P_1, a_1, b_1)^r$ and $(P_2, a_2, b_2)^r$ are red triangles, then $\overline{b_2} \preccurlyeq a_1$ implies $a_1 = \overline{b_2}$ or $a_2 \preccurlyeq \overline{b_1}$*

Proof: So $a_1 - (F - b_2) \in S$, but $a_1 + b_2 - F = (F - P_1 - b_1) + (F - P_j - a_2) - F = F - P_1 - P_2 - b_1 - a_2$. Now as P_1 and P_2 are Pseudo-Frobenius numbers $F - a_2 - b_1 \in S$ (unless $F - P_1 - P_2 - b_1 - a_2 = 0$ i.e. $a_1 - (F - b_2) = 0$). Finally $F - a_2 - b_1 \in S$ means $a_2 \preccurlyeq F - b_1$

Corollary 3.19.1. *$X_1(T) \cup X_2(T) \cup Mi(T)$ is an anti-chain*

Proof: Lemma 3.14 implies that if $x, y \in X_1(T)$ then $x \parallel y$. On the other hand if $x, y \in X_2(T)$. Then say $(P, a_1, \overline{x})$ and $(Q, a_2, \overline{y})$ are the corresponding triangles. Then $P + a + 1 = x$ and $Q + a_2 = y$. If possible assume $x \not\parallel y$ and $x \neq y$. WLoG say $x \prec y$ i.e. $y - x \in S$. But $y - x = y - P - a_1$. $y - x \neq 0$ and P is a Pseudo-Frobenius number therefore $y - a_1 = (y - x) + P \in S$. But this contradicts corollary **??**. $Mi(T) \cup X_2(T)$ is obviously an anti-chain. If possible assume $Mi(T) \cup X_1(T)$ is not an anti-chain so $\exists a \in X_1(T), F - P \in Mi(T)$ s.t. $F - P \prec a$. Say $(a, b) \in Tri(T)$ then by above $F - P \preccurlyeq F - b$ which implies $F - b \in T$ which is a contradiction.

3.4 Normalizations of Order Ideals

Definition 3.20. *If I is an order ideal of B, define its* Lower Normalization *$Nl(I)$ to be the order ideal of B generated $I \cap PF(S)$, $Mi(I)$ and $X_1(I)$*

Note that $Mi(I) = Mi(Nl(I)), I \cap PF(S) = Nl(I) \cap PF(S)$ follow trivially from the definition.

Lemma 3.21. *Given an order ideal I of B, $A(I \cup S) = S$ implies $A(Nl(I) \cup S) = S$.*

Moreover, $Tri(I) \subseteq Tri(Nl(I))$ and $X_1(Nl(I)) \subseteq (I \cap PF(S)) \cup Mi(I) \cup X_1(I)$.

Proof: Firstly, observe that $Nl(I) \subseteq I$ and $X_1(I) \subseteq Nl(I)$ imply $Tri(I) \subseteq Tri(Nl(I))$.

From theorem 3.13 it follows that $A(I \cup S) = S \implies A(Nl(I) \cup S) = S$.

Moreover $(a, b) \in Tri(Nl(I))$ implies $\exists x \in (I \cap PF(S)) \cup Mi(I) \cup X_1(I)$ s.t. $x \preccurlyeq a$. And hence $X_1(Nl(I)) \subseteq (I \cap PF(S)) \cup Mi(I) \cup X_1(I)$. $\square$

Remark 3.22. *We don't necessarily have $Tri(Nl(T)) = Tri(T)$, even if we assume max-embedding dimension*

For e.g. $S =< 7, 29, 16, 31, 25, 26, 34 >$, $T = (3, 5, 9, 10, 12, 17, 18, 19, 24)$, $Nl(T) = [18, 9, 3, 10, 17, 24, 19]$, $Tri(T) = [[3, 5]]$ *and* $Tri(Nl(T)) = [[3, 15], [3, 5]]$

Definition 3.23. *If T is an order ideal of B we define* $Nu(T) = \{x|\forall y \in X_2(T) x \not\preccurlyeq y \text{ and } (x \preccurlyeq P, P \in PF(S) \implies P \in T)\}$

Lemma 3.24. $A(T \cup S) = S \implies A(Nu(T) \cup S) = S$

Proof: Follows from theorem 3.13

Lemma 3.25. $Nl(T) \subseteq T \subseteq Nu(T)$

Lemma 3.26. $Nl(Nl(T)) = Nl(T)$ *and* $Nu(Nu(T)) = Nu(T)$

Proof: It is clear that $Nl(Nl(T)) = Nl(T)$ because $Nl(T) \cap PF(S) = T \cap PF(S)$, $Mi(Nl(T)) = Nl(T)$ and $X_1(T) \subseteq X_1(Nl(T))$

Similarly $Nu(Nu(T)) = Nu(T)$ because $Nu(T) \cap PF(S) = T \cap PF(S)$ and $X_1(T) \subseteq X_1(Nu(T))$

Definition 3.27. *An order ideal T of B is called lower Normalised if $Nl(T) = T$. It is called upper Normalised if $Nu(T) = T$.*

Theorem 3.28. *If $A(T_1 \cup S) = S$ and $Nl(T_1) \subseteq T \subseteq Nu(T_1)$ then $A(T \cup S) = S$*

Proof: We know that $Nl(T_1) \cap PF(S) = T \cap PF(S) = Nu(T_1) \cap PF(S)$. Now given $P \in T \cap PF(S)$

- If $\overline{P} \in T_1$ then $\overline{P} \in Nl(T_1)$ and $\overline{P} \in T$
- If $\overline{P} \notin T_1$ then by theorem 3.13 there is a red triangle (P, a, b) s.t. $a \in T_1$ and $\overline{b} \notin T_1$. Now $a \in Nl(T_1)$ and hence $a \in T$. Also $\overline{b} \notin Nu(T_1)$ so $\overline{b} \notin T$

Corollary 3.28.1. *If T, T_1 are as in the theorem then $T \cap PF(S) = T_1 \cap PF(S)$, $Mi(T_1) \subseteq Mi(T)$ and $Tri(T_1) \subseteq Tri(T)$*

3.5 Differences of Pseudo-Frobenius Numbers

Remark 3.29. *Our study of Numerical Semigroups of type 3 suggests that differences of Pseudo-Frobenius numbers play a key role in determining $P(S)$*

Lemma 3.30. *If $P, Q \in PF(S) \setminus \{F\}$, $P - Q \in B$, moreover $\forall R \in PF(S) \setminus \{P, F\} R - Q \notin B$ and $\exists R_1 \in PF(S) \setminus \{F\}$ s.t. $P - Q \preccurlyeq R_1$. Moreover if we assume that every good numerical set that has R_1 also has $F - R_1$. Then Q cannot satisfy a red triangle.*

Proof: Say Q satisfies a red triangle (Q, a, b) then by corollary 3.15.2 $a, b \preccurlyeq P - Q$. $a \in T \implies P - Q \in T \implies R_1 \in T \implies F - R_1 \in T \implies F - (P - Q) \in T \implies F - b \in T$. So the triangle cannot be satisfied.

Definition 3.31. *A numerical semigroup is called P-minimal if $P(S) = 2^k$.*

Lemma 3.32. *If $P, Q \in PF(S) \setminus \{F\}$, $P - Q \in B$, and $\forall R \in PF(S) \setminus \{Q, F\}$ $P - Q \not\preccurlyeq R$ then S is not P-minimal*

Proof: Q is the only maximal element above $P - Q$, hence $F - Q$ is the only minimal element below $F - (P - Q)$. Let $Y = \{x | x \preccurlyeq F - (P - Q)\}$, $T' = B \setminus Y$. Then T' is an order ideal, all Pseudo-Frobenius numbers except Q have their conjugates in T'. Moreover $(Q, F - P, P - Q)$ is a red triangle, $F - P \in T'$ and $\overline{P - Q} \notin T'$, thus the triangle is satisfied and by theorem 3.13 $A(T' \cup S) = S$

Finally T' is not self dual since $Q \in T'$, $F - Q \notin T'$ ($Q = F - (P - Q)$ iff $F = P$ which is impossible)

Theorem 3.33. *Let $PF(S) = P_1 < P_2 < \cdots < P_{t-1} < F$, If for exactly one pair $i < j$ $P_j - P_i \in B$ then:*

- *If $\nexists k \neq i$ s.t. $P_j - P_i \preccurlyeq P_k$ then $P(S) > 2^k$, S then S is not P-minimal*
- *If $\exists k \neq i$ s.t. $P_j - P_i \preccurlyeq P_k$ then $P(S) = 2^k$ and S is P-minimal*

Proof: The first case follows from lemma 3.32

In the second case P_i is the only Pseudo-Frobenius number with a red triangle by lemma 3.15.2. Moreover P_k does not have a red triangle and hence by lemma 3.30 Q does not satisfy a red triangle either. Therefore $P(S) = 2^k$

Definition 3.34 (DPF-Poset)**.** *DPF-Poset is the poset whose set of vertices is $(PF(S) \cup \{P - Q | P, Q \in PF(S), P - Q \in B\}) \setminus \{F\}$. The poset structure is induced from the B-Poset*

Definition 3.35. $DPF(S) = \{P - Q | P, Q \in PF(S) \setminus \{F\}, P - Q \in B\}$

Lemma 3.36. *Say $P \in PF(S) \setminus \{F\}$, $A \subseteq PF(S) \setminus \{P, F\}$, $A \neq \emptyset$ If $Q \in A \implies P - Q \in B$ and $R \notin A, Q \in A \implies P - Q \not\preccurlyeq R$ then S is not P-minimal.*

Proof: Let $T = \{x | \exists Q \in A, P - Q \preccurlyeq x\}$

If $Q \in T \cap PF(S)$ then $\exists Q' \in A$ s.t. $P - Q' \preccurlyeq Q$. $(Q, P - Q, F - P)$ is a red triangle, $P - Q \in T$ and $P \notin T$. Hence Q satisfies a red triangle and by theorem 3.13 $A(T \cup S) = S$

We prove that T is not self dual. First notice that $P - Q$ s.t. $Q \in A$ are the minimal elements of T ($P - Q_1 \preccurlyeq P - Q_2 \implies Q_2 \preccurlyeq Q_1$), so it has $|A|$ minimal elements. If it is self dual then it has $|A|$ maximal elements and hence $A \subseteq T$. Now let Q be the smallest (according to usual order in $\mathbf{Z}$) element of A, $F - Q \in T \implies F - Q = P - Q'$ for some $Q' \in A$. Therefore $Q = (F - P) + Q' > Q'$ which is a contradiction.

Definition 3.37. *If $Q \in PF(S)$, $Q \neq F$ then $GPF_Q(S)$ is the graph obtained from $GPF(S)$ by deleting all edges involving Q*

Lemma 3.38. *If $P_1 + P_2 = F + Q$, $Q \neq \frac{F}{2}$ and P_1, P_2 are in different components of $GPF_Q(S)$ then S is not P-minimal*

Proof: Let Z be the order ideal generated by the conjugates of Pseudo-Frobenius numbers in the component of P_2 in $GPF_Q(S)$. Note $P_1 \notin Z$. Let $T = Z \cup \{Q\}$, T is also an order ideal. $R \in T \cap PF(S)$, $R \neq Q$ implies $F - R_1 \preccurlyeq R$ for some R_1 in connected component of P_2 of $GPF_Q(S)$, therefore R is in the same component and $F-R \in T$. $(Q, P_1-Q, F-P_1)$ is a red triangle, $P_1 - Q = F - P_2 \in T$ also $P_1 \notin T$ and hence $A(T \cup S) = S$ by theorem 3.13. Moreover T is not self dual because $Q \in T$, $F - Q \notin Z$ and $F - Q \neq Q$. And hence S is not P-minimal.

Lemma 3.39. *$Q \in PF(S) \setminus \{F\}$, Let $C = \{P | P - Q \in DPF(S)\}$. If $\forall P \in C$ $F - (P - Q) \notin PF(S)$ and $\forall P \in C$ $\exists R \in PF(S) \setminus \{F\}$ s.t. $P - Q \preccurlyeq R$ and R cannot satisfy a triangle. We slso assume that each $\forall P \in C$ P cannot satisfy a triangle. Then Q cannot satisfy a triangle either.*

Proof: Say (Q, a, b) is a Red triangle, say $F - P_1 \preccurlyeq b$ and $F - P_2 \preccurlyeq a$. Then by corollary 3.15.2 $a \preccurlyeq P_1 - Q$ and $b \preccurlyeq P_2 - Q$. Also say $P_1 - Q \preccurlyeq R_1$, $P_2 - Q \preccurlyeq R_2$ s.t. R_1 and R_2 cannot satisfy red triangles. Next we know that $F - P_2 \prec P_1 - Q$ (They are not equal) so by lemma 3.15 $F - P_2 \preccurlyeq P_1$

Now $a \in T \implies P_1 - Q \in T \implies R_1 \in T \implies F - R_1 \in T \implies P_2 \in T \implies F - P_2 \in T \implies P_1 \in T \implies F - P_1 \in T \implies R_2 \in T \implies F - R_2 \in T \implies F - (P_2 - Q) \in T \implies F - b \in T$

Lemma 3.40. *$Q \in PF(S) \setminus \{F\}$, Let $C = \{P | P - Q \in DPF(S)\}$. If $\forall P \in C$ $\exists R \in PF(S) \setminus \{F, Q\}$ s.t. $P - Q \preccurlyeq R$ And $\forall P_1, P_2 \in C$ (if $P_1 + P_2 = F + Q$ then P_1, P_2 belong to the same component of $GPF_Q(S)$). Moreover if no Pseudo-Frobenius number other than Q can satisfy a triangle.*

Then Q cannot satisfy a triangle.

Proof: Say (Q, a, b) is a Red triangle, say $F - P_1 \preccurlyeq b$ and $F - P_2 \preccurlyeq a$. Then by corollary 3.15.2 $a \preccurlyeq P_1 - Q$ and $b \preccurlyeq P_2 - Q$. Also say $P_1 - Q \preccurlyeq R_1$, $P_2 - Q \preccurlyeq R_2$ s.t. $R_1 \neq Q$ and $R_2 \neq Q$ so they cannot satisfy red triangles.

First we assume $F + Q \neq P_1 + P_2$ so we know that $F - P_2 \prec P_1 - Q$ (They are not equal) so by lemma 3.15 $F - P_2 \preccurlyeq P_1$. Now $a \in T \implies P_1 - Q \in T \implies R_1 \in T \implies F - R_1 \in T \implies P_2 \in T \implies F - P_2 \in T \implies P_1 \in T \implies F - P_1 \in T \implies R_2 \in T \implies F - R_2 \in T \implies F - (P_2 - Q) \in T \implies F - b \in T$. So the triangle cannot work.

Next if $F + Q = P_1 + P_2$ then $a \preccurlyeq P_1 - Q = F - P_2$ so $a = P_1 - Q = F - P_2$, similarly $b = P_2 - Q = F - P_1$. We know that there is a path in $GPF_Q(S)$ from P_2 to P_1: $P_2, Q_1, Q_2, \ldots, Q_n, P_1$. Now $a = F - P_2 \in T \implies Q_1 \in T \implies F - Q_1 \in T \implies Q_2 \in T \cdots \implies Q_n \in T \implies F - Q_n \in T \implies P_1 \in T$, $P_1 = F - b$ so the triangle cannot work.

Theorem 3.41. *Say S has type 4, $PF(S) = R < Q < P < F$, $GPF(S)$ has k connected components. Then S is P-minimal iff all of the following holds:*

- *not both $P - Q, P - R$ are in B*
- *If $P - Q \in B$ then $P - Q \preccurlyeq R$*

- *If $P-R \in B$ then $P-R \preccurlyeq Q$*
- *If $Q-R \in B$ then $Q-R \preccurlyeq P$*

The only exception being if $F-P = Q-R$, $F \neq 2R$ in which case S is not P-minimal

Proof:
Case 0: None of $P-Q$, $P-R$, $Q-R$ are in B: Then $P(S) = 2^k$ by lemma **??**

Case 1: Exactly one of them is in B
This case has been done in Theorem 3.33

Case 2: $P-Q$ and $P-R$ are in B

- $P-Q \not\preccurlyeq P$, $P-R \not\preccurlyeq P$, so $A = \{Q, R\}$ in lemma 3.36 $P(S) > 2^k$

Case 3: $P-Q, Q-R$ are in B: By lemma 3.15.2 $F-P \preccurlyeq P-Q$ and $F-Q \preccurlyeq Q-R$

- $P-Q \not\preccurlyeq R$ or $Q-R \not\preccurlyeq P$

  then by lemma 3.32 $P(S) > 2^k$

- $P-Q \preccurlyeq R$ and $Q-R \preccurlyeq P$

  By lemma 3.30 R cannot satisfy a red triangle (as P cannot)

  And by a further application of lemma 3.30 Q cannot satisfy a red triangle either Therefore $P(S) = 2^k$

Case 4: $P-R, Q-R$ are in B: Notice that $F-Q \preccurlyeq P-R$ iff $F-P \preccurlyeq Q-R$. $P-R, Q-R$ cannot be above $F-R$ by lemma 3.15.2

- $Q-R \not\preccurlyeq P$ or $P-R \not\preccurlyeq Q$

  Then by Lemma 6.2 $P(S) > 2^k$

- $F = 2R$

  Then $R = F-R$ and every nemerical set is self dual, $P(S) = 2^k$ (Note that $F = 2R \implies Q-R \not\preccurlyeq R \implies Q-R \preccurlyeq P$, similarly $F = 2R \implies P-R \preccurlyeq Q$)

- $P-R \preccurlyeq Q$ and $Q-R \preccurlyeq P$, $F \neq 2R$

  Note that R is the only Pseudo-Frobenius number with a triangle by corollary 3.15.2.

  If $F+R \neq P+Q$ then by lemma 3.39 S is P-minimal

  If $F+R = P+Q$ then $P-(F-Q) = R \notin S$ hence $GPF_R(S)$ is completely disconnected and hence by lemma 3.38 S is not P-minimal ($R \neq \frac{F}{2}$)

  Case 5: $P-Q, P-R, Q-R$ are all in B Let $A = \{Q, R\}$, $P-R \not\preccurlyeq P$ and $P-Q \not\preccurlyeq P$ so by lemma 3.36 $P(S) > 2^k$

Theorem 3.42. *If the graph $GPF(S)$ is completely disconnected then S is not P-minimal iff $\exists R_1, R_2, R_3 \in PF(S) \setminus \{F\}$ s.t. $F + R_3 = R_1 + R_2$ with $R_1 \neq R_2$ $R_3 \neq \frac{F}{2}$*

Proof: First assuming no such R_1, R_2, R_3 exist. Say the Pseudo-Frobenius numbers are $F > P_1 > P_2 > \cdots > P_n$. We proceed by strong induction to show that no P_i can satisfy a red triangle. The base case is clear, P_1 cannot satisfy a red triangle.

If $P_1, \ldots P_{m-1}$ cannot satisfy a red triangle. If $P_m = \frac{F}{2}$ then $F - P_m = P_m$ so it doesn't need a triangle, so now assume $P_m \neq \frac{F}{2}$. Say (P_m, a, b) is a red triangle. $a = F - P_m - b < F - P_m$, so $F - P_i \preccurlyeq a \implies i \leq m-1$, similarly say $F - P_j \preccurlyeq b$ then $j \leq m-1$. If possible assume $i \neq j$. Now by corollary 3.15.2 $b \preccurlyeq P_i - P_m$ and $a \preccurlyeq P_j - P_m$. So $F - P_i \prec P_j - P_m$ ($F - P_i \neq P_j - P_m$ otherwise $F + P_m = P_i + P_j$) so by lemma 3.14 $F - P_i \preccurlyeq P_j$. Now $GPF(S)$ being completely disconnected implies $i = j$, so $F - P_i \preccurlyeq a, b$. Also $GPF(S)$ being completely disconnected implies $a, b \preccurlyeq P_i$. Finally $a \in T \implies P_i \in T \implies F - P_i \in T \implies F - b \in T$ (here we used $F - P_i \preccurlyeq F - b$ which is obtained from conjugation from $b \preccurlyeq P_i$). So the triangle cannot work. And by strong induction S is P-minimal.

Next if $F + R_3 = R_1 + R_2$ with $R_1 \neq R_2$ and $R_3 \neq \frac{F}{2}$. Let Z be the order ideal generated by $F - R_1$ and $T = Z \cup \{R_3\}$. $GPF(S)$ is completely disconnected so $x \in Z \implies x \preccurlyeq R_1 \implies F - R_1 \preccurlyeq F - x \implies F - x \in Z$. $(R_3, F - R_1, F - R_2)$ is a red triangle, $F - R_1 \in T$, $R_2 \notin T$ (as $R_2 \neq R_1$ and $R_2 = R_3 \implies F = R_1$ which is impossible).

Therefore $A(T \cup S) = S$, T is not self dual because $F - R_3 \neq R_3 \implies F - R_3 \notin T$ but $R_3 \in T$. So S is not P-minimal.

Lemma 3.43. *If $\exists P \in PF(S) \setminus \{F\}$, s.t. $\forall P_1, P_2 \in B, (P_1 - P_2 \in B \implies P_1 = P)$ Then S is not P-minimal iff $\exists A \subseteq PF(S) \setminus \{F\}$ s.t. $A \neq \emptyset$ $\forall Q \in A$ $P - Q \in B$ and $\forall Q \in A$ $P - Q \preccurlyeq R \implies R \in A$*

Proof: If such an A exists then by lemma 3.36 S is not P-minimal.

Conversely if no such A exists, say $C = \{Q | P - Q \in B\}$. C does not satisfy the condition of A, so $\exists Q_1 \in C$ s.t. $P - Q_1 \preccurlyeq R$ for some $R \notin C$. It follows that R does not have a red triangle and hence by lemma 3.30 Q_1 cannot satisfy a red triangle either. Now $C_1 = C \setminus \{Q_1\}$ does not satisfy the condition of A, so $\exists Q_2 \in C_1$, $P - Q_2 \preccurlyeq R_2$, $R_2 \notin C_1$ and $R_2 \notin C_1$ implies R_2 cannot satisfy a red triangle, so by lemma 3.30 Q_2 does not satisfy a Red triangle. Continuing this way no Pseudo-Frobenius number satisfies a triangle and hence S is P-minimal.

Lemma 3.44. *If $\exists A \subseteq PDF(S)$ s.t. $P - Q \in A \implies \nexists R$ s.t. $R - P \in A$. Define $C = \{Q | \exists P, P - Q \in A\}$. If we further have that $\forall (P - Q) \in A$ $P - Q \preccurlyeq R \implies R \in C$ then S is not P-minimal.*

Proof: Let T be the order ideal generated by A, then we know that $T \cap PF(S) \subseteq C$ so given $Q \in T \cap PF(S)$ $\exists P$ s.t. $P - Q \in A$, hence $(Q, P - Q, F - P)$ is a red triangle, moreover $P \notin C$ so $P \notin T$. Hence $A(T \cup S) = S$

Conjecture 3.45. *The DPF- Poset determines whether or not S is P-minimal. Here we assume not just the poset structure, but the knowledge of which elements are differences of which Pseudo-Frobenius numbers.*

Conjecture 3.46. *The stronger conjecture is that if we look at the containment poset of Non-Self-Dual Numerical sets that have the given Numerical Set as their associated semigroup. Then the minimal Numerical Sets in that poset are generated by elements of $DPF(S)$*

Remark 3.47. *The DPF-Poset cannot determine $P(S)$ in general, this is because for example in type 4 $P(S)$ can take arbitrary large values, but there are only finitely many DPF-Posets possible.*

Remark 3.48. *A common occurrence in Numerical Semigroups is that the only red triangles involving a Pseudo-Frobenius number that work are of the form $(Q, P-Q, F-P)$. However this is not always the case for e.g. consider $S =< 17, 38, 40, 65, 73, 81 >$, $T = \{x|25 \preccurlyeq x\}$.*

Moreover all examples I could find of numerical semigroups in which triangles not of this form are satisfied have $P_1, P_2, P_3, P_4 \in PF(S)\{F\}$ s.t. $P_1 - P_2, P_3 - P_4 \in B(S)$ and $P_1 - P_2 \preccurlyeq P_3 - P_4$ (which is quite rare)

Definition 3.49. *If $A(T \cup S) = S$ then we define $DP(T) = \{P - Q|P, Q \in PF(S) \setminus \{F\}, P - Q \in B, Q \in T, \overline{Q} \notin T \; \exists$ red triangle $(Q, a, b), a \preccurlyeq P - Q, a \in T, \overline{b} \notin T\}$*

Conjecture 3.50. *If $\forall P_1, P_2, Q \in PF(S) \setminus \{F\}$ $P_1 - Q, P_2 - Q \in DPF(S) \implies P_1 = P_2$. Then given T s.t. $A(T \cup S) = S$ Let T' be the order ideal generated by $DP(T)$ then $A(T' \cup S) = S$*

4 Containment Poset

Definition 4.1. *If $I \subseteq B$, $\overline{I} = \{x \mid \overline{x} \in I\}$. The* adjoint *of I is defined as $I^* = B \setminus \overline{I}$*

Lemma 4.2. *If I is an order ideal then I^* is also an order ideal*

Proof: If $x \preccurlyeq y$, $x \in I^*$ then $x \notin \overline{I}$, i.e. $\overline{x} \notin I$. $\overline{y} \preccurlyeq \overline{x}$ so $\overline{y} \notin I$ i.e. $y \notin \overline{I}$ i.e. $y \in I^*$

Theorem 4.3. *$A(I^* \cup S) = A(I \cup S)$, $I_1 \subseteq I_2 \iff I_2^* \subseteq I_1^*$ and $(I^*)^* = T$*

Proof: $a \in A(I^* \cup S)$ iff $\forall x \in I^* \cup S$ $a + x \in I^* \cup S$ iff $\forall y \notin I \cup S$ $\overline{a + \overline{y}} \notin I \cup S$ iff $\forall \overline{y} \in I^* \cup S$ $\overline{y + a} \in I^* \cup S$

And hence $A(I^* \cup S) = A(I \cup S)$.

$I_1 \subseteq I_2$ iff $\overline{I_1} \subseteq \overline{I_2}$ iff $B \setminus \overline{I_2} \subseteq B \setminus \overline{I_1}$ iff $I_2^* \subseteq I_1^*$

Finally $a \in I$ iff $\overline{a} \in \overline{I}$ iff $\overline{a} \notin I^*$ iff $a \notin \overline{I^*}$ iff $a \in (I^*)^*$. $\square$

Under the adjoint, by the above theorem, we have that the containment poset of numerical sets satisfying $A(T \cup S) = S$, ordered by inclusion, is self dual under the adjoint operation.

Theorem 4.4. *If $I \cup S$ is a Numerical Semigroup then $I^* = I$*

Proof: $F(I \cup S) = F(I) = F$ so $a \in I \implies F - a \notin I \implies a \in I^*$

Theorem 4.5. *If $\forall P \in PF(S) \setminus \{F\}$, for every triangle $(P, a, b)^r$, a, b are above conjugates of Pseudo-Frobenius numbers in the connected component of P in $GPF(S)$, then for any numerical set T satisfying $A(T) = S$, $A((T \cap I) \cup S) = S$ for every self-dual order ideal I.*

This shows that the containment poset is the product of smaller posets consisting of good numerical semigroups inside minimal self dual order ideals.

Proof: Follows from Theorem 3.13

Theorem 4.6. *If F is even then $P(S)$ is even*

Proof: $\frac{F}{2} \in T \iff \frac{F}{2} \notin T^*$ therefore $T \neq T^*$

5 $P(S)$ for Numerical Semigroups with fixed Frobenius Number

Theorem 5.1. *S_0 is a fixed numerical semigroup*
$\sum_{S_0 \subseteq S, F(S) = F(S_0)} P(S) = \#$ *order ideals of* $B(S_0)$

Proof: If T' is an order ideal of $B(S_0)$ then $A(T' \cup S_0)$ is a numerical semigroup that contains S_0 and has the same Frobenius number as S_0.

Conversely, if $S_0 \subseteq S$ and $F(S) = F(S_0)$, $A(T) = S$. Then we must have $T \subseteq S_0 \cup B(S_0)$ because otherwise $\exists a \in T$ s.t. $F - a \in S_0$ now $F - a \in S_0 \subseteq S = A(T)$ so $(F - a) + T \subseteq T$ which implies $F = (F - a) + a \in T$ which is impossible.

It follows that Numerical Sets corresponding to Numerical semigroups containing S_0 and having the same Frobenius number as S_0 are precisely the order ideals of $B(S_0)$ union with S_0 and the result follows.

Theorem 5.2. *Given m, F s.t. $m \not| F$, say $F = mq + r$ with $1 \leq r \leq m - 1$*
$\sum_{m \in S, F(S) = F} P(S) = (q + 2)^{r-1}(q + 1)^{m-r}$

Proof: Let $S_0 =< m, F + 1, F + 2, \ldots, F + m >$ (Note $F(S_0) = F$).

Next if $m \in S$, $F(S) = F$ then $S_0 \subseteq S$. And conversely if $S_0 \subseteq S$ and $F(S) = F$ then $m \in S$

Now the only atom of S_0 less than F is m, so the B-poset is very simple, it is the disjoint union of $r - 1$ chains with $q + 1$ points each and $m - r$ chains with q points each. And hence the number of order ideals is $(q + 2)^{r-1}(q + 1)^{m-r}$

Theorem 5.3. *If $S_1 = S \cup \{Q\}$ B-Poset of S_1 is obtained from the B-Poset of S as follows:*

Remove $Q, F - Q$ from the Void, for each red triangle (Q, a, b) add new relation $a \preccurlyeq F - b$ and $b \preccurlyeq F - a$.

Proof: It is clear that $B(S_1) = B(S) \setminus \{Q, F-Q\}$ moreover if $x, y \in B(S_1)$ and $y - x \in S$ then $y - x \in S_1$. New relations arise when $x, y \in B(S_1)$ and $y - x = Q$ (as Q is the only element of S_1 that is not an element of S). Note that $y - x = Q$ iff $Q + x + (F - y) = F$

Remark 5.4. *This gives a recursive method of computing $P(S)$ for each Numerical semigroup of a fixed Frobenius number. We start with the semigroup $\{0, F+1 \rightarrow\}$. Semigroups above existing semigroup S are $S \cup \{P\}$ for $P \in PF(S) \setminus \{F\}$ s.t. $2P \in S$. The void poset and red triangles of $S \cup \{P\}$ are obtained as stated earlier.*

Now we start with symmetric or pseudo-symmetric semigroups at the top, they have $P(S) = 1$ or2. We then move downwards, for each semigroup S we calculate the number of order ideals in it's void poset and and subtract the $P(S')$ for all S' that contain S (and have the same Frobenius number) to get $P(S)$

Remark 5.5. *We had guessed based on small F that If $P(S) = 2$, S is not Pseudo-Symmetric. Then S has a Pseudo-Frobenius number Q for which $2Q \in S$ and $P(S \cup \{Q\}) \in \{1, 2\}$*
It is False: $< 4, 9, 19 >$ only has Numerical Semigroups with $P(S) = 3$ directly above it
$< 7, 10, 18 >$ only has a Numerical Semigroup with $P(S) = 3$ directly above it
$< 10, 11, 18, 23 >$ only has a Numerical Semigroup with $P(S) = 6$ directly above it

6 Characterising all Good Numerical Sets when there is exactly one PF difference

This section was written quite early and checks red triangles for all points not just pseudofrobenuis numbers

Lemma 6.1. *If $P, Q \in PF(S) \setminus \{F\}$, $P - Q \in B$, $F - P \preccurlyeq P - Q$ and $\forall R \in PF(S) \setminus \{Q, F\}$ $P - Q \not\preccurlyeq R$ then*

Consider the graph $GPF(S)$ and delete all edges involving Q, the component of Q will break into several components

Say the graph now has $k + n + 1$ components ($n \geq 0$)(The point Q is a new component). Construct a set X by not including Q, not including the new component of P and randomly choosing whether or not the remaining $k + n - 1$ components are included.

Let I_1 be the order ideal generated by the conjugates of elements of X.

Let C be the collection of vertices originally connected to Q

Let $B_2 = \{x | F - Q \preccurlyeq x, x \not\preccurlyeq P, x \not\preccurlyeq Q\}$, Construct I to be an order ideal of B_2 that contains $X \cap C$. ($I = X \cap C$ works).

Let $B_1 = \{x | x \preccurlyeq Q$ and $x \not\preccurlyeq P\}$ (Note $P - Q \in B_1$ and B_1 is an order ideal)

Finally let Z be an order ideal of B_1 containing $P - Q$, for e.g. $Z = \{x | P - Q \preccurlyeq x\}$. Say there are s such order ideals ($s \geq 1$)

Finally letting $T_1 = (I \cup I_1 \cup Z)$ $A(T_1 \cup S) = S$ (This gives $\geq 2^{k+n-1}s$ numerical sets, all of which are non self-dual as $Q \in T_1$, $F - Q \notin T_1$)

Proof: First we need T_1 to be an order ideal; this is true because I_1, I and Z are order ideals. (Check that I, Z are actually order ideals of B)

Note that $P \notin T_1$

If $x \in I_1$ then $\exists R \in X$ (so $R \neq P, Q$) s.t. $F - R \preccurlyeq x$. $x \in T_1$ so $x \not\preccurlyeq P$, now if $x \preccurlyeq Q$ then $x \in B_1$, we can therefore assume $x \not\preccurlyeq Q$ (We do the case of B_1 later). Say $x \preccurlyeq R_1$, $R_1 \neq P, Q$,$F - R \preccurlyeq R_1$ implies that R and R_1 are in the same new connected component and hence $R_1 \in X$ and $F - R_1 \preccurlyeq F - x \in I$ and hence $F - x \in T_1$

Next if $x \in I$ then $\exists R \in X \cap C$ (R cannot be Q) s.t. $F - Q \preccurlyeq x \preccurlyeq R$, hence $F - R \preccurlyeq F - x \in I$ and hence $F - x \in T_1$

Lastly, if $x \in B_1$; $(Q, F-P, P-Q)$ is a red triangle, $x \in B_1$ implies $x \preccurlyeq Q$ and $x \not\preccurlyeq P \implies x \not\preccurlyeq F - (P - Q)$ so by lemma 3.15 $(x, y, P - Q)$ is also a red triangle where $y = Q + F - P - x$.

Now $P - Q \in T_1$, this is because $P - Q \in Z$.

Finally we need $F - y \notin T_1$; $F - y = P - Q + x$ implies $P - (F - y) = Q - x \in S$ and hence $F - y \preccurlyeq P$, $F - y \notin T_1$

Therefore $A(T_1 \cup S) = S$

T_1 is not self-dual because $P - Q \in Z$ and $P - Q \preccurlyeq Q$ so $Q \in T$. $F - P \preccurlyeq P - Q \preccurlyeq Q$ so $F - Q \preccurlyeq P$ and hence $F - Q \notin T_1$

Corollary 6.1.1. *If $P - Q = F - P$ then the number of such T_1 is 2^{k-1}*

Proof: Notice that in this case the connected component of Q in $GPF(S)$ is $\{P, Q\}$. So there are $k-1$ ways of choosing X and hence I_1 has 2^{k-1} choices. Also $B_2 = \emptyset$ and hence $I = \emptyset$. Lastly B_1 is the order ideal generated by $P - Q$ so Z must be the order ideal generated by $P - Q$

Lemma 6.2. *If $P, Q \in PF(S) \setminus \{F\}$, $P - Q \in B$, and $\forall R \in PF(S) \setminus \{Q, F\} P - Q \not\preccurlyeq R$ then*

Consider the graph $GPF(S)$ and delete all edges involving Q, so the component of Q will break into several components.

Say the graph now has $k + n + 1$ components ($n \geq 0$) (Note that the point Q is a separate component). Construct a set X by not including Q, including the new component of P and randomly choosing whether or not the remaining $k + n - 1$ components are included.

Let C be the collection of vertices originally connected to Q

Let I_1 be the order ideal generated by the conjugates of elements of X. (Note $F - P \in I_1$) (also note $F - (P - Q) \notin I_1$)

Let $B_2 = \{F - Q \preccurlyeq x, x \not\preccurlyeq F - (P - Q), x \not\preccurlyeq Q\}$. Construct I to be an order ideal of B_2 that contains $X \cap C$ (for e.g. $I = X \cap C$ works)

Let $B_1 = \{x | x \preccurlyeq Q, x \not\preccurlyeq P, x \not\preccurlyeq F - (P - Q)\}$. Construct Z to be an order ideal of B_1 containing Q. Say there are s_2 such ideals ($s_2 \geq 1$)

Finally letting $T_2 = (I \cup I_1 \cup Z)$ $A(T_2 \cup S) = S$ (This gives $\geq 2^{k+n-1}s_2$ numerical sets) (Also note that each T_2 is not self dual because $Q \in T_2$, $F - Q \notin T_2$)

Proof: First we observe that T_2 is an order ideal because I, I_1, Z are order ideals. (Check that I and Z are actually order ideals of B)

Note that $F-(P-Q) \notin T_2$; The only minimal element below $F-(P-Q)$ is $F-Q$ so $F-(P-Q) \notin I_1$. Clearly $F-(P-Q) \notin I, Z$

If $x \in I_1$, say $R \in X$ (so $R \neq 0$), $F-R \preccurlyeq x$. Now if $x \preccurlyeq R_1$ for some $R_1 \neq Q$ then R, R_1 are in the same component of the new graph, $F-R_1 \preccurlyeq F-x$ so $F-x \in I_1$ and hence $F-x \in T_2$. Now assume that Q is the only Pseudo-Frobenius number above x this would mean that $x \in B_1$ which is a case we consider later.

Next if $x \in I$ so $F-Q \preccurlyeq x$, say $x \preccurlyeq R$ (so $R \neq Q$). This means that $R \in X$ and $F-R \preccurlyeq F-x$ so $F-x \in I_1$

Now consider an $x \in B_1$. $(Q, P-Q, F-P)$ is a red triangle, $x \preccurlyeq Q$ and $x \not\preccurlyeq F-(F-P)$. So by lemma 3.15 $(x, y, F-P)$ is a red triangle, where $y = Q+P-Q-x = P-x$. $F-(P-Q)-(F-y) = y-P+Q = P-x-P+Q = Q-x \in S$ so $F-y \preccurlyeq F-(P-Q)$ so $F-y \notin T_2$ and the triangle is satisfied.

This ensures $A(T_2 \cup S) = S$

Moreover T_2 is not self dual because $Q \in Z$, $F-Q \preccurlyeq F-(P-Q)$ so $F-Q \notin T_2$

Corollary 6.2.1. *If $P-Q = F-P$ then there are exactly 2^{k-1} such T_2, moreover these are the same sets as the ones in corollary 6.1.1*

Proof: Notice that in this case the connected component of Q in $GPF(S)$ is $\{P, Q\}$. So there are $k-1$ ways of choosing X and hence I_1 has 2^{k-1} choices. We know that $P \in X$ so I contains the order ideal of $F-P = P-Q$. Also $B_2 = \emptyset$ and hence $I = \emptyset$. Lastly B_1 is the order ideal generated by $P-Q = F-P$ so $Z \subseteq I_1$. Thus $T = I_1$, notice that these were the same sets in corollary 6.1.1

Theorem 6.3. *Let $PF(S) = P_1 < P_2 < \cdots < P_{t-1} < F$, If for exactly one pair $i < j$ $P_j - P_i \in B$ then:*

- *If $\nexists k \neq i$ s.t. $P_j - P_i \preccurlyeq P_k$ then $P(S) > 2^k$ and all numerical sets are given by lemmas 6.1 and 6.2 (and the self dual ones)*

  Moreover If $P_j - P_i = F - P_j$ then $P(S) = 3 \times 2^{k-1}$, the numerical sets from lemmas 6.1 and 6.2 are the same.

  And if $P_j - P_i \neq F - P_j$ then $P(S) \geq 2^k + 2^{k+n}$, the numerical sets obtained from lemmas 6.1 and 6.2 are distinct

- *If $\exists k \neq i$ s.t. $P_j - P_i \preccurlyeq P_k$ then $P(S) = 2^k$*

Proof: Rename $P_j = P$, $P_i = Q$. Note that by corollary 3.15.2 Q is the only Pseudo-Frobenius number that can have a triangle, also $F-P \preccurlyeq P-Q$, it is the only minimal element below $P-Q$.

In the first case Q has the triangle $(Q, P-Q, F-P)$. We show that it cannot satisfy any other triangle, if (Q, a, b) is a triangle then $F-P \prec a, b \prec P-Q$ by corollary 3.15.2. $a, b \neq P-Q$ so by corollary 3.14.1 $a, b \preccurlyeq P$. $a \in T \implies P \in T$, since P does not have a triangle $F-P \in T$. (Q, a, b) is a red triangle, $F-P \prec a$ so by corollary 3.14 $F-P \preccurlyeq F-b$ so $F-b \in T$ and the triangle doesn't work.

Therefore the only triangle that can work is $(Q, P-Q, F-P)$. Remember that it can work in two ways:

- First way is if $P-Q \in T$ and $P \notin T$.

  In this case define $Z = \{x | x \in T_1, x \preccurlyeq Q\}$ (Note $x \in Z \implies x \not\preccurlyeq P$). Z is an order ideal of $B_1 = \{x | x \preccurlyeq Q, x \not\preccurlyeq P\}$ and $P-Q \in Z$.

  Let $I = \{x | x \in T, F-Q \preccurlyeq x, x \not\preccurlyeq Q\}$, it is clear that I is an order ideal of $B_2 = \{x | F-Q \preccurlyeq x, x \not\preccurlyeq P, x \not\preccurlyeq Q\}$.

  Let $X = T \cap (PF(S) \setminus \{Q\})$. If $R \in X$ then R does not have a triangle so $F-R \in T$, let I_1 be the order ideal generated by conjugates of elements of X, so $I_1 \subseteq T$.

  Now if $x \in T \setminus (I \cup I_1 \cup Z)$ then $F-R_1 \preccurlyeq x$ with $R_1 \notin X$, $R_1 \neq Q$ (together meaning $R_1 \notin T$) and $x \not\preccurlyeq Q$, so say $x \preccurlyeq R$ $(R \neq Q)$. So $R \in T$, R does not have a triangle so $F-R \in T$. $F-R_1 \preccurlyeq R \implies F-R \preccurlyeq R_1$ so $R_1 \in T$ which is a contradiction. Therefore $T = (I \cup I_1 \cup Z)$

  Next if $R \in X$ is connected to $R_1 \neq Q$ in $GPF(S)$ then $F-R \preccurlyeq R_1$. $R \in T$, R does not have a red triangle so $F-R \in T$ and hence $R_1 \in T$. This means that if a Pseudo-Frobenius number is in T then all Pseudo-Frobenius numbers connected to it in the new graph are in T. $P \notin T$, so the new component of P cannot be in X.

  We conclude that T is given by lemma 6.1

- Second way is $F-P \in T$ and $F-(P-Q) \notin T$

  Let $Z = \{x | x \in T, x \preccurlyeq Q\}$. $x \in Z \implies x \not\preccurlyeq F-(P-Q) \implies x \not\preccurlyeq P$. It follows that Z is an order ideal of $B_1 = \{x \preccurlyeq Q, x \not\preccurlyeq P, x \not\preccurlyeq F-(P-Q)\}$.

  Let $X = (\{P\} \cup (T \cap PF(S))) \setminus \{Q\}$, if $R \in X\{P\}$ then R does not have a red triangle and hence $F-R \in T$, we also have $F-P \in T$. Let I_1 be the order ideal generated by conjugates of elements of X, it follows that $I_1 \subseteq T$

  Let $I = \{x | x \in T, F-Q \preccurlyeq x, x \not\preccurlyeq Q\}$, it is clearly an order ideal of $B_2 = \{F-Q \preccurlyeq x, x \not\preccurlyeq F-(P-Q), x \not\preccurlyeq Q\}$

  Now if $x \in T \setminus (I \cup I_1 \cup Z)$ then $F-R_1 \preccurlyeq x$ with $R_1 \notin X$, $R_1 \neq Q$ (together meaning $R_1 \notin T \cup \{P\}$) and $x \not\preccurlyeq Q$, so say $x \preccurlyeq R$ $(R \neq Q)$. So $R \in T$, R does not have a triangle so $F-R \in T$. $F-R_1 \preccurlyeq R \implies F-R \preccurlyeq R_1$ so $R_1 \in T$ which is a contradiction. Therefore $T = (I \cup I_1 \cup Z)$

  Next if $R \in X$ is connected to $R_1 \neq Q$ in $GPF(S)$ then $F-R \preccurlyeq R_1$. $R \in T$, R does not have a red triangle so $F-R \in T$ and hence $R_1 \in T$. This means that if a Pseudo-Frobenius number is in X then all Pseudo-Frobenius numbers connected to it in the new graph are in T. And $P \in S$

  Therefore T is given by lemma 6.2

Now if $P-Q = F-P$, then by corollaries 6.1.1 and 6.2.1 we know that both lemmas give the same numerical 2^{k-1} sets. So total number of sets is $2^k + 2^{k-1} = 3 \times 2^{k-1}$

And if $P-Q \neq F-P$, then the ones from lemma 6.1 don't have P in them, the ones from 6.2 have P (proved next)

$(Q, P-Q, F-P)$ is a red triangle $F-P \prec P-Q$ $(F-P \neq P-Q)$ so by lemma 3.14 $F-P \preccurlyeq P$ and hence $P \in T$

For the second case denote such a $P_k = R$. We have $F-P \preccurlyeq P-Q \preccurlyeq R$ and hence $F-R \preccurlyeq F-(P-Q) \preccurlyeq P$. If (Q,a,b) is a triangle then $a,b \preccurlyeq P-Q$ by lemma 3.15.2. So if the triangle is satisfied then $P-Q \in T$, so $R \in T$, so $F-R \in T$ so $F-(P-Q) \in T$ so $F-a, F-b \in T$. And hence the triangle cannot be satisfied. Therefore $P(S) = 2^k$ $\square$

7 Arf Semigroups

Lemma 7.1. *S is an Arf Numerical semigroup of multiplicity m. If $x \in B \setminus PF(S)$ then $x+m \in B$*

Proof: $x \notin PF(S)$ so $\exists s_1 \in S$ s.t. $s_1 \neq 0$ $x+s_1 \notin S$. Now if $x+m \in S$ then $m \leq s_1$ and $m \leq x+m$ so $x+s_1 = s_1+(x+m)-m \in S$ (because S is Arf) which is a contradiction. Next if $F-(x+m) \in S$ then $F-x = (F-(x+m))+m \in S$ which contradicts $x \in B$. Therefore $x+m \in B$

Corollary 7.1.1. *If S is an Arf numerical semigroup then. The width of the B-Poset is $t-1$, where t is the type of S ($t = m-1$ as Arf Semigroups have max embedding dimension)*

Remark 7.2. *The $St(m,n)$ families are always Arf*

Conjecture 7.3 (April Conjecture)**.** *The cover relations of the B posets are always small generators, within the first $\frac{1}{3}$ of the set of generators.*

Remark 7.4. *Approach towards April Conjecture:*

Every Arf Numerical semigroup can be obtained via a sequence (and every semigroup obtained this way is Arf):

$S_0 = \mathbb{N}$, $S_1 = (x_1 + S_0) \cup \{0\}$, $S_2 = (x_2 + S_1) \cup \{0\}$, $\dots$, $S_n = (x_n + S_{n-1}) \cup \{0\}$ *s.t.* $x_i \in S_{i-1}$ *for each* i

Now $B(S_0) = B(S_1) = \cdots = B(S_{k-1})$ *s.t. k is the first entry for which* $x_k \geq 3$.

Next if the denote $Brel(S) = \{y-x | x,y \in B(S), y-x \in S\}$ *then* $r \geq k$ *implies* $Brel(S_{r+1}) = (x_r + Brel(S_r)) \cup \{0\}$

We then need to determine which elements of $Brel(S_r)$ cannot be written as sum of other elements of $Brel(S_r)$

It looks like the cover relations of $B(S)$ are first several consecutive generators of S. And the ratio of the number of generators and the multiplicity (which is also the embedding dimension) is at most $\frac{1}{x_k}$

x_k *being at least 3 leads to the April Conjecture*

8 Families of Semigroups

Chris's Cowardly Conjecture

At approximately 9:15am on June 21, Christopher O'Neill conjectured that the type of a semigroup S and $P(S)$ were related. Some investigation finds us many, many semigroups where $P(S) = 2$, but $T(S) \neq 2$.

Definition 8.1 (Additive Semiclosure)**.** *Given a numerical semigroup S, and a finite set $\{a_i\} \subset \mathbb{N} \setminus S$, the* additive semiclosure *of S with respect to $\{a_i\}$ is the set S' constructed by adjoining a_i, and then iteratively adding elements in order to satisfy additive closure.*

By applying TBUS and the concept of additive semiclosure to semigroups with fixed Frobenius numbers, we identified all of the numerical sets that map to them. In this way, we found some families with $P(S) = 2$, but $T(S) \neq 2$.

Definition 8.2 (Quasisymmetric Semigroups)**.** *A numerical Semigroup for which the size of the B set is 2 is called a Quasisymmetric semigroup.*

Theorem 8.3. *Quasisymmetric Semigroups have $P(S) = 2$ unless $B = \{a, F - a\}$ and $F = 3a$*

Proof: If $B = \{a, F - a\}$, we know that $A(S) = S$ and $A(B \cup S) = S$...

E.g. For an even number $2n$, the semigroup $\{0, n+1, n+2 \ldots 2n, 2n+1 \rightarrow\}$ has $P(S) = 2$ but $T(S) = 3$. In particular, $PF(S) = \{\frac{F(S)\pm 1}{2}, F(S)\}$.

Definition 8.4 (YET UNNAMED SEMIGROUPS)**.** *The semigroup $\{0, n, n+1, \ldots 2n-5, 2n-2, 2n \rightarrow\}$ has $PF(S) = \{2n-4, 2n-3, 2n-1\}$, but $P(S) = 2$.*

Proof: This is a semigroup since all nontrivial elements are greater than $\frac{F(S)}{2}$. The Pseudo-Frobenius numbers are just the gaps larger than $\frac{F(S)}{2}$, i.e. $\{2n-4, 2n-3, 2n-1\}$.

The only numerical sets corresponding to this semigroup are S and $B \cup S$. $B = \{2, 3, 2n-4, 2n-3\}$. If $b \in T$, $\bar{b} \in T$. If $2 \in T$, thus $2n-3 \in T$, and since $2n-6 \in S$, $2n-4 \in T$ so $3 \in T$ which is $B \cup S$. If $3 \in T$, $2n-4 \in T$, and since $2n-6 \in T$, $2n-3 \in T$ and $2 \in T$. Again this is $B \cup S$, so if any element of B is in T, $T = B \cup S$. This shows $P(S) = 2$. $\square$

In fact, when $P(S) = 2$, both $|B|$ and the type of the semigroup can be unbounded, as evidenced by the following families:

Example 8.5 ($3n$ Semigroups)**.** *For $n \in \mathbb{N}$, the family $S_n = \{0, 3, 6, \ldots 3n \rightarrow\}$ has $|B| = n$ and $P(S_n) = 2$.*

Proof: Note that every multiple of 3 is contained in every S_n. For $b \leq F(S)$, if $b \equiv 1 \mod 4$, then $F(S) - b \equiv 1 \mod 4$ so $b, F(S) - b \notin S$, but if $b \equiv 2 \mod 4$, $F(S) - b \in S$. Thus, B is exactly the elements of S that are $1 \mod 4$, so $|B| = n$.

Furthermore, if $A(T) = S$ and $T \neq S$, then $T = B \cup S$. Since $T \neq S$, $b \in T \backslash S$, so $b = 3k+1$. Then, since $b + S \subseteq S$, then for $0 \leq l \in \mathbb{N}$, $3(k+l)+1 \in S$, so every element of B larger than b is also in T. Then, since $3n-2 \in T$, but $3n-2 \notin A(T)$, $1 \in T$. After this, every element of B is also in T, so $T = B \cup S$.

Example 8.6 (2^n Semigroups)**.** *For $n \in \mathbb{N}$, the family $S_n = \{0, m, m+1, \dots m+n, m+n+2, \dots m+2n, \dots 2m-2, 2m \rightarrow\}$, where $m = 2^n + n - 1$ and the Pseudo-Frobenius numbers are $\{2m - 2^{k-1} - k + 1 | 1 \leq k \leq n\}$. $P(S_n) = 2$ and $T(S_n) = n$.*

Proof: First, each S_n is a semigroup, since every nontrivial element of S_n is larger than $\frac{F(S_n)}{2}$. Similarly, each element of $PF(S)$ is larger than $\frac{F(S_n)}{2}$, so $2m - 2^{k-1} - k + 1 + S \subseteq S$.

Now, if $A(T) = S$ and $T \neq S$, then $T = B \cup S$. In this case, B is composed of the Pseudo-Frobenius numbers (except F, where $k = 1$) and their conjugates. If T contains some Pseudo-Frobenius number $PF_k = 2m - 2^{k-1} - k + 1$, it also contains its conjugate $F - PF_k = 2^{k-1} + k - 2$. Since for higher values of k, the gaps are $2^{k-1}+1$ apart, for $k < n$, if $PF_k \in T$, $PF_{k+1} \in T$. If $PF_n \in T$, then its conjugate $2^{n-1} + n - 2 \in T$. Then, since for $n > 2$, $m \leq PF_2 - (2^{n-1} + n - 2) < PF_n$, if $PF_n \in T$, then $PF_2 \in T$. Thus, if one Pseudo-Frobenius number is in T, then all of them are, so the only semigroups with $A(T) = S$ are S and $TBUS$.

8.1 Noble Semigroups

Definition 8.7. *A semigroup is* Noble *if for all $P \in PF(S), b \in B$, we have that $P + b \in B \implies b = F - P$. Otherwise, it is* Ignoble.

Theorem 8.8. *If S is noble, then it is P-Minimal.*

Proof: Let T be a numerical set such that $A(T) = S$; it suffices to show that $T \setminus S$ is self-dual, so let $P \in T \cap PF(S)$. There must be $t \in T \cap B$ such that $P + t \notin T$. If $P + t \notin B$, we have from the proof of Theorem 3.11 that $F - P \in T$. If $P + t \in B$, there is $Q \in PF(S)$ such that $Q - (P + t) \in S$, and so $Q - P = (Q - P - t) + t \in T \cap B$ and $P + (Q - P) = Q \in PF(S)$; we thus have $Q - P = F - P \in T$. Either way $P \in T \implies F - P \in T$, and so $T \setminus S$ is self-dual. $\square$

8.2 P(S) for Semigroups of Type 3

From the symmetric semigroups, we know that if $T(S) = 1$, $P(S) = 2$. From Theorem 3.2, we can see that $T(S) = 2$ implies $P(S) = 2$. In the following section, we will show that $T(S) = 3$ implies that $P(S) = 2, 3, 4$, and $P(S)$ can be arbitrarily large for $T(S) = 4$.

Theorem 8.9. *If $t = 3$ and number of connected components of $GPF(S)$ is 2, then $P(S) = 4$.*

Proof: Let P and Q be the maximal elements of the B Poset (with $P < Q$), so their conjugates are the minimal ones. In order to have two connected components in $GPF(S)$ we must have $F-P \not\preccurlyeq Q$. So $F-P \preccurlyeq P$ and $F-Q \preccurlyeq Q$. Now assume we have a red triangle $P+x+y=F$ then $F-P \not\preccurlyeq x, y$. So $F-Q \preccurlyeq x, y \preccurlyeq Q$. It follows that if either of x, y are in T then $Q \in T$ and hence $F-Q \in T$ and conjugates of both x, y are in T so the triangle cannot work. Hence, $P(S)=4$. $\square$

Theorem 8.10. *If $t=3$, $PF(S)=\{P,Q,F\}$ with $P<Q<F$, $Q-P \notin B$, then S is noble.*

Proof: Follows from corollary 3.15.2

Lemma 8.11. *If $t=3$, $Q-P \in B$, and (P,x,y) is a red triangle, then $x \preccurlyeq Q-P$. (The same applies to y.)*

Proof: Follows from corollary 3.15.2

Lemma 8.12. *If $t=3$, $Q-P \in B$, and $x \preccurlyeq Q-P$ and $x \neq Q-P$ then $x \preccurlyeq Q$.*

Proof: Follows from lemma 3.14

Lemma 8.13. *If $t=3$, $Q-P \in B$, and $b \parallel Q-P$ then $b \preccurlyeq Q$*

Proof: If possible, assume $b \not\preccurlyeq Q$. Then, $b \preccurlyeq P$ i.e. $P-b \in S$. Also $Q-b \notin S$ therefore either $Q-b \in B$ or $Q-b \in Gap \setminus B$

If $Q-b \in B$ then $Q-b$ cannot be below Q, and hence it must be below P i.e. $P-Q+b \in S$ which means $Q-P \preccurlyeq b$, which is a contradiction.

Next assume $Q-b \in Gap \setminus B$, which implies $F-Q+b \in S$, but then $F-(Q-P)=(F-Q+b)+(P-b) \in S$, which is also a contradiction because $Q-P \in B$.

We conclude that $b \preccurlyeq Q$. $\square$

Lemma 8.14. *If $t=3$, $Q-P \in B$, and $\overline{Q-P} \preccurlyeq b$ and $\overline{Q-P} \neq b$ then $F-Q \preccurlyeq b$*

Proof: $F-(Q-P) \preccurlyeq b \implies \overline{b} \preccurlyeq Q-P$ and $\overline{b} \neq Q-P$ hence by lemma 8.12 $\overline{b} \preccurlyeq Q$ i.e. $\overline{Q} \preccurlyeq b$ $\square$

Theorem 8.15. *If $t=3$ then $P(S) \leq 4$*

Proof: The only case remaining is when $GPF(S)$ is connected and $Q-P \in S$. $GPF(S)$ being connected means $F-Q \preccurlyeq P$ and $F-P \preccurlyeq Q$,

Consider T s.t. $A(T)=S$ and let $T'=T \setminus S$. If $T' \neq \emptyset$, then T has at least one Pseudo-Frobenius number. If it has Q, then it has $F-Q$ and hence also P, i.e. $T' \neq \emptyset \implies P \in T'$.

Now if $F-P \in T'$, then $Q \in T'$, which implies $F-Q \in T'$, which implies $T'=B$. Therefore $T' \neq \emptyset, B \implies P \in T'$ and $F-P \notin T'$. Hence P must satisfy a red triangle and by lemma 8.11 we know $Q-P \in T'$

Let $T_1=\{x | Q-P \preccurlyeq x\}$, by lemma 8.12 and lemma 8.13 we see that $T' \neq \emptyset, B, T_1 \implies Q \in T \implies \overline{Q} \in T$ So $F-Q \preccurlyeq x \implies x \in T$

Let $T_2 = \{x|F-Q \preccurlyeq x\}$. If $x \parallel F-(Q-P)$ then $F-x \parallel Q-P$ and hence by Lemma 8.13 $F-x \preccurlyeq Q$, hence $F-Q \preccurlyeq x$ and $x \in T_2$. Next if $F-(Q-P) \preccurlyeq x$, $x \neq F-(Q-P)$ then by Lemma 8.14 $F-Q \preccurlyeq x$ and $x \in T_2$

Finally if $T' \neq \emptyset, B, T_1, T_2$ then $\exists x \in T'$ s.t. $x \preccurlyeq F-(Q-P)$. Hence $F-(Q-P) \in T'$. Now if a red triangle (P,y,x) works i.e. $y \in T'$, $F-x \notin T'$ then by Lemma 8.12, $x \preccurlyeq Q-P$ and hence $F-(Q-P) \preccurlyeq F-x$. It follows that no triangle can work, which is a contradiction.

We have shown that $P(S) \leq 4$. □

Lemma 8.16. *If $t=3$, the graph $GPF(S)$ is connected and $Q-P \in B$ then $A(T_1 \cup S) = S$*

Proof: Firstly T_1 is an order ideal and P is the only Pseudo-Frobenius number in T_1 ($Q \in T_1 \implies Q-P \preccurlyeq Q \implies P = Q-(Q-P) \in S$ which is impossible). Moreover $(P, Q-P, F-Q)$ is a red triangle with $Q-P \in T_1$ and $Q = F-(F-Q) \notin T_1$. Hence by theorem 3.13 $A(T_1 \cup S) = S$

Theorem 8.17. *If t=3 then $P(S)=2$ iff $GPF(S)$ is connected and $Q-P \notin B$*

Proof: Firstly if $GPF(S)$ is connected and $Q-P \notin B$ then by Theorem 8.10, S is noble and hence $P(S)=2$.

Conversely assuming $P(S)=2$, if $GPF(S)$ is not connected then $P(S)=4$ so $GPF(S)$ must be connected. And if $Q-P \in B$ then by lemma 8.16 $A(T_1)=S$ and $P(S) \geq 3$

Lemma 8.18. *If $t=3$, $GPF(S)$ is connected, $Q-P \in B$ then $A(T_2 \cup S) = S$*

Proof: Let $T_2 = \{x|F-Q \preccurlyeq x\}$. Note that $P \in T_2$ and Q may or may not be in it.

$(P, F-Q, Q-P)$ is a red triangle, $F-Q \in T_2$. Moreover $(F-(Q-P)) - (F-Q) = P \notin S$ and hence $\overline{Q} \not\preccurlyeq \overline{Q-P}$ i.e. $\overline{Q-P} \notin T_2$. Therefore the triangle is satisfied.

If $Q \in T_2$ then we know that $\overline{Q} \in T_2$.

Therefore by theorem 3.13 $A(T_2 \cup S) = S$

Theorem 8.19. *if t=3, GPF(S) is connected and $Q-P \in B$ then:*
if $F = 2Q-P$ then $P(S)=3$, otherwise $P(S)=4$

Proof: $T_1 = T_2$ iff $Q-P = F-Q$

Remark 8.20. *Note That $T_1^* = T_2$*

8.3 Chris's Courageous Conjecture

Definition 8.21. *Given a Numerical Semigroup S and $\beta \geq 2$, f s.t. $\beta \not| f$, $f > \beta(F(S)+2m(S))$ define $M(S,\beta,f) = \beta S \cup \{0, f+1 \rightarrow\}$*

Conjecture 8.22. *If we fix S and β then $P(M(S,\beta,f))$ is eventually a quasi-polynomial is f*

Notation: denote by F the Frobenius number of S, by m the multiplicity of S (as opposed to those of $M(S,\beta,f)$)

Remark 8.23. *$St(m,n) = M(\mathbb{N}, m, nm-1)$ and*
$St(l,m,n) = M(\{0, l \rightarrow\}, m, m(l+n)-1)$

Definition 8.24. *Given a Numerical Semigroup S and $\beta \geq 2$ we define the βS-Poset to be the Poset whose elements are $\mathbb{N} \setminus (\beta S)$ and $x \preccurlyeq y$ iff $y - x \in \beta S$ (Note that it has infinitely many elements)*

Definition 8.25. *Given a numerical Semigroup S the S-poset is the poset whose elements are $\mathbb{N}$ and $x \preccurlyeq y$ iff $y - x \in S$.*

The Gap-Poset is the poset whose elements are $\mathbb{N} \setminus S$ and $x \preccurlyeq y$ iff $y - x \in S$

Remark 8.26. *Note that the B-Poset is obtained from the Gap-Poset by deleting everything that is below the Frobenius number in the poset.*

Lemma 8.27. *The βS-Poset has the following description:*

Let $C_i = \{x | x \equiv i (\mathrm{mod}\ \beta)\}$ if $1 \leq i \leq \beta - 1$ and $C_\beta = \{x | x = \beta t, t \notin S\}$. Note that the sets C_i are mutually parallel.

If $i \leq \beta - 1$, then C_i is isomorphic to the S poset, with $q_1\beta + i \preccurlyeq q_2\beta + i$ in the βS-Poset iff $q_1 \preccurlyeq q_2$ in the S-Poset. In addition, C_β is isomorphic to the *Gap*-Poset, $\beta t_1 \preccurlyeq \beta t_2$ in βS-Poset iff $t_1 \preccurlyeq t_2$ in the *Gap*-Poset.

Corollary 8.27.1. *If $S = \{0, k, \rightarrow\}$ the βS-Poset has the following description:*

The sets C_i are mutually parallel.
For $i \leq \beta - 1$, $x, y \in C_i$ then $x \preccurlyeq y$ iff $y - x \geq \beta k$.
C_β is a Chaos Poset of size $k - 1$.

We describe the structure of C_i as a poset ($i \neq \beta$) by arranging them in towers. The first layer has those elements that are between $1 \leq x < m\beta$, the second layer those between $m\beta \leq x < 2m\beta$ and so on.

Note that we cannot have edges within a layer, this is because if $q_1\beta + i$ and $q_2\beta + i$ are in the same layer then $q_2 - q_1 < m$ and hence $q_2 - q_1 \notin S$. Let a be the largest atom of S then we can never have a direct edge from the l^{th} layer to the l_2^{th} layer with $l_2 - l_1 \geq \lceil \frac{a}{m} \rceil + 1$. This is because if such an edge exists, say between points $q_1\beta + i$ and $q_2\beta + i$, with $(l_1 - 1)m \leq q_1 < ml_1$ and $(l_2 - 1)m \leq q_2 < ml_2$. Note that by the lemma 8.27 this is equivalent to there being a direct edge from q_1 to q_2 in the S poset i.e. $q_2 - q_1$ is an atom (generator) of S. But $q_2 - q_1 \geq m((l_2 - 1) - l_1) \geq m(\lceil \frac{a}{m} \rceil) > a$ ($m \not| a$) which is impossible.

Also note that elements in the l^{th} layer are obtained by adding $(l-1)m\beta$ to elements in the 1^{st} layer. Note also that $x + m\beta \cdots \preccurlyeq x + (l-1)m\beta$. Thus, the edges between the l^{th} and $l + 1^{th}$ layers are in natural correspondence with edges between 1^{st} and 2^{nd} layer and the edges between l^{th} and $l + 2^{th}$ layer are in natural correspondence with edges between 1^{st} and 3^{rd} layers. Continuing this process, edges between l^{th} and $l + \lceil \frac{a}{m} \rceil^{th}$ layer are in natural correspondence with edges between 1^{st} and $\lceil \frac{a}{m} \rceil + 1^{th}$ layers.

Lemma 8.28. *The B-Poset of $M(S,\beta,f)$ (assuming $f > \beta F(S)$) has the following description:*

It is a sub-Poset of the βS-Poset.

Say $f \equiv r(\text{mod } \beta)$ $1 \leq r < \beta$,

Then from C_j ($j \neq r$, $j \neq \beta$) we remove all elements $\geq f$ Let $D_j = \{x | x \in C_j, x < f\}$. From C_r we remove everything except $D_r = \{f - \beta g, g \in \mathbb{N} \setminus S\}$ ($f > \beta F(S)$). Note that D_r as a Poset is the dual of the *Gap*-Poset of S. C_β remains as it is ($f > \beta F(S)$).

Corollary 8.28.1. *In case of $S = \{0, k \rightarrow\}$ the Gap-Poset of $\{0, k \rightarrow\}$ is the chaos poset of $k-1$ elements and hence the B-Poset is $M(\{0, k \rightarrow\}, \beta, f)$ (we assume $f > \beta k$) is the disjoint union of $\beta - 1$ cut-off S-Posets and two chaos Posets of size $k-1$ each.*

Definition 8.29. *The S cut off at n Poset is the poset whose elements are natural numbers less than n with $x \preccurlyeq y$ iff $y - x \in S$*

Remark 8.30. *In the B-Poset of $M(S,\beta,f)$. If $f \equiv r(\text{mod } \beta)$, with $1 \leq r < \beta$ Then for $j < r$ then D_j is naturally isomorphic to S cut off at $\lceil \frac{f}{\beta} \rceil$ and for $j > r$ D_j is naturally isomorphic to S cut off at $\lfloor \frac{f}{\beta} \rfloor$*

Lemma 8.31. *The maximal elements of S cut off at n Poset are $n-1$ and $n-1-x$ where x is a minimal element of the Gap-Poset of S*

Corollary 8.31.1. *$PF(M(S,\beta,f+\beta)) \setminus C_\beta = \beta + (PF(M(S,\beta,f)) \setminus C_\beta)$*

And of course $PF(M(S,\beta,f+\beta)) \cap C_\beta = PF(M(S,\beta,f)) \cap C_\beta$

Corollary 8.31.2. *Type of $M(S,\beta,f)$ is $t(S)+(\beta-1)(1+\#\{\text{minimal elements of Gap Poset}\}) = t(S) + (\beta-1)m(S)$*

Lemma 8.32. *If $P \in PF(M(S,\beta,f)) \setminus (\{f\} \cup C_\beta)$ and (P,a,b) is a red triangle of $M(S,\beta,f)$ then $P + m\beta \geq f$ and hence $a + b < m\beta$ and a, b and $\overline{P} = a + b$ are in the bottom layer.*

Corollary 8.32.1. *If $f \equiv r(\text{mod } \beta)$ then D_r has no elements in the bottom layer if $f > (F+m)\beta$.*

And hence the $a, b, \overline{P}$ from the lemma cannot be in D_r

Lemma 8.33. *If $P \in PF(M(S,\beta,f)) \setminus (\{f\} \cup C_\beta)$ then (P,a,b) is a red triangle of $M(S,\beta,f)$ iff $(P+\beta), a, b)$ is a red triangle of $M(S,\beta,f+\beta))$*

Note that a, b were not in D_r of $PF(M(S,\beta,f))$ and hence so a, b are the B-Poset of $M(S,\beta,f+\beta)$

Also note that if at least one of x, y (say x) is a newly added element of the B-Poset of $M(S,\beta,f+\beta)$ (i.e. it was not in the B-Poset of $M(S,\beta,f)$) then x is in the top layer of the B-poset and hence $(P+\beta, x, y)$ is not a red triangle of $M(S,\beta,f+\beta)$.

Remark 8.34. *We fix S and β, move f within a particular equivalence class mod β (while ensuring $f > (F+m)\beta$). For $j \neq \beta$ We denote the numerically largest element of D_j by $P(j,0)$ (so $P(r,0)$ is the Frobenius number) and we define $P(j,gp) = P(j,0) - gp$ for $gp \in \mathbb{N} \setminus S$*

We also define $P(\beta,p) = \beta p$ for $p \in PF(S)$.

Note that these are all the Pseudo-Frobenius numbers.

Remark 8.35. *We divide order ideals of the B-Poset of $M(S,\beta,f)$ into categories depending on which elements of the first layer are in the order ideal and which elements in the first layer have their conjugates in the order ideal, which elements of C_β are in the order ideal, which elements of D_r are in the order ideal.*

Lemma 8.36. *For $P \in T \cap PF(M(S,\beta,f)) \setminus (\{f\} \cup C_\beta)$ note that whether or not $\overline{P} \in T$ and whether or not P satisfies a triangle is determined by which category P is in.*

Lemma 8.37. *For $P \in C_\beta \cap PF(M(S,\beta,f))$ if P has a red triangle (P,a,b) with $a \in C_\beta \cup D_r$ Then whether or not an order ideal satisfies the red triangle is determined by the category.*

Proof: If $a \in C_\beta$ then $f - b = P + a \equiv 0 (mod\ \beta)$ and $f - b \in C_\beta$ and the category determines whether or not this order ideal is satisfied.

If $a \in D_r$ then $f - b = P + a \equiv r (mod\ \beta)$. Therefore whether or not $a \in T$ and $f - b \in T$ is determined by the category.

Lemma 8.38. *If $i \neq r, \beta$ and $x \in D_i$ then the set $\{y | x < y, y \in D_i, x \parallel y\}$ is the same as the set $\{x + gp\beta | gp \in \mathbb{N} \setminus S, gp\beta < f - x\}$ with $x < f - F\beta$ then the set $\{y | x < y, y \in D_i, x \parallel y\}$ is the same as the set $\{x + gp\beta | gp \in \mathbb{N} \setminus S\}$ and as a poset is isomorphic to the Gap-Poset of S*

Lemma 8.39. *If $i \neq r, \beta$ and $x \in D_i$ then the set $\{y | y < x, y \in D_i, x \parallel y\}$ is the same as the set $\{x - gp\beta | gp \in \mathbb{N} \setminus S, gp\beta < x\}$ with $x > F\beta$ then the set $\{y | x < y, y \in D_i, x \parallel y\}$ is the same as the set $\{x - gp\beta | gp \in \mathbb{N} \setminus S\}$ and as a poset is isomorphic to the duel of the Gap-Poset of S*

Notation: For each pair of disjoint subsets A, B of the set of maximal elements of C_β let $\gamma_{\infty,A,B}$ be the number of order ideals of *Gap* poset of S for which $\forall p \in A$ either $\frac{p}{\beta}$ is in the order ideal or there is a pair of elements of the poset that differ by $\frac{p}{\beta}$, the smaller element is in the order ideal, the larger element is not. Moreover $\forall p' \in B$ such a pair does not exist and $\frac{p'}{\beta}$ is not in the order ideal.

And let $\gamma_{n,A,B}$ be the number of order ideals of poset obtained from *Gap* poset of S by throwing away everything numerically bigger than n, we only count order ideals that satisfy: $\forall p \in A$ either $\frac{p}{\beta}$ is in the order ideal or there is a pair of elements of the poset that differ by $\frac{p}{\beta}$, the smaller element is in the order ideal, the larger ideal is not. And for $\forall p' \in B$ such a pair does not exist and $\frac{p'}{\beta}$ is not in the order ideal.

If in a category we had chosen an element but excluded an element above it then the category has no order ideals and we throw it away.

For the remaining categories we count the number of good numerical sets in them:

Counting Good Numerical Sets with Towers

Fix a category. Let A be the set of elements of maximal elements of C_β that are chosen in the category, while their conjugates are not chosen.

If the category was not thrown out then it has three kinds of towers (D_i, $i \neq r$ are called towers).

1. At least one element of first layer and all elements whose conjugates are in first layer are chosen.
2. No element of first layer is chosen and at least one element whose conjugate is in the first layer is not chosen.
3. No element of the first layer is chosen, all elements whose conjugates are in the first layer are chosen.

In a tower of the first kind, all but finitely many elements are above the chosen minimal elements (the set of the remaining ones does not change when we change f within an equivalence class).

In a tower of the second kind, all but finitely many elements are below one of the maximal elements that is not chosen (the set of the remaining ones does not change when we change f within an equivalence class)

We divide the category into sub-categories by randomly choosing which of the remaining elements of towers of the first and second kind are to be included in the order ideal.

If while making the subcategory we picked an element but missed something above it, then the subcategory has no order ideals in it and we throw it away.

If the subcategory survives then some of the elements of A might satisfy a triangle within the decided elements (decided elements are those that are chosen or excluded). We remove those elements from A and create a modified A set.

Now we have a subcategory and a modified A set. We still have towers of third kind to consider.

Given a tower of the third kind, say D_i. If $i < r$ it has $\lceil \frac{f}{\beta} \rceil$ elements and if $r < i$ it has $\lfloor \frac{f}{\beta} \rfloor$ elements; in either case, denote the number of elements in D_i by n. The first m of these are in the first layer and have been excluded, while the last m have their conjugates in the first layer and have been included. The remaining $n - 2m$ elements have to be decided. Suppose the smallest (numerically) element among these that is included in the order ideal is x; then everything above x is included, and everything numerically smaller than x is thrown away. Note that the set $\{y | x < y, y \in D_i, x \parallel y\}$ remains to be decided.

Suppose we have picked the x in each tower of the third kind. If there are s_1 towers of the third kind with $i < r$ and s_2 towers of third kind with $i > r$, there are $(\lceil \frac{f}{\beta} \rceil - 2m)^{s_1}(\lceil \frac{f}{\beta} \rceil - 2m - 1)^{s_2}$ ways of picking the elements x from each tower.

Note that any $p \in A$ (the modified A) cannot satisfy a triangle with the decided elements, because if $x + p$ is either undecided, or in the top layer and hence chosen or not in the B-Poset at all. And if $x \preccurlyeq a$ then $a + p$ is either chosen or not in the B-Poset at all.

Now we need to decide the remaining elements, so we first split the subcategory into several divisions. Each division is a tuple $D = (\sigma_{gp1}, \sigma_{gp2}, \ldots, \sigma_F, r_1, r_2)$ where $gp1, gp2, \ldots$ are the Gaps of S, σ_{gp} is how many towers (of third kind) have the poset of undecided elements naturally isomorphic to Gap-Poset of S cut off at gp. r_1 is how many towers D_i (of third kind) with $i < r$ have $f - x > F\beta$ (and hence the poset of undecided elements naturally isomorphic to Gap-Poset of S) and r_2 is how many towers (of third kind) with $i > r$ $f - x > F\beta$. Each division D has a coefficient a_D which is the number of ways of partitioning the towers of the third kind into $g + 2$ parts s.t. all towers D_i in the $g + 1^{th}$ part have $i < r$ and all towers D_i in the $g + 2^{th}$ part have $i > r$.

Denote the number of towers of the third kind by d.

Note that $\sigma_{gp1} + \sigma_{gp2} + \cdots + \sigma_F + r_1 + r_2 = d$ otherwise the division has $a_D = 0$ and can be ignored.

Lastly we further split each division into several Partitions. Each partition is a tuple $(A_1, A_2, \ldots, A_d)$ s.t. $A_1 \cup A_2 \cup \cdots \cup A_d = A$. Define $B_i = A \setminus A_i$. We define function g on the components of the tuple, g maps the first σ_{gp1} components to $gp1$, then next σ_{gp2} components to $gp2$, … further σ_F components to F and last $r_1 + r_2$ components to ∞

The number of good numerical sets in a partition is $a_d \prod_{i=1}^{d} \gamma_{g(A_i), A_i, B_i}(\lceil \frac{f}{\beta} \rceil - 2m - F)^{s_1}(\lceil \frac{f}{\beta} \rceil - 2m - 1 - F)^{s_2}$. □

Theorem 8.40. *If we fix S and β then $P(M(S, \beta, f))$ is eventually a quasi-polynomial is f with period β*

Proof: Once we have fixed S, β and which equivalence class mod β f is in, we can determine all the categories, all of their subcategories, all of their divisions and all of their partitions. Once we do this we have a polynomial in $\lceil \frac{F}{\beta} \rceil$ as the number of good numerical sets within a partition.

Once we sum these polynomials over all partitions of all divisions of all subcategories of all categories we get the polynomial expression of $P(M(S, \beta, f))$, the polynomial depends on which equivalence class mod β f is in.

Corollary 8.40.1. *The degree of the polynomial is the largest d among all good categories (that have a good numerical set). d is the number of towers of the third kind.*

Proof: Once we have such a category we can pick a subcategory and then take the division $(0, 0, \ldots, 0, s_1, s_2)$ and then all of its partitions have polynomials of degree $s_1 + s_2 = d$

8.3.1 Staircase $St(m,n)$ families

Definition 8.41. $St(m,n) = \{0, m, 2m \ldots, nm, \rightarrow\}$

Lemma 8.42. *The B-Poset of $St(m,n)$ has a simple structure, it is the disjoint union of $m-2$ chains of length n each. The r^{th} chain ($1 \leq r \leq m-2$) is $r \preccurlyeq r+m \preccurlyeq r+2m \preccurlyeq \cdots \preccurlyeq r+(n-1)m$*

Lemma 8.43. *If P is a Pseudo-Frobenius number of $St(m,n)$, (P,a,b) is a red triangle then a, b are minimal elements of the B-Poset*

Theorem 8.44. *For a fixed m there is a polynomial $g_m(x)$ s.t. $P(St(m,n)) = g_m(n)$*

Proof: We partition the numerical sets into categories, created as follows:

We partition equivalence classes mod m, $1 \leq r \leq m-2$ into 3 kinds: those included completely, those not included at all, those included partially (in a way that ensures it is an order ideal, i.e. where if an element is included, so are all the elements above it).

There are finitely many ways of making those selections. Note that these selections do not depend on the value of n.

Note that if one order ideal in a category has $A(T \cup S) = S$, then T satisfies the condition of theorem 3.13 for each Pseudo-Frobenius number in T then all order ideals in that category satisfy the condition of theorem 3.13. This is because if (P,a,b) is a red triangle, $P \in PF(S) \setminus \{F\}$, then a, b are minimal elements of the B-Poset by lemma 8.43 and hence $a \in T$ means the entire equivalence class of a is in T amd $\bar{b} \notin T$ means no element in the equivalence class of $\bar{b}$ is in T. And therefore either all order ideals in the category satisfy a triangle or none do.

Note that whether or not a selection satisfies the condition of theorem 3.13 does not depend on the value of n

If a category has d equivalence classes in the third kind then it has $(n-1)^d$ order ideals.

Now if $a_{m,d}$ is the number of categories that satisfy condition of theorem 3.13 and have d equivalence classes in the third kind.

Then $P(S) = \sum_{d \geq 0} a_{m,d}(n-1)^d$. (Note that this is a finite sum as $d \leq m-2$)$\square$

We next describe a way to compute the polynomials $g_m(x)$

Theorem 8.45. *Fix m, consider the numerical sets for which $A(T \cup St(m,1)) = St(m,1)$.*

We create a diagram with them, place a set in the h^{th} row if T has h elements ($h \geq 0$). Now if $T_1 \subseteq T_2$ and $\forall x \in T_2$ $x + T_1 \not\subseteq T_2 \cup St(m,1)$ then we draw an edge from T_1 to T_2. Length of the edge is size of $T_2 \setminus T_1$

If there are $b_{m,d}$ edges of length d then $a_{m,d} = b_{m,d}$ and $g_m(x) = \sum_{d \geq 0} b_{m,d}(x-1)^d$

Proof: Consider a category that satisfies condition of theorem 3.13. Say the equivalence classes of the first kind are $r_1, \ldots, r_l$ $(1 \leq r_i \leq m-2)$. Then $T_1\{r_1, r_2 \ldots r_l\}$ is a good numerical set of $St(m,1)$ as it satisfies the condition of Theorem 3.13. Next if the equivalence classes of the third kind are $s_1, \ldots, s_d$ then $T_2\{r_1, \ldots, r_l\} \cup \{s_1, \ldots s_d\}$ is also a good set of $St(m,1)$ as it satisfies the condition of theorem 3.13

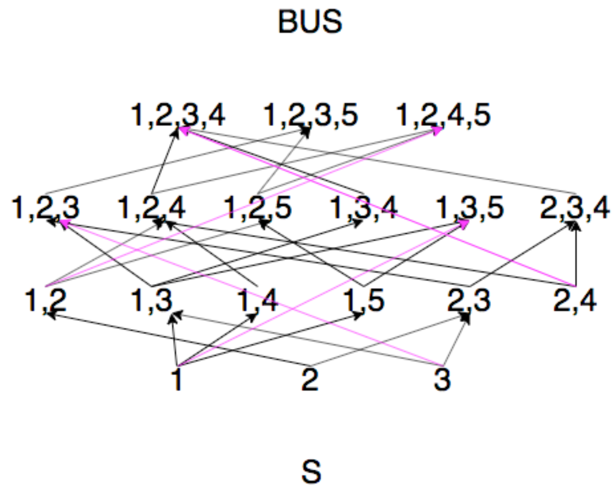
Moreover for each $x \in T_2$, the class of x has Pseudo-Frobenius number $P_i = x + m(n-1) \in T$ if it satisfies a triangle (P_i, a, b) (so $1 \leq a, b \leq m-2$) $a \in T$ means the class of a is in the first kind. $F - b = mn - 1 - b \notin T$ means the class of $m - b - 1$ is of the second kind. Also $F = P_i + a + b$ means $mn - 1 = (x + mn - m) + a + b$ so $x + a = m - 1 - b \notin T_2 \cup St(m,1)$ (as $m - 1 - b \leq m - 2$) and hence $x + T_1 \nsubseteq T_2 \cup St(m,1)$. Next if it does not satisfy a triangle then $F - P_i \in T$, $F - P_i = m - 1 - x$ $(1 \leq m - 1 - x \leq m - 2)$, $F - P_i \in T$ means $m - 1 - x \in T_1$ and hence $x + T_1 \nsubseteq T_2 \cup St(m,1)$

And course the size of $T_2 \setminus T_1$ is the number of equivalence classes in the category.

Conversely if T_1, T_2 are good numerical sets of $St(m,n)$ s.t. $T_1 \subseteq T_2$ and $\forall x \in T_2$ $x + T_1 \nsubseteq T_2 \cup St(m,1)$.

Construct a category by having the classes of T_1 in the first kind, classes of $T_2 \setminus T_1$ in third category and the remaining classes in the second category. We will show that an order ideal in this class satisfies the condition of theorem 3.13. Let T be an order ideal in the category. Say P is a Pseudo-Frobenius number in T, it is in the class of x $(1 \leq x \leq m-2)(P = x + m(n-1))$, the class of x is in the first or third kind so $x \in T_2$. $x + T_1 \notin T_2 \cup St(m,1)$ i.e. $\exists y \in T_1$ s.t. $x + y \notin T_2 \cup St(m,1)$ that means $x + y \leq m - 1$ and $x + y \notin T_2$. $y \in T_1$ means the class of y is in the first kind and hence $y \in T$. First consider the case if $x + y = m - 1$ so $P + y = x + mn - m + y = mn - 1 = F$ i.e. $y = F - P \in T$. Next consider the case $x + y \neq m - 1$, so $1 \leq x + y \leq m - 2$, $x + y \notin T_2$ means the class of $x + y$ is of the second kind. $z = F - P - y = (mn - 1) - (x + mn - m) - y = m - 1 - (x + y)$ so $1 \leq z \leq m - 2$, (P, y, z) is a red triangle, $y \in T$, $F - z = (mn - 1) - (m - 1 - (x + y)) = mn - m + (x + y)$ which is in the class of $x + y$ which is in the second kind and hence $F - z \notin T$ and the condition of theorem 3.13 is satisfied.

Also again the number of classes in the third kind is the size of $T_2 \setminus T_1$ $\square$



Example 8.46. $P(St(2,n)) = 1$

$P(St(3,n)) = 2$

$P(St(4,n)) = 3$

$P(St(5,n)) = 6 + 2(n-1) = 2n + 4$

$P(St(6,n)) = 10 + 8(n-1) = 8n + 2$

$P(St(7,n)) = 20 + 26(n-1) + 4(n-1)^2 = 4n^2 + 18n - 2$

$P(St(8,n)) = 37 + 70(n-1) + 28(n-1)^2$

$P(St(9,n)) = 74 + 179(n-1) + 122(n-1)^2 + 10(n-1)^3$

$P(St(10,n)) = 140 + 414(n-1) + 403(n-1)^2 + 106(n-1)^3 + 2(n-1)^4$

$P(St(11,n)) = 280 + 969(n-1) + 1218(n-1)^2 + 546(n-1)^3 + 40(n-1)^4$

$P(St(12,n)) = 542 + 2150(n-1) + 3327(n-1)^2 + 2206(n-1)^3 + 464(n-1)^4 + 12(n-1)^5$

$P(St(13,n)) = 1084 + 4839(n-1) + 8816(n-1)^2 + 7710(n-1)^3 + 2850(n-1)^4 + 274(n-1)^5 + 6(n-1)^6$

$P(St(14,n)) = 2118 + 10492(n-1) + 21952(n-1)^2 + 23728(n-1)^3 + 12699(n-1)^4 + 2598(n-1)^5 + 106(n-1)^6$

$P(St(15,n)) = 4236 + 23060(n-1) + 54306(n-1)^2 + 69446(n-1)^3 + 48618(n-1)^4 + 16206(n-1)^5 + 1804(n-1)^6 + 42(n-1)^7$

$P(St(16,n)) = 8337 + 49444(n-1) + 129225(n-1)^2 + 190086(n-1)^3 + 163972(n-1)^4 + 77174(n-1)^5 + 16016(n-1)^6 + 952(n-1)^7 + 14(n-1)^8$

$P(St(17,n)) = 16647 + 107099(n-1) + 307386(n-1)^2 + 509320(n-1)^3 + 518866(n-1)^4 + 315277(n-1)^5 + 100766(n-1)^6 + 12956(n-1)^7 + 452(n-1)^8 + 6(n-1)^9$

$P(St(18,n)) = 32963 + 227682(n-1) + 710703(n-1)^2 + 1305834(n-1)^3 + 1526512(n-1)^4 + 1131718(n-1)^5 + 494043(n-1)^6 + 107072(n-1)^7 + 8430(n-1)^8 + 116(n-1)^9$

$P(St(19,n)) = 6592 + 6487946(n-1) + 1646834(n-1)^2 + 3319058(n-1)^3 + 4362414(n-1)^4 + 3796502(n-1)^5 + 2100180(n-1)^6 + 662816(n-1)^7 + 96906(n-1)^8 + 4646(n-1)^9 + 68(n-1)^{10}$

$P(St(20,n)) = 130787 + 1031600(n-1) + 3738425(n-1)^2 + 8183350(n-1)^3 + 11902732(n-1)^4 + 11829600(n-1)^5 + 7884416(n-1)^6 + 3289314(n-1)^7 + 746888(n-1)^8 + 72022(n-1)^9 + 2022(n-1)^{10} + 16(n-1)^{11}$

$P(St(21,n)) = 261574 + 2192679(n-1) + 8497908(n-1)^2 + 20074322(n-1)^3 + 31959848(n-1)^4 + 35588411(n-1)^5 + 27632358(n-1)^6 + 14345136(n-1)^7 + 4541440(n-1)^8 + 742606(n-1)^9 + 47647(n-1)^{10} + 922(n-1)^{11} + 6(n-1)^{12}$

$P(St(22,n)) = 520095 + 4613914(n-1) + 19027321(n-1)^2 + 48188560(n-1)^3 + 83180055(n-1)^4 + 102214578(n-1)^5 + 90121675(n-1)^6 + 55675764(n-1)^7 + 22668899(n-1)^8 + 5424436(n-1)^9 + 628142(n-1)^{10} + 26024(n-1)^{11} + 348(n-1)^{12}$

Theorem 8.47. *The diagrams described in theorem 8.45 have the following property: the diagram of $St(m,1)$ is contained in $St(m+1,1)$*

Proof: First we prove that if $A(T \cup St(m,1)) = St(m,1)$ then $A(T \cup St(m+1,1)) = St(m+1,1)$. The B-Poset of both is the chaos poset, so $A(T \cup St(m+1,1)) \subseteq St(m+1,1)$. Now, given $x \in T$ we know that $x \notin A(T \cup St(m,1))$ so $\exists y \in T \cup St(m,1)$ s.t. $x+y \notin T \cup St(m,1)$. $y \in St(m,1) \implies x+y \in St(m,1) \cup \{x\}$ therefore $y \in T$. It follows that $x + T \not\subseteq T \cup St(m+1,1)$ i.e. $x \notin A(T \cup St(m+1,1))$ and hence $A(T \cup St(m+1,1)) = St(m+1,1)$

Moreover if there was an edge from T_1 to T_2 in the m^{th} diagram then $T_1 \subseteq T_2$ and $\forall x \in T_2\ x + T_1 \not\subseteq T_2 \cup St(m,1)$ which implies $\forall x \in T_2\ x + T_1 \not\subseteq T_2 \cup St(m+1,1)$ and hence there is an edge from T_1 to T_2 in the $m+1^{th}$ diagram as well.

Corollary 8.47.1. $degree(g_m(x)) \leq degree(g_{m+1}(x))$

Corollary 8.47.2. $a_{m,d} \leq a_{m+1,d}$

Lemma 8.48. *Given T_1, T_2 s.t. $A(T_1 \cup St(m,1)) = A(T_2 \cup St(m,1)) = St(m,1)$ and there is an edge between T_1 and T_2. If $|T_1| = k-1$ then $|T_2| \leq m-2-\lfloor \frac{m-2}{k} \rfloor$*

Proof: Let $G = \mathbb{N} \setminus (T_2 \cup St(m,1))$, $|G| = g+1$. Also say $m-1 = qk+r$, $0 \leq r \leq k-1$. We know that $\forall x \in T_2\ \exists y \in T_1$ s.t. $x+y \in G$. Given $z \in G$ it can appear as $x+y$ for at most $k-1$ $x \in T_2$. Of course the same x could have multiple z correspond to it, so there could be some double counting. Therefore $|T_2| \leq (k-1)|G|$ and $|T_2| \leq (k-1)|G| - a$ (for some $a \geq 0$). Finally note that $|T_2| + |G| = m-1$ i.e. $k|G| - a = m-1$.

Now if $r = 0$ then $|G| \geq \frac{m-1}{k} = 1 + \lfloor \frac{m-2}{k} \rfloor$, so $|T_2| = m-1-|G| \leq m-2-\lfloor \frac{m-2}{k} \rfloor$

And if $r > 0$ then $k | r+a$ so $a \geq k-r$ and hence $k|G| = m-1+a \geq m-1+k-r = k(q+1)$ and hence $|G| \geq q+1$. Remember $m-1 = kq+r$, so $r \geq 1 \implies \lfloor \frac{m-2}{k} \rfloor = q$. Lastly $|T_2| = m-1-|G| \leq m-2-q = m-2-\lfloor \frac{m-2}{k} \rfloor$

Lemma 8.49. *Given $2 \leq k \leq m-1$. Let $T_1 = \{y | 1 \leq y \leq k-1\}$ and $T_2 = \{x | 1 \leq x \leq m-2, k \nmid x\}$.*

Then $A(T_1 \cup St(m,1)) = A(T_2 \cup St(m,1)) = St(m,1)$ and there is an edge from T_1 to T_2 of length $m-2-\lfloor \frac{m-2}{k} \rfloor$

Proof: Firstly we know that $St(m,1) \subseteq A(T_1 \cup St(m,1))$ and $St(m,1) \subseteq A(T_2 \cup St(m,1))$.

Next, given $y \in T_1$ we see that $k-y \in T_1$, $y+(k-y)=k \notin T_1 \cup St(m,1)$ therefore $y \notin A(T_1 \cup St(m,1))$ and $A(T_1 \cup St(m,1)) = St(m,1)$

Next, given $x \in T_2$ Case 1: $x+k-1 \leq m-1$. Now say $x \equiv y (\text{mod } k)$ $(1 \leq y \leq k-1)$. We see that $k-y \in T_2$ and $k-y \in T_1$ $(1 \leq k-y \leq k-1)$, and $x+k-y \notin T_2 \cup St(m,1)$ which means $x \notin A(T_2 \cup St(m,1))$ and $x+T_1 \nsubseteq T_2 \cup St(m,1)$

Case 2: $x+k-1 > m-1$ i.e.$(m-1)-x < k-1$ so $(m-1)-x \in T_1$ and $(m-1)-x \in T_2$ and $x+((m-1)-x) = m-1 \notin T_2 \cup St(m,1)$ which means $x \notin A(T_2 \cup St(m,1))$ and $x+T_1 \nsubseteq T_2 \cup St(m,1)$.

Combining we see that $A(T_2 \cup St(m,1)) = St(m,1)$ and that there is an edge from T_1 to T_2.

Theorem 8.50. *$degree(g_m(x)) = m-1-\lceil\sqrt{m-2}\rceil - \lfloor\frac{m-2}{\lceil\sqrt{m-2}\rceil}\rfloor$*

Proof: Given an edge from T_1 to T_2 of length d, if $|T_1| = k-1$ then by lemma 8.48 $|T_2| \leq m-2-\lfloor\frac{m-2}{k}\rfloor$ so $d \leq m-1-k-\lfloor\frac{m-2}{k}\rfloor$ and by elementary calculus we see that $d \leq m-1-\lceil\sqrt{m-2}\rceil - \lfloor\frac{m-2}{\lceil\sqrt{m-2}\rceil}\rfloor$.

Finally picking $k = \lceil\sqrt{m-2}\rceil$ in lemma 8.49 we find an edge of length $m-1-\lceil\sqrt{m-2}\rceil - \lfloor\frac{m-2}{\lceil\sqrt{m-2}\rceil}\rfloor$. Therefore it is the length of the longest edge and hence the degree of $g_m(x)$

Remark 8.51. *The sequence $deg(g_m(x))$ can be described combinatorially as follows: let $n = m-1$ we draw a lattice spiral of n points $(0,0),(0,1),(1,1),(1,0),(0,2),(1,2),(2,2),(2,1),(2,0),$ Let the number of lattice squares (of area 1) formed be d_n. Note that each point leads to a new square except for k^2+1^{th} point $(k \geq 0)$ and k^2+k+1^{th} point $(k \geq 1)$. Say $(a-1)^2 < n-1 \leq a^2$ (So $a = \lceil\sqrt{n-1}\rceil$)*

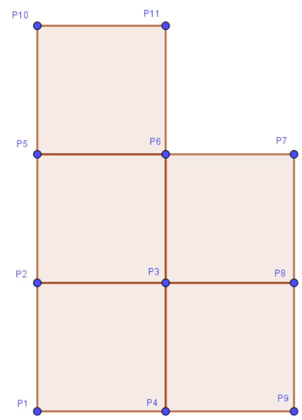
- *If $n-1 \neq a^2$. Then $\#\{k^2+1 | 0 \leq k, k^2+1 \leq n\} = a$.*

  Now if $n < (a-1)^2+(a-1)+1$ (i.e. $n-1 < (a-1)a$) then $\#\{k^2+k+1 | 1 \leq k, k^2+k+1 \leq n\} = a-2 = \lfloor\frac{n-1}{a}\rfloor$. (Notice that $a-2 = \frac{a^2-2a}{a} < \frac{a^2-2a+1}{a} \leq \frac{n-1}{a} < a-1$). Hence $d_n = n-a-\lfloor\frac{n-1}{a}\rfloor$

  On the other hand if $n \geq (a-1)^2+(a-1)+1$ (i.e. $n-1 \geq (a-1)a$) then $\#\{k^2+k+1 | 1 \leq k, k^2+k+1 \leq n\} = a-1 = \lfloor\frac{n-1}{a}\rfloor$ (as $a-1 \leq \frac{n-1}{a} < \frac{a^2}{a} = a$). Therefore $d_n = n-a-\lfloor\frac{n-1}{a}\rfloor$

- *If $n-1 = a^2$. Then $\#\{k^2+1 | 0 \leq k, k^2+1 \leq n\} = a+1$*

  Also $n > (a-1)^2+(a-1)+1$ (i.e. $n-1 \geq (a-1)a$) and hence $\#\{k^2+k+1 | 1 \leq k, k^2+k+1 \leq n\} = a-1 = \lfloor\frac{n-1}{a}\rfloor - 1$ (as $\frac{n-1}{a} = a$). Therefore $d_n = n-a-\lfloor\frac{n-1}{a}\rfloor$



8.3.2 Transposed Staircase $St(l,m,n)$ Families

Definition 8.52. $St(l,m,n) = \{0, lm, (l+1)m, \ldots (l+n)m \rightarrow\}$.

In general, for constant l and m, the P values these semigroups follow the same pattern as the corresponding $St(m,n)$ staircase for n large enough.

Theorem 8.53. *When $m = 2$, $P(S)$ is constant. In particular, with l constant, as sufficiently large n grows, the size of the void and the structure of the red triangles stays the same.*

Proof: Consider $S = 2l, 2l+2, \ldots 2m \rightarrow$, with $F = 2m-1$. Then, $B(S) = \{2, 4, \ldots 2l-2, F-2l+2, \ldots F-2\}$. Then, $|B(S)|$ does not depend on $2m$. Furthermore, if $(a,b,c) \in B$ is a red triangle, then $a+b+c = F$. However, the first half of B is even, and the second half is odd, so without loss of the generality, let $c = F-2k$, a, b even. Then for a different semigroup in the family, S' with Frobenius number F', all red triangles $(a, b, F-2k)$ of S correspond to red triangles of S', $(a, b, F'-2k)$.

Let $T \subseteq B$ have $A(T \cup S) = S$. Then, for semigroup S' with $S \subseteq S'$, define T' as T except for $a > \frac{F}{2}$, $a' = a + F' - F \in T'$. Then, $T' \subseteq B'$ and $A(T' \cup S') = S'$.

If $s \in T'$, then if $s < \frac{F'}{2}$, $s \in T$, so it must cancel. If $s > \frac{F'}{2}$, $s + F - F' \in T$, so either $F' - s \in T$ or there is a triangle with two even elements (which are the same in T and T' so it cancels.

If $s \in S \subseteq S'$, $s + TT \cup S$. For $t \in T$ with $t < \frac{F}{2}$, t is even so $s + t \in S'$. If $t > \frac{F}{2}$, it is shifted down along with F', so $s + t' \subseteq T' \cup S'$.

If $s \in S' \setminus S$, let $t' \in T'$. If $t' > \frac{F}{2}$, and $s' < F'$, $s' + t'$ is even so it is in S. If $t' < \frac{F'}{2}$, it is equivalent to $t \in T$. Then, if $s + t \notin T' \cup S'$, $s + t' < F'$, but then $s + F - F' + t' < F$, but $s + F - F' \in S$, which is a contradiction.

Thus, $A(T' \cup S') = S'$.

For $m = 3$, it is also constant. To map one T numerical set to another, if $a \in T$ is $0 \mod 3$, keep it the same. If $a \equiv 1 \mod 3$, and $F - 1 \in T$, delete a if

$a > F'$ and leave it otherwise; if $F-1 \notin T$, replace it with $a-F+F'$. If $a \equiv 2 \mod 3$, replace with $a-F+F'$.

We can also look at the P values for transposed staircase semigroups where the conductor is not necessarily a multiple of m. For example, the $3n$ staircase $\{0,6,9,12,15,17 \rightarrow\}$.

Example 8.54. *The P values for some of these families:*

$$S=\{0,10,15,20,25,\ldots c\rightarrow\}\colon P(S)=\begin{cases}26\lfloor\frac{c}{5}\rfloor+58 & c\equiv 0,1 \mod 5\\ 26\lfloor\frac{c}{5}\rfloor+54 & c\equiv 3 \mod 5\\ 100 & c\equiv 2,4 \mod 5\end{cases}$$

$$S=\{0,15,20,25,30,35\ldots c\rightarrow\}\colon P(S)=\begin{cases}532\lfloor\frac{c}{5}\rfloor+1096 & c\equiv 0,1 \mod 5\\ 532\lfloor\frac{c}{5}\rfloor+998 & c\equiv 3 \mod 5\\ 2184 & c\equiv 2,4 \mod 5\end{cases}$$

$$S=\{0,12,18,24,30,\ldots c\rightarrow\}\colon P(S)=\begin{cases}200\lfloor\frac{c}{6}\rfloor+115 & c\equiv 0,1 \mod 6\\ 100\lfloor\frac{c}{6}\rfloor+132 & c\equiv 2 \mod 6\\ 150\lfloor\frac{c}{6}\rfloor+160 & c\equiv 3 \mod 6\\ 100\lfloor\frac{c}{6}\rfloor+166 & c\equiv 4 \mod 6\\ 150\lfloor\frac{c}{6}\rfloor+326 & c\equiv 5 \mod 6\end{cases}$$

$$S=\{0,14,21,28,35,\ldots c\rightarrow\}\colon P(S)=\begin{cases}172\lfloor\frac{c}{7}\rfloor^2+834\lfloor\frac{c}{7}\rfloor+716 & c\equiv 0,1 \mod 7\\ 86\lfloor\frac{c}{7}\rfloor^2+597\lfloor\frac{c}{7}\rfloor+667 & c\equiv 2 \mod 7\\ 86\lfloor\frac{c}{7}\rfloor^2+780\lfloor\frac{c}{7}\rfloor+642 & c\equiv 3 \mod 7\\ 86\lfloor\frac{c}{7}\rfloor^2+552\lfloor\frac{c}{7}\rfloor+544 & c\equiv 4 \mod 7\\ 86\lfloor\frac{c}{7}\rfloor^2+736\lfloor\frac{c}{7}\rfloor+808 & c\equiv 5 \mod 7\\ 86\lfloor\frac{c}{7}\rfloor^2+927\lfloor\frac{c}{7}\rfloor+1501 & c\equiv 6 \mod 7\end{cases}$$

9 Max Embedding Dimension

Definition 9.1. *Given a Numerical Semigroup S, we define the void-height of S as following:*

Say Apery set of S is $(0,P_1,P_2,\ldots,P_{m-1})$ s.t. $P_i \equiv i (mod\, m)$.

Then the void-height of S is the smallest element of the set $D=\{\frac{P_i+P_j-P_r}{m} | i+j\equiv r (mod\, m) 1\le i,j,r\le m-1\}$. It is denoted by $h(S)$

Lemma 9.2. *S is of maximum embedding dimension iff $h(S)\geq 1$*

Definition 9.3. *Given a numerical semigroup S with Apery set $(0,P_1,P_2,\ldots,P_{m-1})$ s.t. $P_i\equiv i(mod\, m)$. We define $E(S,n)$ to be the numerical semigroup generated by $\{m,P_1+mn,P_2+mn,\ldots,P_{m-1}+mn\}$*

Remark 9.4. *Note that $E(E(S,n_1),n_2)=E(S,n_1+n_2)$ and $E(S,n)\subseteq S$.*

Lemma 9.5. *$h(E(S,n))=h(S)+n$, $F(E(S,n))=F(S)+nm$*

Corollary 9.5.1. *If $n\geq 1$ then $E(S,n)$ has max embedding dimension.*

Lemma 9.6. *If $x \geq 0$ then $mn + x \in E(S,n)$ iff $x \in S$*

Lemma 9.7. *$x \in E(S,n)$ iff $m|x$ or $x - mn \in S$*

In the next few lemmas we describe how to obtain the B-Poset of $E(S,n)$ given the B-Poset of S

Lemma 9.8. *If the Apery Poset of S is $(0, P_1, P_2, \ldots, P_{m-1})$ s.t. $P_i \equiv i (mod\ m)$ and $P_b = Max\{P_i\}$ (i.e. $F(S) = P_b - m$) then $B(E(S,n)) = B(S) \cup \{P_i + am | 0 \leq a \leq n-1, b \neq i\}$*

Proof: If $x \in B(S)$ then $x \notin S$ and hence $x \notin E(S,n)$. If $F(E(S,n)) - x \in E(S,n)$ then $F(S) + mn - x \in E(S,n)$ and by lemma 9.6 $F(S) - x \in S$ which is a contradiction. Therefore $x \in B(E(S,x))$ and $B(S) \subseteq B(E(S,x))$.

If $x \in Gap(S) \setminus B(S)$ then $F(S) - x \in S$ and hence $F(E(S,n)) - x = F(S) - x + mn \in E(S,n)$ and $x \notin B(E(S,n))$

Finally if $x \in Gap(E(S,n)) \cap S$ then $x = P_i + ml$ for some i, $0 \leq l \leq n-1$. $F(E(S,n)) - x = F(S) + mn - (P_i + ml) = F(S) - P_i + m(n-l)$

Case 1: $i = b$, then $F(E(S,n)) - x = (P_i - m) - P_i + m(n-l) = m(n-1-l) \in E(S,n)$ and $x \notin B(E(S,n))$

Case 2: $i \neq b$, then $m \not| F(E(S,n)) - x$ therefore by lemma 9.7 $F(E(S,n)) - x \in E(S,n)$ iff $F(E(S,n)) - x - mn \in S$, but $F(E(S,n)) - x - mn = F(S) - P_i - ml$ and if it was in S then $F(S) = (F(S) - P_i - ml) + (P_i + ml) \in S$ which is impossible. Therefore $F(E(S,n)) - x \notin E(S,n)$ and hence $x \in B(E(S,n))$ (remember $x \in Gap(E(S,n))$) $\square$

Lemma 9.9. *If $x \in B(E(S,n)) \setminus B(S)$, $y \in B(E(S,n))$ then $x \preccurlyeq y \implies m|y-x$ ($\preccurlyeq$ is of $B(E(S,n))$)*

Proof: By lemma 9.8 $x \in B(E(S,n)) \setminus B(S)$ implies $x = P_i + am$ with $i \neq b$ and $0 \leq a \leq n-1$. $x \preccurlyeq y$ means that $y - x \in E(S,n)$. By lemma 9.7 either $m|y-x$ or $y - x - mn \in S$.

If $m \not| y - x$ then $y - x - mn \in S$. Note that $x \in S$ and hence $y - mn \in S$ and hence $y \in E(S,n)$ which is a contradiction. $\square$

Lemma 9.10. *If $x \in B(S)$, $y \in B(E(S,n))$ and $m \not| y - x$ then*
$x \preccurlyeq y$ in $B(E(S,n))$ implies $y - mn \in B(S)$

Proof: We know that $y - x \in E(S,n)$ and $m \not| y - x$ so by lemma 9.7 $y - x - mn \in S$. $B(S) + S \in S \cup B(S)$ therefore $y - mn = x + (y - x - mn) \in S \cup B(S)$. Now if $y - mn \in S$ then $y \in E(S,n)$ which is a contradiction. Therefore $y - mn \in B(S)$

Lemma 9.11. *If $z \in B(S)$ then $z + mn \in B(E(S,n))$*

Proof: We know that $B(S) + S \subseteq S \cup B(S)$ so $z + mn \in S \cup B(S)$. $z + mn \in B(S) \implies z + mn \in B(E(S,n))$, on the other hand if $z + mn \in S$ then $z + mn = P_i + am$ for some i, $a \geq 0$. So $z = P_i - (n-a)m$, since $z \notin S$ we must have $n - a \geq 1$. Moreover we have $(P_i - m) - z = (n-a-1)m \in S$ so $P_i - m \neq F(S)$ i.e. $i \neq b$ and hence by lemma 9.8 $z + mn \in B(E(S,n))$

Lemma 9.12. *If $x \in B(S)$, $y \in B(E(S,n))$ and $m \not| y - x$ then*
$x \preccurlyeq y$ in $B(E(S,n))$ iff $x \preccurlyeq y - mn$ in $B(S)$

Proof: We know that $m \not| y - x$ so by lemma 9.7 $y - x \in E(S,n)$ iff $y - x - mn \in S$

Definition 9.13. *Say the Apery Set of S is $(0, P_1, P_2, \ldots, P_{m-1})$ we define $L(S)$ to be the Numerical Semigroup generated by $\{m, P_1 - m, P_2 - m, \ldots, P_{m-1} - m\}$*

Lemma 9.14. *$L(E(S,1)) = S$*
Moreover if all $P_i > 2m$ and $h(S) \geq 1$ then $E(L(S),1) = S$

Lemma 9.15. *If $h(S) = h$ and $m \geq 3$ then $P_l > hm$ for each l*

If $P_l \neq F + m$ then $\exists l' \neq 0$ s.t. $P_{l+l'(mod\ m)} = F + m$ and hence $hm \leq P_l + P_{l'} - (F + m) < P_l$ (As $P_{l'} < F + m$). Moreover if $P_l = F + m$ then pick an $P_a \neq F + m$ (such a exists as $m > 2$) then $km < P_a < F + m$

Corollary 9.15.1. *If $h(S) \geq 2$ then $E(L(S),1) = S$*

Corollary 9.15.2. *Say $h(S) = h$, $S_1 = L(S)$, $S_2 = L(S_1)$,... $S_{h-1} = L(S_{h-2})$. Then $E(S_{h-1}, h-1) = S$ and $h(S_{h-1}) = 1$*

Lemma 9.16. *As always let the Apery set of S be $(0, P_1, P_2, \ldots, P_{m-1})$, assume $h(S) \geq 1$, let $S' = E(S,n)$. If $(P_i + (n-1)m, a, b)$ is a red triangle of $E(S,n)$ then $a, b \in B(S)$*

Proof: If $a \notin B(S)$ then $a \in B(E(S,n)) \setminus \{B(S)\}$ and by lemma 9.8 $a = P_j + lm$ s.t. $P_j - m \neq F(S)$ and $0 \leq l \leq n-1$.

$F(E(S,n)) = P_i + (n-1)m + a + b$, hence $b = F(E(S,n)) - P_i - (n-1)m - a = F(S) + mn - P_i - mn + m - P_j - lm = F(S) - P_i - P_j + m - lm$. Now $h(S) \geq 1$ means that $P_i + P_j - m \in S$ and hence $P_i + P_j - m + lm \in S$ and $b = F(S) - (P_i + P_j - m + lm) \in Gap(S) \setminus B(S)$ and by lemma 9.8 this contradicts the fact that $b \in B(E(S,n))$

Lemma 9.17. *If $h(S) \geq 1$, $(0, P_1, P_2, \ldots, P_{m-1})$ is the Apery Set of S as always. Then $(P_i + (n-1)m, a, b)$ is a red triangle of $E(S,n)$ iff $(P_i - m, a, b)$ is a red triangle of S*

Proof: Firstly note that if $(P_i + (n-1)m, a, b)$ is a red triangle of $E(S,n)$ then $a, b \in B(S)$. Also of course if $(P_i - m, a, b)$ is a red triangle of S then $a, b \in B(S)$. Therefore in both directions we can assume $a, b \in B(S)$

Now $(P_i + (n-1)m, a, b)$ is a red triangle of $E(S,n)$ iff $F(E(S,n)) = P_i + nm - m + a + b$ iff $F(S) = P_i - m + a + b$ iff $(P_i - m, a, b)$ is a red triangle of S $\square$

Lemma 9.18. $F(E(S,n)) - (P_i + (n-1)m) = F(S) - (P_i - m) \in B(S)$

Definition 9.19. *Assume $h(S) \geq 1$ We define categories among subsets of $B(E(S,n))$. If T is an order ideal of $B(E(S,n))$ the category of T is $(T \cap B(S), \{x | x \in B(S), F(E(S,n)) - x \in T\})$*

Lemma 9.20. *Assume $h(S) \geq 1$. If T is an order ideal of $B(E(S,n))$ and T' is an order ideal of $B(E(S,n'))$ s.t. T and T' have the same category.*

Then $A(T \cup E(S,n)) = E(S,n)$ iff $A(T' \cup E(S,n')) = E(S,n')$

Proof: Assume $A(T \cup E(S,n)) = E(S,n)$. Now if $P_i + (n'-1)m \in T'$, note that $x = F(E(S,n')) - (P_i + (n'-1)m) = F(S) - (P_i - m) \in B(S)$. Now T and T' have the same category so $F(E(S,n)) - x \in T$. Note that $F(E(S,n)) - x = F(E(S,n)) - F(S) + P_i - m = P_i + (n-1)m$

Now we know from theorem 3.13 that either $F(E(S,n)) - (P_i + (n-1)m) \in T$ or there is a triangle $(P_i + (n-1)m, a, b)$ of $B(E(S,n))$ for which $a \in T$ and $F(E(S,n)) - b \notin T$

Case 1: $F(E(S,n)) - (P_i + (n-1)m) \in T$, then note that $F(E(S,n)) - (P_i + (n-1)m) = F(S) - (P_i - m) \in B(S)$. Now since T and T' have the same category $F(S) - (P_i - m) \in T'$. Finally note that $F(S) - (P_i - m) = F(E(S,n')) - (P_i + (n'-1)m)$

Case 2: $(P_i + (n-1)m, a, b)$ is a triangle of $B(E(S,n))$ for which $a \in T$ and $F(E(S,n)) - b \notin T$. Lemma 9.17 tells us that $(P_i - m, a, b)$ is a red triangle of S, $a, b \in B(S)$. A further application of lemma 9.17 tells us that $(P_i + (n'-1)m, a, b)$ is a red triangle of $E(S,n')$. Moreover $a, b \in B(S)$, $a \in T$, $F(E(S,n)) - b \notin T$ so T and T' having the same category implies that $a \in T'$ and $F(E(S,n')) - b \notin T'$

With theorem 3.13 we conclude that $A(T' \cup E(S,n')) = E(S,n')$

Lemma 9.21. *If T is an order ideal of $B(E(S,n))$ and $T' \subseteq B(E(S,n'))$ and T, T' have the same category and $x, y \in B(E(S,n'))$, $m|y-x$, $x \in T'$ implies $y \in T'$*

then T' is an order ideal of $B(E(S,n'))$

Proof: Let $x \in T'$, $x \preccurlyeq y$ in B-Poset of $E(S,n')$. Now if $m|y-x$ then $y \in T'$. So now assume $m \not| y-x$, lemma 9.9 says $x \in B(S)$. Next lemma 9.10 says $y - mn' \in B(S)$ and lemma 9.12 says $x \preccurlyeq y - mn'$ in B-Poset of S. A further application of lemma 9.12 says $x \preccurlyeq y - mn' + mn$ in the B-Poset of $E(S,n)$.

$x \in B(S)$, T and T' have the same categories, hence $x \in T$ which implies $y - mn' + mn \in T$.

$F(E(S,n)) - (y - mn' + mn) = F(S) + mn - y + mn' - mn = F(S) - (y - mn')$. We know that $y - mn' \in B(S)$ which means $F(S) - (y - mn')$, T and T' have the same categories, therefore $F(E(S,n')) - (F(E(S,n)) - (y - mn' + mn)) \in T'$. But $F(E(S,n')) - (F(E(S,n)) - (y - mn' + mn)) = F(E(S,n')) - (F(S) - (y - mn')) = F(S) + mn' - F(S) + y - mn' = y$. Thus $y \in T'$ and T' is an order ideal.

Lemma 9.22. *The number of good numerical sets of $E(S,n)$ within a fixed category is eventually a polynomial of n.*

Moreover its degree is then number of P_i s.t. $\forall y \in B(S)$ $y \equiv (F(S) - (P_i - m))(mod\ m)$ implies y is in the second component of the category and $\forall x \in B(S)$ $x \preccurlyeq P_i - m$ implies x is not in the first component of the category

Remark 9.23. *In most examples it seems that $P(E(S,n))$ is not just eventually a polynomial, but a polynomial from the start ($h(S) \geq 1$).*

However I am not entirely sure if this is true, $S =< 14, 34, 43, 54, 63, 72, 74, 83, 92, 94, 101, 103, 121, 123 >$ might be a counter e.g.

$P(E(S,0)) = 1214$, $P(E(S,1)) = 22180$, $P(E(S,2)) = 136690$, $P(E(S,3)) = 517844$, $P(E(S,4)) = 1488694$, $P(E(S,5)) = 3580084$, $P(E(S,6)) = 7595690$

Remark 9.24. *Remember these definitions for the next theorem:*

given an order ideal T of $B(S)$, $Tri(T) = \{(a,b)|a,b \in B(S), \exists P \in T \cap PF(S), P+a+b = F(S), a \in T, F(S)-b \notin T\}$, $X_1(T) = \{a|\exists b, (a,b) \in Tri(T)\}$ and $X_2(T) = \{y|\exists a, (a, F(S)-y) \in Tri(T)\}$

Remark 9.25. *We will show that $P(E(S,1)) \geq P(S)$ if $h(S) \geq 1$. In order to do this we define an injective map from good numerical sets of S to good numerical sets of $E(S,1)$. Given T s.t. $A(T \cup S) = S$ define $f_1(T) = \{x+m|x \in T\} \cup \{x|x \in T, \forall z \in X_2(T), x \not\equiv z (mod\ m)\}$*

Lemma 9.26. *If $h(S) \geq 1$ and T is an order ideal of $B(S)$ then $f_1(T)$ is an order ideal of $E(S,1)$*

Proof: if $x \in f_1(T)$ and $x \preccurlyeq y$ in $B(E(S,1))$

- if $x-m \in T$ and $x \equiv y(mod\ m)$ then $x-m, y-m \in B(S)$, $x-m \leq y-m$ so $x-m \preccurlyeq y-m$ in $B(S)$ which implies $y-m \in T$ which implies $y \in f_1(T)$
- if $x \in T, \forall z \in X_2(T), x \not\equiv z(mod\ m)$ and $x \equiv y(mod\ m)$; then $y \in T$ ($x \leq y$). And $\forall z \in X_2(T), y \not\equiv z(mod\ m)$. Therefore $y \in f_1(T)$
- if $x \not\equiv y(mod\ m)$ then $y - x \in E(S,1) \implies (y-m)-x \in S$. Now if $x \in B(S)$ then $x \in T$ and hence $y-m \in T$ and $y \in f_1(T)$. On the other hand if $x \notin B(S)$ then $x-m \in PF(S)$ which implies $x \in PF(E(S,1))$ which implies $y = x$

It follows that $f_1(T)$ is an order ideal of $B(E(S,1))$ $\square$

Theorem 9.27. *If $h(S) \geq 1$ and $A(T \cup S) = S$ then $A(f_1(T) \cup E(S,1)) = E(S,1)$*

Proof: If $P \in f_1(T) \cap PF(E(S,1))$ then $P-m \in T$ which means $P-m \in T \cap PF(S)$. Now theorem 3.13 implies that either $F(S)-(P-m) \in T$ or $\exists (a,b) \in Tri(T)$ s.t. $P-m+a+b = F(S)$

- if $F(S)-(P-m) \in T$; If $\exists z \in X_2(T)$ s.t. $z \equiv F(S)-(P-m)(mod\ m)$ then $F(S)-(P-m) \preccurlyeq z$ which implies $z \in T$ and we have a contradiction. Therefore $\forall z \in X_2(T)$ $z \not\equiv F(S)-(P-m)(mod\ m)$ and hence $F(S)-(P-m) \in f_1(T)$ Finally note that $F(S)-(P-m) = F(E(S,1))-P$
- Next if $\exists (a,b) \in Tri(T)$ s.t. $P-m+a+b = F(S)$. By corollary 3.19.1 $\forall z \in X_2(T)$ $z \not\equiv a(mod\ m)$ and hence $a \in f_1(T)$. And $F(E(S,1))-b-m = F(S)-b \notin T$, $F(E(S,1))-b-m \in X_2(T)$, hence $F(E(S,1))-b \notin f_1(T)$

It follows from theorem 3.13 that $A(f_1(T) \cup E(S,1)) = E(S,1)$ $\square$

Corollary 9.27.1. *$f_1(T) \cap PF(E(S,1)) = ((T \cap PF(S)) + m)$, $Mi(T) = Mi(f_1(T))$ and $Tri(T) \subseteq Tri(f_1(T))$*

Theorem 9.28. *If $h(S) \geq 1$ then $P(E(S,1)) \geq P(S)$*

Proof: We just need to show that the map is injective. Let T_1, T_2 be good numerical sets of S s.t. $f_1(T_1) = f_1(T_2)$. Then T_1 and T_2 have the same Pseudo-Frobenius numbers and the conjugates of the same Pseudo-Frobenius numbers.

If $x \in T_1$ and $x - m \in T_1$ then $x \in (f_1(T_1) \cap B(S)) = (f_1(T_2) \cap B(S))$ and hence $x \in T_2$

If $x \in T_1$ and $\forall z \in X_2(T_1)$ $z \not\equiv x (mod\ m)$ then $x \in (f_1(T_1) \cap B(S)) = (f_1(T_2) \cap B(S))$ and hence $x \in T_2$

If $(a,b) \in Tri(T_1)$, say $a + b + P = F(S)$ for $P \in T_1 \cap PF(S)$. Then $a \in f_1(T_1)$ as seen above and hence $a \in T_2$. Next $F(S) - b + m \notin f_1(T_1)$ because $F(S) - b \notin T_1$ and $F(S) - b + m \equiv F(S) - b (mod\ m)$ Now if $F(S) - b \in T_2$ then $F(S) - b + m \in f_1(T_2) = f_1(T_1)$ which is a contradiction. Therefore $Tri(T_1) = Tri(T_2)$

Finally if $x \in T_1$, $x - m \notin T_1$ and $\exists z \in X_2(T_1)$ $z \equiv x (mod\ m)$. Then $x + m \in f_1(T_1) = f_1(T_2)$. Now $z \in X_2(T_2)$ as $Tri(T_1) = Tri(T_2)$ therefore $x \in T_2$

We conclude that $T_1 = T_2$ and the map is injective. $\square$

Definition 9.29. *We define a new map $f_2(T) = \{x+m | x \in T\} \cup \{x | x \in T, \exists a \in X_1(T), x \equiv a (mod\ m)\} \cup Mi(T)$*

Lemma 9.30. *If $h(S) \geq 1$ and T is an order ideal of $B(S)$ then $f_2(T)$ is an order ideal of $B(E(S,1))$*

Proof: if $x \in f_2(T)$ and $x \preccurlyeq y$ in $B(E(S,1))$

- if $x - m \in T$ and $x \equiv y (mod\ m)$ then $x - m, y - m \in B(S)$, $x - m \leq y - m$ so $x - m \preccurlyeq y - m$ in $B(S)$ which implies $y - m \in T$ which implies $y \in f_2(T)$
- if $x \in T, \exists a \in X_1(T), x \equiv a (mod\ m)$ and $x \equiv y (mod\ m)$; then $y \in T$ $(x \leq y)$. And $y \equiv a (mod\ m)$. Therefore $y \in f_2(T)$
- if $x \in Mi(T)$ and $x \equiv y (mod\ m)$; then either $x = y$ in which case we are done or $x \preccurlyeq y - m$ in $B(S)$ which implies $y - m \in T$ which implies $y \in f_2(T)$
- if $x \not\equiv y (mod\ m)$ then $y - x \in E(S,1) \implies (y - m) - x \in S$. Now if $x \in B(S)$ then $x \in T$ and hence $y - m \in T$ and $y \in f_2(T)$. On the other hand if $x \notin B(S)$ then $x - m \in PF(S)$ which implies $x \in PF(E(S,1))$ which implies $y = x$

It follows that $f_2(T)$ is an order ideal of $B(E(S,1))$ $\square$

Theorem 9.31. *If $h(S) \geq 1$ and $A(T \cup S) = S$, then $A(f_2(T) \cup E(S,1)) = E(S,1)$*

Proof: If $P \in f_2(T) \cap PF(E(S,1))$ then $P - m \in T$ which means $P - m \in T \cap PF(S)$. Now theorem 3.13 implies that either $F(S) - (P - m) \in T$ or $\exists (a,b) \in Tri(T)$ s.t. $P - m + a + b = F(S)$

- if $F(S) - (P - m) \in T$; then $F(S) - (P - m) \in Mi(T) \subseteq f_2(T)$ Finally note that $F(S) - (P - m) = F(E(S,1)) - P$
- Next if $\exists (a,b) \in Tri(T)$ s.t. $P - m + a + b = F(S)$. Then $a \in X_1(T)$ and hence $a \in f_2(T)$. $F(S) - b \notin T$; $F(E(S,1)) - b - m \in B(S) \implies F(E(S,1)) - b \notin Mi(T)$. Also $F(E(S,1)) - b - m = F(S) - b \notin T$, by corollary 3.19.1 $\not\exists a' \in X_1(T)\ a' \equiv F(S) - b (mod\ m)$ and hence $F(E(S,1)) - b \notin f_2(T)$.

It follows from theorem 3.13 that $A(f_2(T) \cup E(S,1)) = E(S,1)$ □

Corollary 9.31.1. *$f_2(T) \cap PF(E(S,1)) = ((T \cap PF(S)) + m)$, $Mi(T) = Mi(f_2(T))$, $Tri(T) \subseteq Tri(f_2(T))$*

Lemma 9.32. *If $h(S) \geq 1$, then f_2 is an injective map.*

Proof: If $f_2(T_1) = f_2(T_2)$, then T_1 and T_2 have the same Pseudo-Frobenius numbers and $Mi(T_1) = Mi(T_2)$.

If $x \in T_1$ and $x - m \in T_1$, then $x \in (f_2(T_1) \cap B(S)) = (f_2(T_2) \cap B(S))$ and hence $x \in T_2$.

If $x \in T_1$ and $\exists a \in X_1(T_1)$ with $a \equiv x (mod\ m)$, then $x \in (f_2(T_1) \cap B(S)) = (f_2(T_2) \cap B(S))$, and hence $x \in T_2$.

If $x \in T_1$ and $x \in Mi(T_1)$ then $x \in T_2$.

If $(a,b) \in Tri(T_1)$, say $a + b + P = F(S)$ for $P \in T_1 \cap PF(S)$. Then $a \in f_2(T_1)$ and hence $a \in T_2$. Next $F(S) - b + m \notin f_2(T_1)$ because $F(S) - b \notin T_1$, $F(S) - b + m \notin Mi(T)$ and $\forall a' \in X_1(T_1) : F(S) - b \not\equiv a' (mod\ m)$. Now if $F(S) - b \in T_2$ then $F(S) - b + m \in f_2(T_2) = f_2(T_1)$ which is a contradiction. Therefore $Tri(T_1) = Tri(T_2)$.

Finally if $x \in T_1$, $x - m \notin T_1$, $x \notin Mi(T_1)$ and $\forall a' \in X_1(T_1)\ x \not\equiv a' (mod\ m)$. Then $x + m \in f_2(T_1) = f_2(T_2)$. Now $x + m \notin Mi(T_2)$, $\forall a' \in X_1(T_2)$ $x + m \not\equiv a' (mod\ m)$ as $Tri(T_1) = Tri(T_2)$ therefore the fact that $x + m \in f_2(T_2)$ implies $x \in T_2$.

We conclude that $T_1 = T_2$ and the map is injective. □

Lemma 9.33. *$h(S) \geq 1$, T is an order ideal of $B(S)$ then $f_2(T) \subseteq f_1(T)$*

Theorem 9.34. *If $h(S) \geq 1$, If $\exists T$ s.t. $A(T \cup S) = S$ and $f_1(T) \neq f_2(T)$ then $P(E(S,1)) > P(S)$*

Proof: If possible assume that $P(E(S,1)) = P(S)$. Then the maps f_1 and f_2 are both surjective. Now consider the set $Z = \{T' | f_1(T_1) = T' = f_2(T_2), T_1 \neq T_2\}$. The assumption implies that f_1 and f_2 are not identical functions and hence

Z is non empty. Now consider a maximal element of Z (under containment), say it is $T' = f_1(T_1) = f_2(T_2)$ with $T_1 \neq T_2$. Now $f_1(T_1) = f_2(T_2) \subseteq f_1(T_2)$ by lemma 9.33. Moreover the fact that $T_1 \neq T_2$ and f_1 being injective imply that $f_1(T_2)$ is strictly bigger than T' (under containment). But the maximality of T' implies that $f_1(T_2) = f_2(T_2)$ which is a contradiction.

Remark 9.35. *The previous theorem tell us that if $h(S) \geq 1$ and $P(E(S,1)) = P(S)$ then for every good numerical set of S $f_1(T) = f_2(T)$ which means that $\{x(mod\ m)|x \in T\} \setminus \{y(mod\ m)|y \in X_2(T)\} = \{a(mod\ m)|a \in X_1(T)\} \cup \{z(mod\ m)|z \in Mi(T)\}$*

Definition 9.36. *If $h(S) \geq 1$ and T' is a good numerical set of of $E(S,1)$. Then define $g_1(T') = (T' \cap B(S)) \cup \{x - m|x \in T', \exists z \in X_2(T')x \equiv z(mod\ m)\}$ (note that $g_1(T')$ is not always an order ideal of $B(S)$, but it is always a subset)*

Lemma 9.37. *If $h(S) \geq 1$ and T is an order ideal of of $B(E(S,1))$. Then $g_1(Nu(T))$ is an order ideal of $B(S)$*

Proof: Say $x \in g_1(Nu(T))$ and $x \preccurlyeq y$ in $B(S)$. (Also assume $x \neq y$ as otherwise we have nothing to prove)

- If $x \in Nu(T)$; $y-x \in S$ and hence $y-x+m \in E(S,1)$ which implies $y+m \in Nu(T)$. If possible assume $y \notin Nu(T)$ then either $\exists P \in PF(E(S,1)) \setminus T$ s.t. $y \preccurlyeq P$ in $B(E(S,1))$ or $\exists z \in X_2(T)$ s.t. $y \preccurlyeq z$.

  Note that we also have $x \not\equiv y(mod\ m)$ (because $x \equiv y(mod\ m)$ and $x < y$ imply $y \in Nu(T)$). Now $y + m - x \in E(S,1)$, $y - x \notin E(S,1)$ imply $y + m - x \in Ap(E(S,1))$

  - If $\exists P \in PF(E(S,1)) \setminus T$ s.t. $y \preccurlyeq P$ in $B(E(S,1))$; then $y + m \not\preccurlyeq P$ in $B(E(S,1))$. Therefore $y \not\equiv P(mod\ m)$, $P - y \in E(S,1)$ and $P - y - m \notin E(S,1)$ i.e. $P - y \in Ap(E(S,1))$

    Finally $h(E(S,1)) \geq 2$, so $P - y, y + m - x \in Ap(E(S,1))$ imply $(P-y)+(y+m-x)-2m \in E(S,1)$. $(P-y)+(y+m-x)-2m = P - x$ meaning $x \preccurlyeq P$ in $B(E(S,1))$ and hence $P \in T$ which is a contradiction
  - $\exists z \in X_2(T)$ s.t. $y \preccurlyeq z$; Replace P with z in the previous argument and it will work here.

  So $y \in Nu(T) \cap B(S)$ and hence $y \in g_1(Nu(T))$

- If $x \notin Nu(T)$; then $x \in g_1(T)$ implies $\exists z \in X_2(T)$ $x \equiv z(mod\ m)$ and $x + m \in Nu(T)$.

  Now $x + m \in Nu(T)$ implies $z \leq x$

  If $z < x$ then by corollary 3.14.2 $x \in Nu(T)$ which is a contradiction. And hence $z = x$

  Note that we also have $x \not\equiv y(mod\ m)$ (because $x \equiv y(mod\ m)$ and $x < y$ imply $y \in Nu(T)$).

By corollary 3.14.2 $z = x \prec y + m$ (in $B(E(S,1))$) implies $y + m \in T \subseteq Nu(T)$

If possible assume $y \notin g_1(Nu(T))$ then $y \notin Nu(T)$ and $\nexists z' \in X_2(T)$ $y + m \equiv z'(mod\ m)$

Now $y \notin Nu(T)$ so either $\exists P \in PF(E(S,1)) \setminus T$ s.t. $y \preccurlyeq P$ in $B(E(S,1))$ or $\exists z_1 \in X_2(T)$ s.t. $y \preccurlyeq z_1$

We combine the two cases, denote either P or z_1 by α, note that $\alpha \notin Nu(T)$

$\alpha - y \in E(S,1)$ and $\alpha - (y+m) \notin E(S,1)$, so $\alpha - y \in Ap(E(S,1))$

$y + m - x \in E(S,1)$, so $y + m - x = \beta + lm$ for some $\beta \in Ap(E(S,1))$ and $l \geq 0$.

Now $h(E(S,1)) \geq 2$ so $(\alpha - y) + \beta - 2m \in E(S,1)$ which implies $(\alpha - y) + (\beta + lm) - 2m \in E(S,1)$. $(\alpha - y) + (\beta + lm) - 2m = (\alpha - y) + (y + m - x) - 2m = \alpha - x - m$ i.e. $x + m \preccurlyeq \alpha$ in $B(E(S,1))$ which contradicts $\alpha \notin Nu(T)$

Lemma 9.38. *If $h(S) \geq 1$ and T is an order ideal of of $B(E(S,1))$. $(g_1(Nu(T)) \cap PF(S)) + m = Nu(T) \cap PF(E(S,1)) = T \cap PF(E(S,1))$*

Proof: Firstly it is clear that $((g_1(Nu(T)) \cap PF(S)) + m) \subseteq Nu(T) \cap PF(E(S,1))$

Next if $P \in Nu(T) \cap PF(E(S,1))$ and $P - m \notin g_1(Nu(T))$ then $P - m \notin Nu(T)$ and $\nexists z \in X_2(T)$ s.t. $z \equiv P(mod\ m)$.

$P - m \notin Nu(T)$ implies either $\exists Q \in PF(E(S,1)) \setminus T$ s.t. $P - m \preccurlyeq Q$ in $B(E(S,1))$ or $\exists z \in X_2(T)$ s.t. $P - m \preccurlyeq z$

Obviously $P - m \neq Q$; $z = P - m$ would imply $x \equiv z(mod; m)$ which is not the case.

Now we combine the two cases by denoting by α either Q or z. We have $P - m \prec \alpha$ in $B(E(S,1))$ (they are not equal). $\alpha \notin Nu(T) \implies \alpha \neq P$ and hence $P - m \not\equiv \alpha(mod\ m)$. It follows that $P - m \prec \alpha - m$ in $B(S)$ which is impossible as $P - m \in PF(S)$. $\square$

Theorem 9.39. *If $h(S) \geq 1$ and $A(T \cup E(S,1)) = E(S,1)$. Then $A(g_1(Nu(T)) \cup S) = S$*

Proof: Firstly we have shown that $g_1(Nu(T))$ is an order ideal of $B(S)$

Given $P \in g_1(Nu(T)) \cap PF(S)$ we know that $P + m \in T \cap PF(S)$. And by theorem 3.13 either $F(E(S,1)) - (P+m) \in T$ or there is a red triangle $(P+m, a, b)$ s.t. $a \in T$ and $F(E(S,1)) - b \notin T$.

- If $F(E(S,1)) - (P+m) \in T$; $F(E(S,1)) - (P+m) = F(S) - P \in Nu(T) \cap B(S)$ which implies $F(S) - P \in g_1(Nu(T))$
- If there is a red triangle $(P+m, a, b)$ s.t. $a \in T$ and $F(E(S,1)) - b \notin T$; $a \in T \implies a \in NU(T) \cap B(S) \implies a \in g_1(Nu(T))$.

  Also $F(E(S,1)) - b = F(S) - b + m \notin NU(T)$ implies $F(S) - b \notin g_1(Nu(T))$

Corollary 9.39.1. *If $h(S) \geq 1$ and $A(T \cup E(S,1)) = E(S,1)$. Then $Mi(Nu(T)) = Mi(g_1(Nu(T)))$ and $Tri(Nu(T)) = Tri(g_1(Nu(T)))$*

Proof: It is clear that $Mi(Nu(T)) = Mi(g_1(Nu(T)))$ and $Tri(Nu(T)) \subseteq Tri(g_1(Nu(T)))$. If $(a,b) \in Tri(g_1(Nu(T))) \setminus Tri(Nu(T))$ then either $a \notin Nu(T)$ or $F(E(S,1)) - b \in Nu(T)$

- If $a \notin Nu(T)$ then $a + m \in Nu(T)$ and $\exists z \in X_2(Nu(T))$ s.t. $z \equiv a (mod\, m)$.

  $a + m \in Nu(T) \implies z \leq a$

  If $z < a$ then by corollary 3.14.2 $a \in Nu(T)$ which is not the case. Therefore $z = a$

  But $z - m \in X_2(g_1(Nu(T)))$ and $a \in X_1(g_1(Nu(T)))$ and we get a contradiction to corollary 3.19.1

- If $a \in Nu(T)$ then $F(E(S,1)) - b \in Nu(T)$ and $F(S) - b \notin g_1(Nu(T))$ which implies that $F(S) - b \notin Nu(T)$ and $\not\exists z \in X_2(Nu(T))$ s.t. $z \equiv a (mod\, m)$

  $F(S) - b \notin Nu(T)$ implies either $\exists P \in PF(E(S,1)) \setminus T$ s.t. $F(S) - b \preccurlyeq P$ or $\exists z \in X_2(Nu(T))$ s.t. $F(S) - b \preccurlyeq z$

  In the second case $z \neq F(S) - b$; In the first case $P \neq F(S) - b$ as $F(S) - b + m \in B(E(S,1))$

  We combine the two cases by denoting either P or z by α, so $F(S) - b \prec \alpha$ in $B(E(S,1))$ and $\alpha \notin Nu(T)$

  $\alpha - F(S) + b \in E(S,1)$, $\alpha - F(S) + b - m \notin E(S,1)$ i.e. $\alpha - F(S) + b \in Ap(E(S,1))$

  Say Q was the Pseudo-Frobenius number of S for which $Q + a + b = F(S)$ so $F(S) - b = Q + a$. Thus $\alpha - F(S) + b = \alpha - Q - a = \alpha - a + m - (Q + m)$ i.e. $\alpha - a + m = (Q + m) + (\alpha - F(S) + b)$ i.e. $\alpha - a + 2m = (Q + 2m) + (\alpha - F(S) + b)$. And $h(E(S,1)) \geq 2$ implies $\alpha - a + 2m - 2m \in E(S,1)$ which implies $\alpha \in Nu(T)$ which is a contradiction.

Corollary 9.39.2. *If $h(S) \geq 1$ and $A(T \cup E(S,1)) = E(S,1)$. Then $f_1(g_1(Nu(T))) \subseteq Nu(T)$*

Proof: It follows from the previous corollary and the definitions of g_1 and f_1

Definition 9.40. *If $h(S) \geq 1$ and T' is a good numerical set of of $E(S,1)$. Then define $g_2(T') = (T' \cap B(S)) \cup \{x - m | x \in T' \setminus X_1(T), F(S) - x \notin PF(S)\}$ (note that $g_2(T')$ is not always an order ideal of $B(S)$)*

Lemma 9.41. *If $h(S) \geq 1$ and T is an order ideal of of $B(E(S,1))$. Then $g_2(Nl(T))$ is an order ideal of $B(S)$*

Proof: Say $x \in g_2(Nl(T))$ and $x \preccurlyeq y$ in $B(S)$. (Also assume $x \neq y$ as otherwise we have nothing to prove)

- If $x \in Nl(T)$; $y - x \in S$ and hence $y - x + m \in E(S,1)$ which implies $y + m \in Nl(T)$. Now $y \in B(S) \implies F(S) - (y + m) \notin PF(S)$. By lemma 3.14 (and the fact that $x \prec y + m$ in $B(E(S,1))$ we know that $y + m \notin X_1(Nl(T))$ and hence $y \in g_2(Nl(T))$.

- If $y \equiv x(mod\ m)$ then $y \in g_2(Nl(T))$

- If $y - x \in E(S,1)$ then $x + m \prec y + m$ in $B(E(S,1))$ so $y + m \in Nl(T)$ and by lemma 3.14 $y + m \notin X_1(Nl(T))$. Also $y \in B(S) \implies F(S) - (y + m) \notin PF(S)$ and hence $y \in g_2(Nl(T))$

- If $x \notin Nl(T)$ then $x + m \in Nl(T) \backslash X_1(Nl(T))$. Now $x + m \notin X_1(Nl(T)) \implies x + m \notin X_1(T)$ and $x \in B(S) \implies x + m \notin Mi(T)$, moreover $x \prec y$ in $B(S)$ implies $x \notin PF(S) \implies x + m \notin PF(E(S,1))$. It follows that $\exists \alpha \in X_1(T) \cup Mi(T)$ s.t. $\alpha \prec x + m$ in $B(E(S,1))$. Now $\alpha \not\equiv x(mod; m)$ (otherwise $x \in Nl(T)$). Observe that $x + m - \alpha \in E(S,1), x - \alpha \notin E(S,1)$ i.e. $x + m - \alpha \in Ap(E(S,1))$.

  We can assume $y \not\equiv x(mod\ m)$ and $y - x \notin E(S,1)$ (otherwise we are back to a previous case). Also $y - x \in S \implies y - x + m \in E(S,1)$ and hence $y + m - x \in Ap(E(S,1))$ (we have already done the case when $y - x \in E(S,1)$)

  Next $h(E(S,1)) \geq 2$ so $(y + m - x) + (x + m - \alpha) - 2m \in E(S,1)$ i.e. $y - \alpha \in E(S,1)$. $\alpha \in X_1(T) \cup Mi(T) \implies \alpha \in Nl(T) \implies y \in Nl(T) \implies y \in g_2(Nl(T))$

Lemma 9.42. $h(S) \geq 1$. *If T is an order ideal of $B(E(S,1))$ then $T \cap PF(E(S,1)) = (g_2(T) \cap PF(S)) + m$*

Lemma 9.43. *If $h(S) \geq 1$ and T is a good numerical set of $E(S,1)$. Then $A(g_2(Nl(T)) \cup S) = S$*

Proof: Firstly we have shown that $g_2(Nl(T))$ is an order ideal of $B(S)$

Say $P \in g_2(Nl(T)) \cap PF(S)$ then $P + m \in Nl(T) \cap PF(S) \implies P + m \in T \cap PF(S)$. So either $F(E(S,1)) - (P + m) \in T$ or there is a a red triangle $(P + m, a, b)$ s.t. $a \in T$ or $F(E(S,1)) - b \notin T$

- If $F(E(S,1)) - (P + m) \in T$; Note $F(E(S,1)) - (P + m) = F(S) - P \in Nl(T) \cap B(S) \implies F(S) - P \in g_2(Nl(S))$

- If there is a a red triangle $(P + m, a, b)$ s.t. $a \in T$ or $F(E(S,1)) - b \notin T$. So $P + a + b = F(S)$, $a \in Nl(T) \cap B(S)$ and hence $a \in g_2(Nl(T))$.

  $F(S) - b + m = F(E(S,1)) - b \notin Nl(T) \implies F(S) - b \notin g_2(Nl(T))$

So by theorem 3.13 $A(g_2(Nl(T)) \cup S) = S$

Corollary 9.43.1. *If $h(S) \geq 1$ and $A(T \cup E(S,1)) = E(S,1)$. Then $Mi(Nl(T)) = Mi(g_2(Nl(T)))$ and $Tri(Nl(T)) = Tri(g_2(Nl(T)))$*

Proof: Firstly we know $(g_2(Nl(T))\cap PF(S))+m = Nl(T)\cap PF(E(S,1)) = T\cap PF(E(S,1))$. And $Mi(Nl(T)) \subseteq Mi(g_2(Nl(T))), Tri(Nl(T)) \subseteq Tri(g_2(Nl(T)))$.

First we show that $Mi(g_2(Nl(T))) = Mi(Nl(T))$. If $P \in (g_2(Nl(T)) \cap PF(S))$ s.t. $F(S) - P \in g_2(Nl(T))$ but $F(S) - P \notin Nl(T)$. This means that $F(S) - P + m \in Nl(T)$ and $F(S) - P + m \in (X_1(Nl(T)) \cup Mi(T))$

Obviously $F(S) - P + m \notin Mi(T)$ so $F(S) - P + m \in X_1(T)$

Now $x \preccurlyeq F(S)-P+m$ in $B(E(S,1))$ and $x \neq F(S)-P, F(S)-P+m$ imply $x \preccurlyeq F(S)-P$ in $B(S)$ which is impossible. Therefore $F(S)-P+m$ is a minimal element of $Nl(T)$ and hence belongs to $X_1(T) \cap Mi(T) \cap (T \cap PF(E(S,1)))$.

We know $F(S) - P + m \notin (X_1(T) \cap Mi(T))$, hence $F(S) - P + m \in PF(E(S,1)) \cap T$. Which means that $F(S) - P + m \notin B(S)$ but this contradicts $F(S) - P + m \in X_1(T)$

Therefore $Mi(g_2(Nl(T))) = Mi(Nl(T))$.

Next we show that $Tri(Nl(T)) = Tri(g_2(Nl(T)))$. If possible say $(a,b) \in Tri(g_2(Nl(T))) \setminus Tri(Nl(T))$. So $a \in g_2(Nl(T))$, $F(S) - b \notin g_2(Nl(T))$ and either $a \notin Nl(T)$ or $F(S) + m - b \in Nl(T)$

- If $a \notin Nl(T)$; then $a \in g_2(Nl(T))$ implies $a + m \in Nl(T)$ and $a + m \in X_1(Nl(T)) \cup Mi(Nl(T))$

  Obviously $a+m \notin Mi(Nl(T))$, so $a+m \in X_1(Nl(T))$. But $X_1(Nl(T)) \subseteq X_1(g_2(Nl(T)))$ so both $a, a+m \in X_1(g_2(Nl(T)))$ which is a contradiction to corollary 3.19.1.

- And if $a \in Nl(T)$ then $F(S)+m-b \in Nl(T)$. Now $F(S)-b \notin g_2(Nl(T))$ implies $F(S) + m - b \in X_1(Nl(T)) \cup Mi(Nl(T))$

  Obviously $F(S)-b+m \notin Mi(Nl(T))$, So $F(S)-b+m \in X_1(Nl(T))$ but then $F(S)-b+m \in X_1(g_2(Nl(T)))$ and $F(S)-b \in X_2(g_2(Nl(T)))$ which contradicts corollary 3.19.1

Therefore $Tri(g_2(Nl(T))) = Tri(Nl(T))$

Corollary 9.43.2. *If $h(S) \geq 1$ and $A(T \cup E(S,1)) = E(S,1)$. Then $Nl(T) = f_2(g_2(Nl(T)))$*

Proof: Follows from previuos corollary and the definitions of g_2 and f_2

Remark 9.44. *Summarising several of the previous lemmas and theorems:*

If $h(S) \geq 1$ and T is a good numerical set of $E(S,1)$ then $g_2(Nl(T))$ and $g_1(Nu(T))$ are good numerical sets of S, $f_2(g_2(Nl(T)) = Nl(T)$ and $f_1(g_1(Nu(T)) = Nu(T)$

Consider the following sets of Numerical Sets:

$Z_1(T) = \{T' \subseteq B(S) | A(T' \cup S) = S, g_2(Nl(T)) \subseteq T' \subseteq g_1(Nu(T)), f_2(T') \subseteq T\}$

$Z_2(T) = \{T' \subseteq B(S) | A(T' \cup S) = S, g_2(Nl(T)) \subseteq T' \subseteq g_1(Nu(T)), T \subseteq f_1(T')\}$

We know that $g_2(Nl(T)) \in Z_1$ and $g_2(Nu(T)) \in Z_2$, so they are non empty.

Remark 9.45. *By continuing in this direction by picking a large* $T_1 \in Z_1$ *and a small* T_2 *in* Z_2 *hope to show* $T_1 = T_2$ *and* $f_2(T_1) \subseteq T \subseteq f_1(T_1)$

Possible approach: Try to find conditions under which for a given T_1 $\exists T'$ *s.t.* $f_2(T_1) = f_1(T')$

Conjecture 9.46. *For each good numerical set* T *of* $E(S, 1)$ *there is a good numerical set* T_1 *of* S *s.t.* $f_2(T_1) \subseteq T \subseteq f_1(T_1)$

Remark 9.47. *Consequences of the conjecture:*

- $P(S) = P(E(S, 1))$ *iff* f_1, f_2 *are identical functions*
- $P(S) = P(E(S, 1))$ *iff* $P(E(S, 1)) = P(E(S, 2))$

Another possible consequence might be that $P(E(S, n))$ *is a polynomial from the start.*

Lemma 9.48. *If* $h(S) \geq 1$ *and* $A(T \cup E(S, 1)) = E(S, 1)$ *then* $Mi(Nl(T)) \cup X_1(Nl(T)) = Mi(T) \cup X_1(T)$

Proof: We know that $Mi(Nl(T)) = Mi(T)$, $Tri(T) \subseteq Tri(Nl(T))$.

If $(a, b) \in Tri(Nl(T)) \backslash Tri(T)$, then using the fact that $Nl(T)$ is generated by $Mi(T) \cup X_1(T) \cup (T \cap PF(E(S, 1)))$ corollary **??** implies $a \in Mi(T) \cup X_1(T) \cup (T \cap PF(E(S, 1)))$. But $a \in T \cap PF(E(S, 1))$ is impossible as then $a \notin B(S)$. Therefore $X_1(Nl(T)) \subseteq X_1(T) \cap Mi(T)$

Definition 9.49 (term could be changed later or removed)**.** *Call a max embedding dimension semigroup covered if for each good numerical set of it* $f_1(T) = f_2(T)$.

Note that this is iff $\{x(mod\ m) | x \in T\} \subseteq \{a(mod\ m) | a \in X_1(T)\} \cup \{z(mod\ m) | z \in Mi(T)\} \cup \{y(mod\ m) | y \in X_2(T)\}$

Remark 9.50. *Note that* f_1, f_2 *are identical iff* S *is covered*

Theorem 9.51. *For a fixed multiplicity* m, *and natural number* P *if there is no max embedding dimension, non-bad (bad semigroups are defined in a later section) semigroup s.t.* $m(S) = m$, $P(S) = P$, $P(E(S, 1)) = P(S)$

then numerical semigroups with $P(S) = P$ *have density* 0 *(within semigroups of multiplicity* m*)*

Proof: under conditions of the theorem $\#\{S | m(S) = m, P(S) = P, S not bad, F(S) \leq F\} \leq \#\{S | m(S) = m, h(S) = 1, F(S) \leq F\}$ as $P(S) = P$ at most once on each ray. Finally semigroups with height 1 have density 0

Conjecture 9.52. *For each multiplicity the set* $\{P | \exists S, m(S) = m, P = P(S) = P(E(S, 1)), h(S) \geq 1, S\ is\ not\ bad\}$ *is finite*

9.1 Semigroups Along a Ray on a face of the Kunz Polyhedron

Semigroups whose Kunz tuples all lie on the same face of the Kunz Polyhedron have the same Apery Poset. Furthermore, we can find the Apery sets for all semigroups that lie along the same ray on a facet of the Kunz Polyhedron.

Lemma 9.53. *If S_0 with Apery set $\{0, a_1, a_2, \ldots, a_{m-1}\}$ is the first semigroup on a ray, semigroups S_k with Apery sets $\{0, a_1+mka_1, a_2+mka_2, \ldots, mka_{m-1}\}$ lie on the same ray.*

Proof: $\{0, a_1, a_2, \ldots, a_{m-1}\}$ is the first integer tuple on the ray, and the ray is then $(\lfloor\frac{a_1}{m}\rfloor + \frac{1}{m}, \lfloor\frac{a_2}{m}\rfloor + \frac{2}{m}, \ldots, \lfloor\frac{a_{m-1}}{m}\rfloor + \frac{m-1}{m})$. To get another integer value, if the greatest common divisor of the a_is is 1, we must add m times the ray to the first tuple, which gives a tuple of $(\frac{ma_1+a_1-1}{m}, \frac{ma_2+a_2-2}{m} \ldots \frac{ma_{m-1}+a_{m-1}-m+1}{m})$, corresponding to Apery set $\{0, a_1+mka_1, a_2+mka_2, \ldots, a_{m-1}+mka_{m-1}\}$.

If the greatest common divisor is not 1, the semigroups of this form do still lie along the ray, but if $d = gcd$, then adding $\frac{m}{d}$ times the ray to the initial semigroup will also produce integer points.

Lemma 9.54. *If the void of S_0 is B_0, then B_k the void of S_k is constructed by B_0 by noting for each $b \in B_0$, for $0 \leq l \leq mk$, $b + mbk + ml \in B_k$.*

Proof: Let a_f be the largest element of the Apery set, so $a_f - m = F$, the Frobenius number. Then, the Frobenius number of B_k is $a_f + ma_fk - m = F+m(F+m)k$. Note also that for $i, j < m$ where $i+j \equiv f \mod m$, all elements of the void set in the i equivalence class are between a_i and $F - (a_j - m)$.

The largest element in equivalence class i is $a_i - m$, and the smallest element is $F - a_j + m$. Then, the largest element of B_k in equivalent class i is $a_i + mka_i - m = (a_i - m) + m(a_i - m)k + m^2k$ which is satisfied by letting $l = mk$, and the smallest element is $F + m(F+m)k - (a_j + mka_j) = (F - a_j + m) + m(F - a_j + m)k$ which is reached when $l = 0$. □

Note that this means the void set grows by $m|B_0|$ as we move along the line. If the greatest common divisor is not 1, then the void grows by $\frac{m|B_0|}{d}$ between semigroups.

Considering *only* cases where the greatest common divisor of the elements of the ray is 1, we see that the structure of the void poset for semigroups further along the line can be constructed from the first one.

Note that the void elements of B_k corresponding to b are unique. If there is some element of B_k that corresponds to both b and b', $b + mbk + ml = b' + mb'k + ml'$, note that $b \equiv b'$, so $b' = b + ma$, so

$$b + ma + m(b+ma)k + ml' = b + mbk + ml$$

$$b + ma + mbk + m^2ak + ml' = b + mbk + ml$$

$$a + mak + l' = l$$

$$a(1+mk) = l - l'$$

But l and l' must be within mk, and so $a = 0$, so $b = b'$.

Thus every element in B_k corresponds to exactly one element of B_0. Furthermore, if we denote the generators of S_0 as $\langle m, g_2, \dots g_n \rangle$ and S_k as $\langle m, g_2 + mkg_2, \dots g_n + mkg_n \rangle$, we know from Lemma 2.19 that every cover relation in the B poset is a generator.

Lemma 9.55. *For every element b in B_0, if its cover relations are some subset of the generators, then g_i is an upper cover of b if and only if the corresponding generator of S_k, $g_i + mkg_i$ is an upper cover of the corresponding void element, $b + mbk + ml$.*

If g_i is an upper cover of b, $b + g_i = c \in B_0$. Then $b + mbk + ml + g_i + mkg_i = c + mck + ml$ for every $0 \leq l \leq mk$, so $g_i + mkg_i$ is an upper cover for elements of B_k corresponding to b.

If $g_i + mkg_i$ is an upper cover for $b + mk + ml$, then $b + mbk + ml + g_i + mkg_i = c + mck + ml'$, though l' is not necessarily equal to l. Then $b + g_i \equiv c \mod m$, so $b + g_i + ma = c$. Substituting, we get $m + mbk + ml + g_i + mkg_i = b + g_i + ma + m(b + g_i + ma)k + ml'$. Then, $l + kg_i = a + g_ik + mak + l'$, so $l - l' = a(1 + mk)$. If $a \neq 0$, $|l - l'| > mk$ which is impossible, so $a = 0$ and $b + g_i = c$, so g_i is an upper cover for b. $\square$

Lemma 9.56. *The red triangles of B_k correspond exactly to the red triangles of B_0.*

If (P, a, b) form a red triangle in B_0, $P + a + b = F_0$, then $((1 + mk)P + m^2k, (1 + mk)a, (1 + mk)b)$ also forms a red triangle. The Frobenius number $F_k = F_0 + (mF_0 + m^2)k$ because for a_i the maximal element of the Apery set of S_0, $F_k = m(ma_ik + a_i) - m = (1 + mk)F_0 + m^2k$. Thus, $(1 + mk)P + m^2k + (1 + mk)a + (1 + mk)b = (1 + mk)F_0 + m^2k = F_k$, so this forms a red triangle. Note that $(1 + mk)a$ and $(1 + mk)b$ are both minimal elements of B_k.

Pseudo-Frobenius numbers of B_k are of the form $(1+mk)P+m^2k$ because they are the largest elements corresponding to the maximal elements of B_0. Thus, if $((1 + mk)P + m^2k, (1 + mk)a + ml, (1 + mk)b + ml')$ is a red triangle in B_k, (P, a, b) is a triangle in B_0.

Note also that if I is an order ideal in B_0, then the set of all elements in B_k corresponding to the elements of I form an order ideal in B_k.

Theorem 9.57. *For semigroups formed in this way, the P value is a polynomial as we travel along the ray.*

Proof: First, we prove that an order ideal of B_k is a numerical set if and only if every "slice" of its poset is a numerical set in B_0. The lth slice of the B_k poset is just the B_0 poset, but for every $b \in B_0$, we change it to $b_k = b + mbk + ml$.

If there is some slice l that is not a numerical set, either it is not an order ideal, in which case B_k would also not be an order ideal, or the slice contains a Pseudo-Frobenius number and neither its conjugate nor a red triangle. If it contains some bad Pseudo-Frobenius number P_0 in B_0, then since the corresponding Pseudo-Frobenius number of B_k is above $P+mPk+ml$, $P_k \in T$, but its conjugate is below the conjugate of P_0, so $\overline{P_k} \notin T$. Similarly, P_0 cannot satisfy any red triangles (P_0, a_0, b_0), but since the red triangles of B_0 and B_k correspond exactly, $a_k \preccurlyeq a_0+ma_0k+ml$ and $\overline{b_0}+m\overline{b_0}k+ml \preccurlyeq \overline{b_k}$, so a_k cannot be included, which would mean the original order ideal of B_k is not a numerical set.

In the other direction, if every slice of the order ideal I_k is a numerical set (and itself an order ideal), then every Pseudo-Frobenius number in that slice satisfies either its conjugate or some red triangle in B_0. Assume for the sake of contradiction that I_k is not a numerical set. If I_k contains a Pseudo-Frobenius number P_k in B_k, then the top slice contains the corresponding Pseudo-Frobenius number P_0 where $P_k = (1+mk)P_0 + m^2k$. Since every slice is a numerical set, then the top slice must either contain the conjugate $\overline{P}_0$,

if Consider the slice where $l = 0$, and suppose the numerical set contains a Pseudo-Frobenius number P_0. Then, $P_0(1+mk) \in I_k$, and since P_k is above that and I_k is an order ideal, $P_k \in I_k$.

Now, to count the number of good numerical sets, we just need to stack slices chosen from the good numerical sets of B_0 in such a way that the result is an order ideal in B_k.

Remark 9.58. *This behavior also appears to apply to semigroups along rays in a face that do not start from the vertex; the void set grows by the same amount each time, the void poset maintains the same general structure, and the P values grow at the same rate.*

9.2 Asymptotics for $P(S) = 2, 3$

Theorem 9.59. *If S is of max embedding dimension and $m(S) \geq 5$ then $P(S) \geq 4$.*

Proof: Denote $m(S) = m$. Let the Apery set be $(0, P_1, P_2, \ldots, P_{m-1})$ s.t. $P_i \equiv i(mod\ m)$

Let $h(S) = h$, say $P_i+P_j = P_r+hm$. (Note $h \geq 1$ as S has max embedding dimension)

Note that $P_l > hm$ for each l by lemma 9.15 ($m > 2$)

Now $P_j > hm$ so $P_i < P_r$ and similarly $P_j < P_r$ and hence $P_i \neq F - m$ and $P_j \neq F - m$

By corollary 9.15.2 there is an S' s.t. $E(S', h-1) = S$. $h(S') = 1$, S' is of max embedding dimension. Apery Set of S' is $(0, P_1 - (h-1)m, \ldots, P_{m-1} - (h-1)m)$. Hence $P_i - hm \in B(S') \subseteq B(S)$ by lemma 9.8

Next if $P_i - hm \preccurlyeq x$ in the void then $x - (P_i - hm)$ is a multiple of m by lemma 9.12 (Remember that $P_i - hm$ is a Pseudo-Frobenius number of S')

Therefore the order ideal generated by $P_i - hm$ is $T_1 = \{P_i - nm | 1 \leq n \leq h\}$. Similarly $P_j - hm \in B$ and the order ideal generated by it is $T_2 = \{P_j - nm | 1 \leq n \leq h\}$. Let $T = T_1 \cup T_2$, it is an order ideal, the Pseudo-Frobenius numbers in it are $P_i - m, P_j - m$.

Case 1: If $P_r = F + m$ then $\overline{P_i - m} = F - (P_i - m) = F + m - P_i = (P_i + P_j - hm) - P_i = P_j - hm \in T$ and similarly $\overline{P_j - m} = P_i - hm \in T$. Therefore $A(T \cup S) = S$, also note that T has at most 2 Pseudo-Frobenius numbers (maybe just one if $i = j$) but the void has $m-2 \geq 3$ Pseudo-Frobenius numbers and hence $T \neq B$ and $P(S) \geq 3$.

In this case T is self dual, $T \neq \emptyset, B$ means that we have at least 2 connected components in $GPF(S)$ and hence $P(S) \geq 4$

Case 2: If $P_r \neq F + m$ then $P_r - m \in B$ and $(P_i - m, P_j - hm, \overline{P_r - m})$ is red triangle, $P_j - hm \in T$ and $P_r - m \notin T$ so $P_i - m$ satisfies a red triangle, similarly $P_j - m$ satisfies the red triangle $(P_j - m, P_i - hm, \overline{P_r - m})$. Therefore $A(T \cup S) = S$, $T \neq B, \emptyset$ so $P(S) \geq 3$.

If $T = T^*$ then $a \in T \iff \overline{a} \notin T$. Since the only Pseudo-Frobenius numbers in T are $P_i - m$ and $P_j - m$ the rest of the Pseudo-Frobenius numbers have their conjugates in T. But the only possible minimal elements in T are $P_i - hm$ and $P_j - hm$. Now if $i = j$ then $t-1 \leq 2$ which is impossible as $m \geq 5$. Therefore $i \neq j$ and $t - 1 \leq 4$. Therefore $m \leq 6$

If $m = 5$ then $PF(S) = \{P_i - m, P_j - m, P_r - m, F\}$. $P_r - m \notin T \implies F - (P_r - m) \in T \implies F - (P_r - m) \in \{P_i - hm, P_j - hm\}$. WLoG say $F - (P_r - m) = P_j - hm$ and $P_j + P_r = (F + m) + hm$ and hence we are back to case 1.

If $m = 6$ then $PF(S) = \{P_i - m, P_j - m, P_r - m, P_l - m, F\}$ and $\{F - (P_r - m), F - (P_l - m)\} = \{P_i - hm, P_j - hm\}$. WLoG say $F - (P_r - m) = P_j - hm$ i.e. $P_j + P_r = (F + m) + hm$ and we are back to case 1.

Therefore $T \neq T^*$ and $P(S) \geq 4$

Corollary 9.59.1. *If $m(S) \geq 5$ and S has max embedding dimension and $P(S) = 4$ then one of the following happens*

- *$GPF(S)$ has two connected components and all the good numerical sets are self-dual.*
- *$GPF(S)$ is connected, there is exactly one (un-ordered)triple (i, j, r) s.t. $P_i + P_j = P_r + hm$, $P_r - m \neq F$, let T be the order ideal generated by $P_i - hm, P_j - hm$ and the only good numerical sets are $\emptyset, B, T, T^*$*

Remark 9.60. *The following result (told to us by Chris) leads to the next theorem*

For a fixed multiplicity m

$$\lim_{F\to\infty} \frac{\#\{S|m(S) = m, F(S) \leq F, S \text{ has max embedding dimension}\}}{\#\{S|m(S) = m, F(S) \leq F\}} = 1$$

Theorem 9.61. *For fixed multiplicity $m \geq 5$*

$$\lim_{F\to\infty} \frac{\#\{S|m(S) = m, F(S) \leq F, P(S) = 1\}}{\#\{S|m(S) = m, F(S) \leq F\}} = 0$$

and

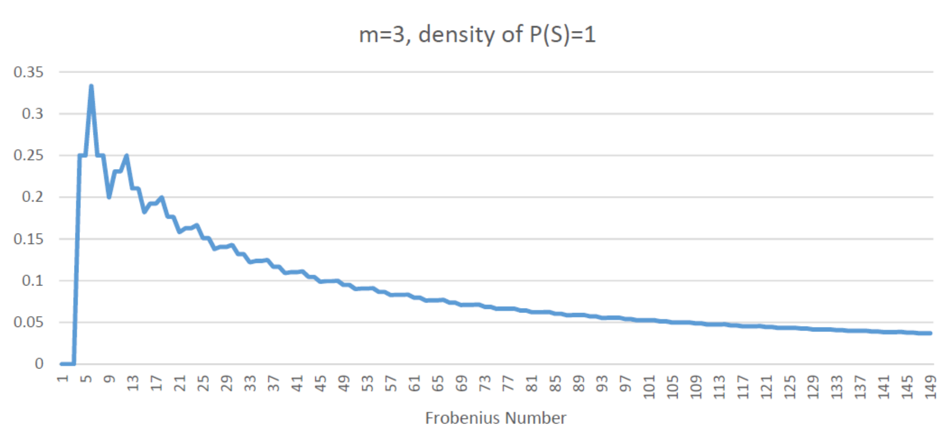
$$\lim_{F\to\infty} \frac{\#\{S|m(S) = m, F(S) \leq F, P(S) = 2\}}{\#\{S|m(S) = m, F(S) \leq F\}} = 0$$

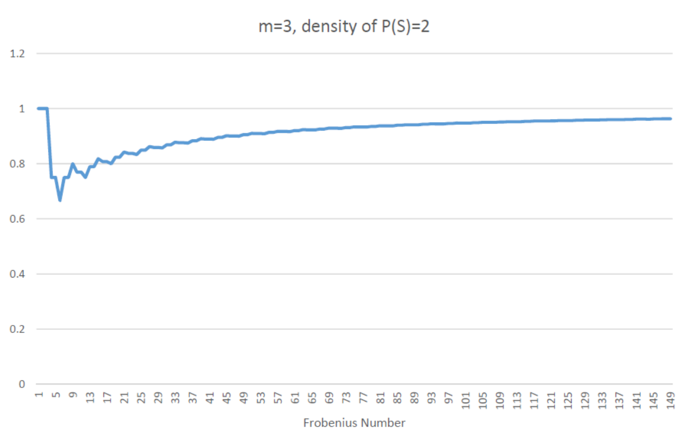
and
$$\lim_{F\to\infty}\frac{\#\{S|m(S)=m, F(S)\le F, P(S)=3\}}{\#\{S|m(S)=m, F(S)\le F\}}=0$$

Remark 9.62. *For multiplicity $m=2$, all semigroups have $P(S)=1$*
For multiplicity $m=3$, all max E.D. semigroups have $P(S)=2$ and hence

$$\lim_{F\to\infty}\frac{\#\{S|m(S)=3, F(S)\le F, P(S)=1\}}{\#\{S|m(S)=3, F(S)\le F\}}=0$$

$$\lim_{F\to\infty}\frac{\#\{S|m(S)=3, F(S)\le F, P(S)=2\}}{\#\{S|m(S)=3, F(S)\le F\}}=1$$





Lemma 9.63. *If $m(S)=4$ and S has max embedding dimension. Say the Apery set is $(0, P_1, P_2, P_3)$*

- *If $P_3+P_1>2P_2$ then $P(S)=2$*
- *If $P_3+P_1=2P_2$ then $P(S)=3$*

- *If $P_3 + P_1 < 2P_2$ then $P(S) = 4$*

  Proof:

  Case 1: $F = P_3 - 4$; $P_1 + P_2 \geq P_3 + 4$ i.e. $P_1 + P_2 - P_3 - 4 \in S$ (it is divisible by 4). Which means $(P_1 - 4) + (P_2 - 4) - F \in S$ and $GPF(S)$ is connected. $P_1 - P_2 \equiv 3(mod\ 4) \implies (P_1 - 4) - (P_2 - 4) \notin B$. $P_2 < 2P_1 \implies P_2 - P_1 < P_1 \implies (P_2 - 4) - (P_1 - 4) \notin S$. Therefore $(P_2 - 4) - (P_1 - 4) \in B$ iff $F - (P_2 - P_1) \notin S$ iff $(P_3 - 4) - P_2 + P_1 < P_2$ iff $P_3 + P_1 < 2P_2 + 4$ iff $P_3 + P_1 \leq 2P_2$.

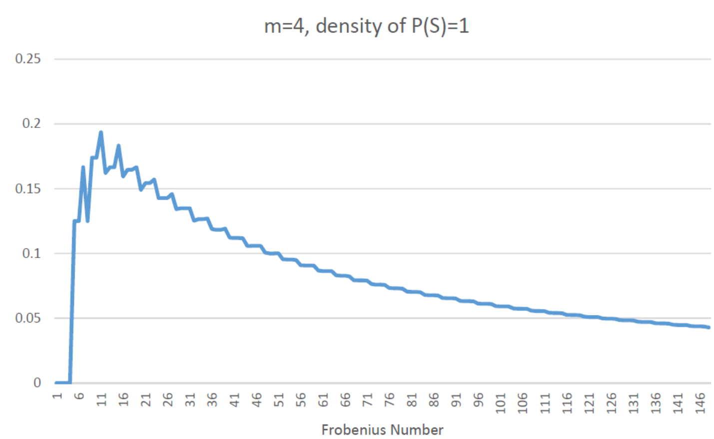
  It follows that $P_1 + P_3 > 2P_2$ implies $(P_1 - 4) - (P_2 - 4), (P_2 - 4) - (P_1 - 4) \notin B$ and hence $P(S) = 2$

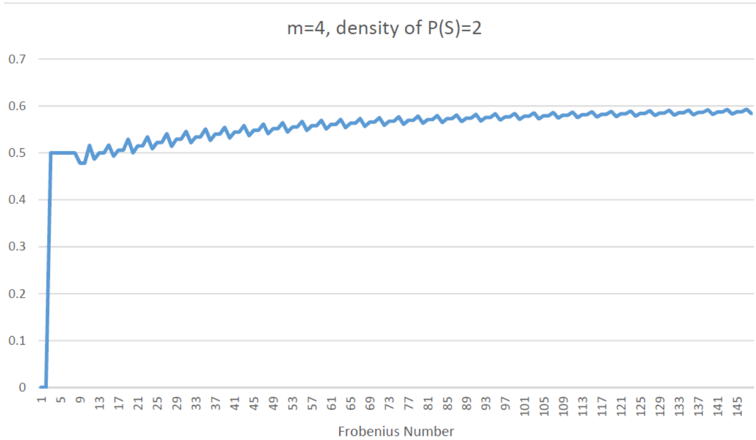
  On the other hand if $P_1 + P_3 \leq 2P_2$, then $(P_2 - 4) - (P_1 - 4) = P_2 - P_1 \in B$. Now $P(S) = 3$ if $P_2 - P_1 = F - (P_2 - 4)$ (i.e. $P_3 + P_1 = 2P_2$) and $P(S) = 4$ otherwise.

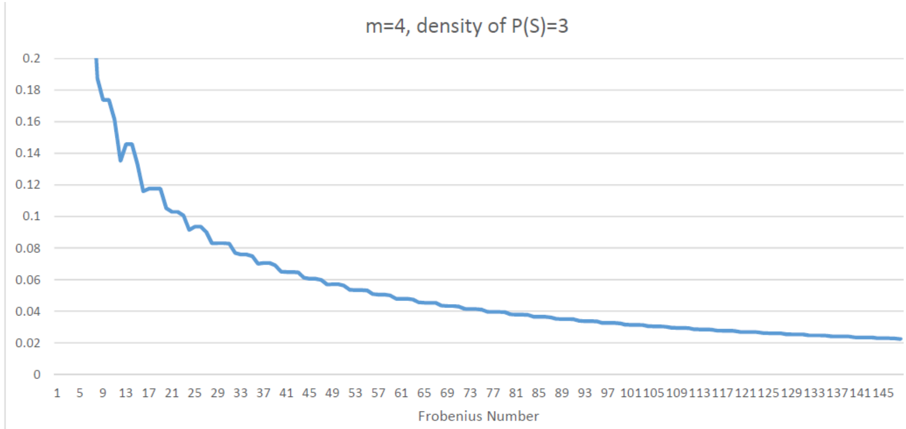
  Case 2: $F = P_2 - 4$ then $P_1 + 1 \leq P_2$ and $P_3 + 3 \leq P_2$ therefore $P_1 + P_3 + 4 \leq 2P_2$ which implies $P_1 + P_3 - 4 < 2P_2$ i.e. $(P_1 - 4) + (P_3 - 4) - (P_2 - 4) < P_2$ which means $(P_1 - 4) + (P_3 - 4) - (P_2 - 4) \notin S$ and hence $GPF(S)$ is not connected and $P(S) = 4$

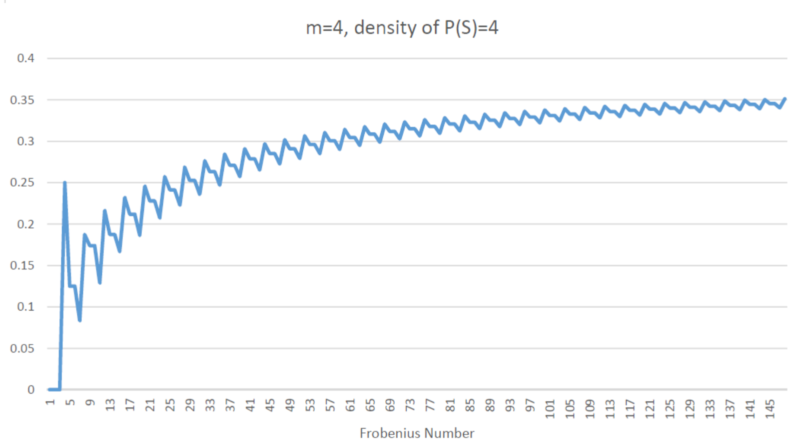
  Case 3: $F = P_1 - 4$; is similar to Case 1

Corollary 9.63.1. *For Numerical semigroups with multiplicity 4 density of $P(S) = 1$ and $P(S) = 3$ is 0. Density of $P(S) = 2$ is ≈ 0.38 and density of $P(S) = 4$ is ≈ 0.62 (exact values can be determined by computing volumes)*




m=4, density of P(S)=2
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1 5 9 13 17 21 25 29 33 37 41 45 49 53 57 61 65 69 73 77 81 85 89 93 97 101 105 109 113 117 121 125 129 133 137 141 145
Frobenius Number


m=4, density of P(S)=3
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
1 5 9 13 17 21 25 29 33 37 41 45 49 53 57 61 65 69 73 77 81 85 89 93 97 101 105 109 113 117 121 125 129 133 137 141 145
Frobenius Number


m=4, density of P(S)=4
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
1 5 9 13 17 21 25 29 33 37 41 45 49 53 57 61 65 69 73 77 81 85 89 93 97 101 105 109 113 117 121 125 129 133 137 141 145
Frobenius Number

9.3 Rays with $P(S) = 4$

In this section we show that if $E(S, n) = 4$ for a particular n then it is true for all n (assuming S is of max E.D.)

Definition 9.64. *Assume $h(S) \geq 1$ We define new categories for order ideals T of $B(E(S, n))$, (A_1, A_2, X) where $A_1 = \{P_i - m | P_i + (n-1)m \in T\}$, $A_2 = \{F(S) - (P_i - m) | F(S) - (P_i - m) \in T\}$. Let $Tri(A_1) = \{(a, b) | a, b \in B(S), F(S) - a - b \in A_1\}$ $X = \{(a, b) \in Tri(A_1) | a \in T, F(E(S, n)) - b \notin T\}$.*

Also note that X contains un-ordered pairs, we denote $X_1 = \{\exists b, (a, b) \in X\}$, $X_2 = \{\_\exists a, (a, b) \in X\}$

Lemma 9.65. *If $(P_i - m, a_1, b_1)$ and $(P_j - m, a_2, b_2)$ are red triangles in $B(S)$. Then $F(S) - b_2 \preccurlyeq a_1$ implies $a_1 = F(S) - b_2$ or $a_2 \preccurlyeq F(S) - b_1$*

Proof: For this paragraph denote $F(S)$ by F. So $a_1 - (F - b_2) \in S$, but $a_1 + b_2 - F = (F - (P_i - m) - b_1) + (F - (P_j - m) - a_2) - F = F - (P_i - m) - (P_j - m) - b_1 - a_2$. Now as $P_i - m$ and $P_j - m$ are Pseudo-Frobenius numbers $F - a_2 - b_1 \in S$ (unless $F - (P_i - m) - (P_j - m) - b_1 - a_2 = 0$ i.e. $a_1 - (F - b_2) = 0$). Now $a_2 \preccurlyeq F - b_1$ in $B(S)$

Lemma 9.66. *If $h(S) \geq 1$ and T is an order ideal of $E(S, n)$ then $\forall n'$ there is an order ideal T' of $E(S, n')$ s.t. $A_1(T') = A_1(T)$, $A_2(T) \subseteq A_2(T')$ and $X(T) \subseteq X(T')$*

Proof: For notation we denote $A_1 = A_1(T)$, $A_2 = A_2(T)$, $X = X(T)$, $A'_1 = A_1(T')$, $A'_2 = A_2(T')$, $X' = X(T')$

Let T' be the order ideal of $E(S, n')$ generated by $(A_1 + mn') \cup A_2 \cup X_1$.

- Clearly $A_1 \subseteq A'_1$. Conversely if $P_i - m \in A'_1$ then $\exists x \in (A_1 + mn') \cup A_2 \cup X_1$ s.t. $x \preccurlyeq P_i + (n' - 1)m$ in $B(E(S, n'))$
  - If $x \in (A_1 + mn')$ then $P_i - m \in A_1$
  - If $x \in A_2 \cup X_1$ then $x \in B(S) \cap T$. Now $x \preccurlyeq P_i + (n' - 1)m$ in $B(E(S, n'))$ means that $P_i - m - x + n'm \in E(S, n')$
    - If $m \not| P_i - m - x + n'm$ then $P_i - m - x + n'm \in E(S, n')$ implies $P_i - m - x \in S$ which implies $P_i - m - x + nm \in E(S, n)$ i.e. $x \preccurlyeq P_i + (n-1)m$ in $B(E(S, n))$ which implies $P_i + (n-1)m \in T$ i.e. $P_i - m \in A_1$
    - If $m | P_i - m - x + n'm$, then in $B(S)$ either $P_i - m \prec x$ or $x \preccurlyeq P_i - m$ according to $P_i - m < x$ or $x \leq P_i - m$. But $P_i - m \prec x$ is impossible and hence $x \leq P_i - m$. Therefore $x \leq P_i + (n-1)m$ and $x \preccurlyeq P_i + (n-1)m$ in $B(E(S, n))$ which implies $P_i + (n-1)m \in T$ i.e. $P_i - m \in A_1$
- Clearly $A_2 \subseteq A'_2$
- If $(a, b) \in X$, say $P_i - m + a + b = F(S)$ and $P_i - m \in A_1$. Then $a \in X_1 \in T'$. If possible assume $F(E(S, n')) - b \in T'$. Therefore $\exists x \in (A_1 + mn') \cup A_2 \cup X_1$ s.t. $x \preccurlyeq F(E(S, n')) - b$ in $B(E(S, n'))$

  - If $x \in (A_1 + mn')$, say $x = P_i + (n'-1)m$; then $x \preccurlyeq F(E(S,n')) - b$ means $P_i + (n'-1)m = F(E(S,n')) - b \implies P_i - m = F(S) - b \implies P_i + (n-1)m = F(E(S,n)) - b$. Now $P_i - m \in A_1$ means that $P_i + (n-1)m \in T$ which contradicts the fact that $F(E(S,n)) - b \notin T$
  - If $x \in A_2 \cup X_1$ then $x \in B(S) \cap T$. Now $x \preccurlyeq F(E(S,n')) - b$ in $B(E(S,n'))$ means that $F(S) - b - x + n'm \in E(S,n')$
    * If $m \not| F(S) - b - x - n'm$; then $F(S) - b - x + n'm \in E(S,n')$ implies $F(S) - b - x \in S$ which implies $F(S) - b - x + nm \in E(S,n)$ i.e. $x \preccurlyeq F(E(S,n)) - b$ in $B(E(S,n))$. This implies $F(E(S,n)) - b \in T$ which is a contradiction.
    * If $m | F(S) - b - x - n'm$
      · If $x \leq F(S) - b$ then $x \leq F(S) - b + nm = F(E(S,n)) - b$ which implies $F(E(S,n)) - b - x \in E(S,n)$ i.e. $x \preccurlyeq F(E(S,n)) - b$ in $B(E(S,n))$. This implies $F(E(S,n)) - b \in T$ which is a contradiction.
      · If $F(S) - b < x$ then $F(S) - b \prec x$ in $B(S)$ and hence $x \notin A_2$, $x \in X_1$. So say $P_j - m \in A_1$, $x + y + P_j - m = F(S)$, $x \in T$ and $F(E(S,n)) - y \notin T$. Now by lemma 9.65 we know that $a \preccurlyeq F(S) - y$ in $B(S)$ i.e. $F(S) - y - a \in S$ which implies $F(E(S,n)) - y - a \in E(S,n)$ i.e. $a \preccurlyeq F(E(S,n)) - y$ in $B(E(S,n)$ and hence $F(E(S,n)) - y \in T$ which is a contradiction.

It follows that $X \subseteq X'$

Corollary 9.66.1. *If $P_i - m \in A_2' \setminus A_2$ in the above construction then $h(S) = 1$, $n' = 0$ and $\exists P_j - m \in A_1$ s.t. $P_i + P_j = F(S) + 2m$*

Proof: If $F(S) - (P_i - m) \in T'$ then $\exists x \in (A_1 + mn') \cup A_2 \cup X_1$ s.t. $x \preccurlyeq F(S) - (P_i - m)$ in $B(E(S,n'))$. Then of course $x = F(S) - (P_i - m)$

- If $x \in A_2 \cup X_1$ then $x \in A_2$
- If $x \in (A_1 + mn')$, say $F(S) - (P_i - m) = P_j + (n'-1)m$, $P_j - m \in A_1$. This means $(P_j + n'm) + (P_i + n'm) = F(S) + 2m + n'm = (F(E(S,n')) + m) + m$ which implies $h(E(S,n')) \leq 1$. But $h(S) \geq 1 \implies h(S) = 1, n' = 0$ and hence $P_i + P_j = F(S) + 2m$

Theorem 9.67. *If $h(S) \geq 1$, $m(S) \geq 5$ and $P(E(S,n_1)) = 4$ for some $n_1 \geq 0$. Then $P(E(S,n)) = 4$ for all $n \geq 0$*

Proof: Firstly by theorem 9.59 $P(E(S,n)) \geq 4$. Now by corollary 9.59.1

- Case 1: $GPF(E(S,n_1))$ has two connected components and the only good numerical sets of $E(S,n_1)$ are the self-dual ones. Moreover in this case $(P_i + n_1 m) + (P_j + n_1 m) = F(E(S,n_1)) + m + h(E(S,n_1))m \implies (P_i - m) + (P_j - m) = F(S) + (h(S) - 1)m$, Let T_1 be the order ideal of $B(E(S,n_1))$

generated by $P_i - h(S)m, P_j - h(S)m$. then the good numerical sets of $E(S, n_1)$ are $\emptyset, B(E(S, n_1)), T_1, T_1^* = T_1^c$.

Now if T is a good numerical set of $E(S, n)$. Then consider the corresponding order ideal T' of $E(S, n_1)$ given by lemma 9.66.

If $h(S) > 1$ or $n_1 > 0$ then $A_2' = A_2$ by corollary 9.66.1 and hence T is self dual and $P(S) = 4$. Therefore now assume $h(S) = 1$ and $n_1 = 0$

- $T' = \emptyset \implies T = \emptyset$
- $T' = T_1$ then $A_1 = \{P_i - m, P_j - m\}$. As $h(S) = 1$ $T \cap B(S) \subseteq \{P_i - m, P_j - m\}$ therefore the only possible red triangle (of $B(E(S, n))$) that $P_i + (n-1)m$ can satisfy is $(P_i + (n-1)m, P_i - m, b)$ which means $2P_i - 2m + b = F(S)$. We know that $i + j \equiv F(mod\ m)$ which implies $i \equiv b(mod\ m)$ which implies $b = P_i - m$, $F(E(S, n)) - b = P_j + (n-1)m$. Therefore the triangle cannot be satisfied. Similarly $P_j + (n-1)m$ cannot satisfy a triangle and hence T is self dual.
- $T' = B(S)$ This means $X' = \emptyset$ which implies $X = \emptyset$ and hence $T = B(E(S, n))$
- $T' = T_1^* = T_1^c$; So $A_1 = PF(S) \setminus \{F, P_i - m, P_j - m\}$. Conjugates of $P_i + (n-1)m$ and $P_j + (n-1)m$ are not in T. If possible assume T is not self dual, therefore at least one Pseudo-Frobenius number does not have its conjugate. It follows that T^* has $P_i + (n-1)m, P_j + (n-1)m$ and at least one more Pseudo-Frobenius number. Therefore $(T^*)' = B(S)$ and as seen earlier $T^* = B(E(S, n))$ which is impossible.

Therefore in Case 1 we see that $P(E(S, n)) = 4$

- Case 2: $GPF(E(S, n_1))$ is connected (which means $GPF(E(S, n_2))$ is connected for each n_2), there is exactly one (un-ordered)triple (i, j, r) s.t. $P_i + P_j = P_r + hm$ (here $h = h(S)$), $P_r - m \neq F$, let T_1 be the order ideal of $B(E(S, n_1))$ generated by $P_i - hm, P_j - hm$ and the only good numerical sets of $E(S, n_1)$ are $\emptyset, B, T_1, T_1^*$

  By corollary 9.15.2 there is a S' s.t. $E(S', h(S) - 1) = S$, $h(S') = 1$

  Let T be a good order ideal of $E(S, n)$, consider the corresponding T' of $E(S, n_1)$ given by lemma 9.66

  - $T' = \emptyset$ implies $T = \emptyset$
  - $T' = T_1$; So $A_1 = \{P_i - m, P_j - m\}$, $A_2' = \emptyset \implies A_2 = \emptyset$. $\{P_i - hm, P_j - hm\} \subseteq X' \subseteq T_1 \cap B(S') = \{P_i - hm, P_j - hm\}$ hence $X' = \{P_i - hm, P_j - hm\}$. If $\exists x \in T$ s.t. $P_i - hm \not\preccurlyeq x$ and $P_j - hm \not\preccurlyeq x$ in $B(E(S, n))$. We must have $x \equiv P_i\ or\ P_j(mod\ m)$, Say $x = P_i - (h + s)m$. Then $x \in B(S)$, The order ideal of $B(S)$ generated by $x, P_j - hm$ in $B(E(S, n_1))$ is not a good numerical set so it must have a Pseudo-Frobenius number other that $P_i - m, P_j - m$ i.e. $x \preccurlyeq P_u + (n_1 - 1)m$ for some u in $B(E(S, n_1))$. Therefore $x \preccurlyeq P_u + (n-1)m$ in $B(E(S, n))$ which is a contradiction.

- $T' = T_1^*$; Say $i + \alpha \equiv F(mod\ m)$ and $j + \beta \equiv F(mod\ m)$ So $A_1' = \{P_s - m | P_s - m \neq F\}$ and $A_2' = \{F(S) - (P_s - m) | P_s - m \neq F, s \neq i, j\}$. Therefore by corollary 9.66.1 $A_2 = A_2'$ and hence $(T^*)' = T_1$ and by the previous part T^* is generated by $P_i - hm, P_j - hm$
- $T' = B(S)$. $A_1 = \{P_s - m | P_s - m \neq F(S)\}$, $A_2' = \{P_s - m | P_s - m \neq F(S)\}$ and by corollary 9.66.1 $\{F(S) - (P_s - m) | P_s - m \neq F, s \neq i, j\} \subseteq A_2$. Therefore $(T^*)'\emptyset$ *or* T_1. If $(T^*)' = \emptyset$ then $T^* = \emptyset$ and $T = B(E(S, n))$. And if $(T^*)' = T_1$ then T^* is generated by $P_i - hm, P_j - hm$ which is a contradiction.

9.4 Red Triangles and bad hyperplanes in Max Embedding Dimension

Remark 9.68. *From this section on P_i will be used to denote Pseudo-Frobenius numbers and A_i used to denote Apery set elements*

Remark 9.69. *The P values in the polyhedron suggests that certain hyper-planes divide the polyhedron into regions of distinct P values.*

Definition 9.70 (Bad Hyper-planes). *Hyper-planes of the form of the form $A_i + A_j = A_k + A_l$ where A_i, A_j, A_k, A_l are in the apery set (or equivalently in $PF(S)$) (and $i + j \equiv k + l(mod\ m)$) are called bad hyper-planes.*

A Numerical Semigroup is called bad if $\exists P, Q, R \in PF(S)$ s.t. $F + Q = P + R$. Note that all bad semigroups lie on a bad hyper-plane.

Lemma 9.71. *S has max embedding dimension. Say $P_i, P_j \in PF(S)$ and $i - j \equiv k(mod\ m)$, $k + l \equiv F(mod\ m)$ then $P_i - P_j \in B(S)$ iff $F - P_l \leq P_i - P_j$*

Proof: Firstly $P_i - P_j \in B(S) \iff F - P_l \preccurlyeq P_i - P_j \preccurlyeq P_k$. But $P_i - P_j \preccurlyeq P_k$ follows from the fact that S has max embedding dimension.

Moreover $P_i - P_j \equiv F - P_l(mod\ m)$ implies $F - P_l \preccurlyeq P_i - P_j \iff F - P_l \leq P_i - P_j$ $\square$

Theorem 9.72. *If S is of max embedding dimension then S is ignoble iff $\exists P_i, P_j, P_l \in PF(S) \setminus \{F\}$ s.t. $i - j \equiv F - l(mod\ m)$ s.t. $F - P_l \leq P_i - P_j$*

Proof: Follows from lemma 9.71

Remark 9.73. *It follows that noble semigroups of max E.D. are geometrically living in certain smaller polyhedrons of the kunz polyhedron*

Moreover they have positive density, which can be calculated by computing volumes.

Lemma 9.74. *S has max embedding dimension. Suppose $P_i, P_l \in PF(S) \setminus \{F\}$, and $i + l \not\equiv F(mod\ m)$, $i + l \not\equiv 2F(mod\ m)$.*

Pick j s.t. $i - j \equiv F - l(mod\ m)$. Then:

- *$F - P_l > P_i - P_j$ implies $F - P_l \not\preccurlyeq P_i$ i.e. P_l and P_i are not connected in $GPF(S)$. In this case $P_i - P_j$ not in B*

- *If $F - P_l = P_i - P_j$ then $F - P_l \not\preccurlyeq P_i$ i.e. P_l and P_i are not connected in $GPF(S)$. In this case $P_i - P_j$ is in B*

- *If $F - P_l < P_i - P_j$ then $F - P_l \preccurlyeq P_i$ i.e. P_i and P_j are not connected in $GPF(S)$. In this case $P_i - P_j$ is in B*

Proof: The conditions ensure $P_j \in PF(S) \setminus \{F\}$
$P_i - (F - P_l) \equiv j(mod\, m)$ so $F - P_l \preccurlyeq P_i$ iff $P_i - (F - P_l) \geq A_j = P_j + m$ iff $P_i - (F - P_l) > P_j$ iff $P_i - P_j > F - P_l$

Corollary 9.74.1. *If S has max E.D. and S is not bad then given $P_i, P_l \in PF(S) \setminus \{F\}$, $P_i + P_l \not\equiv F, 2F(mod\, m)$. Then $j \equiv i + l - F(mod\, m)$*
P_i, P_l are connected in $GPF(S)$ iff $P_i - P_j \in B$ iff $P_l - P_j \in B$

Corollary 9.74.2. *We can restate the result as follows: If S has max E.D. and S is not bad then given $P_i, P_j \in PF(S) \setminus \{F\}$ s.t. $i \neq j$, $i - j \not\equiv F(mod\, m)$ then $l \equiv F - (i - j)(mod\, m)$ implies*
$P_i - P_j \in B$ iff P_i, P_l are connected in $GPF(S)$ iff $P_l - P_j \in B$

Theorem 9.75. *If S is a noble semigroup of max embedding dimension then*

- *The B poset has the simple structure: $x \preccurlyeq y$ iff $m|y - x$ and $x \leq y$*

- *If m is odd then $P(S) = 2^{\frac{m-1}{2}}$*

- *If m is even and F is odd then $P(S) = 2^{\frac{m-2}{2}}$*

- *If m and F are both even then $P(S) = 2^{\frac{m}{2}}$*

Proof: If possible assume there are $x \preccurlyeq y$ in B-Poset s.t. $m \not| y - x$. Say $F - P_l \equiv x(mod\, m)$ and $P_i \equiv x(mod\, m)$. Then P_i, P_l are connected in $GPF(S)$ and $P_i + P_l \not\equiv F(mod\, m)$

Note that Noble Semigroups are not bad. Next if $P_i + P_l \equiv 2F(mod\, m)$ then $P_i + P_l - F \equiv F(mod\, m)$ and $P_i, P_j < F \implies P_i + P_j - F < F$ which contradicts $P_i + P_j - F \in S$.

Therefore by previous lemma $P_i - P_j \in B$ and we have a contradiction.

It follows that the only edges on the Pseudo-Frobenius graph are when $P_i + P_l \equiv F(mod\, m)$ (which are indeed edges)

This means that the graph mostly consists of components of size to except when $2P_i \equiv F(mod\, m)$ in which case P_i is an isolated point.

Now if m is odd then the graph has $m - 2$ vertices, there is exactly one i for which $2P_i \equiv F(mod\, m)$. Hence the number of connected components is $1 + \frac{m-3}{2}$

Next if m is even and F is odd then there is no i for which $2P_i \equiv F(mod\, m)$ hence there are $\frac{m-2}{2}$ connected components.

Finally if m and F are both even then there are two i for which $2P_i \equiv F(mod\, m)$ hence there are $2 + \frac{m-4}{2}$ connected components. □

Remark 9.76. *This is not necessarily true in noble semigroups that are not of max embedding dimension. For e.g. if* $S = < 6, 7, 8, 11 >$ *then* $B(S) = \{1, 5, 9\}$*, it is noble and* $1 \preccurlyeq 9$

Another e.g. $S = < 6, 8, 11, 13, 15 >$ *is noble,* $B(S) = \{1, 3, 5, 7, 9\}$*, cover relations are* $1 \preccurlyeq 7, 1 \preccurlyeq 9, 3 \preccurlyeq 9$

Corollary 9.76.1. *If we look at numerical semigroups with a fixed multiplicity* m *on the kunz polyhedron then:*

- *If* m *is odd then numerical semigroups with* $P(S) = 2^{\frac{m-1}{2}}$ *have a positive density.*
- *If* m *is even then numerical semigroups with* $P(S) = 2^{\frac{m-2}{2}}$ *and those with* $P(S) = 2^{\frac{m}{2}}$ *both have positive densities*

Conjecture 9.77. *If* S *is of max embedding dimension and not bad then* $P(S)$ *is even*

The natural path towards proving this is to prove $T \neq T^*$ *for each good numerical set of* S

9.5 Multiplicity 5

Remark 9.78. *These are the observations we will prove in this section. We assume* $m(S) = 5$ *and* S *has max embedding dimension for each*

- $P(S)$ *is even.*
- $P(E(S, 1)) = P(S) \implies P(S) \in \{4, 6, 8\}$
- $P(E(S, 1)) \neq P(S) \implies P(E(S, 1)) = P(S) + 2$
- $P(S) = 4, 8$ *have positive density, all other values of* P *have zero density.*

Remark 9.79. *We have a Numerical Semigroup of max embedding dimension and multiplicity 5 (this assumption is maintained throughout this section even if I forget to mention it in some lemma/theorem)*

Say $F \equiv 2k(mod\ 5)$*, the Pseudo-Frobenius numbers of* S *are* P_k, F, P_{3k}, P_{4k} *(so the Apery set of* S *is* $(P_k + 5, F + 5, P_{3k} + 5, P_{4k} + 5)$*)*

In $GPF(S)$ P_{3k} *and* P_{4k} *are connected, so the graph has at most 2 connected components.*

$F - P_k \equiv P_k(mod\ 5)$ *therefore* $F - P_k \preccurlyeq P_k$ *and* P_k *is connected to itself in* $GPF(S)$

Also $P_{3k} + P_k - F \equiv F(mod\ 5)$ *and* $P_{3k} + P_k - F > F$ *so* P_k *and* P_{3k} *cannot be connected.*

$T_1 = \emptyset$, $T_1^* = B(S)$ *are good numerical sets*

Theorem 9.80. *Let* S *be a numerical semigroup of maximum embedding dimension such that* $m(S) = 5$*.Then* $A(T \cup S) = S \implies T \neq T^*$ *and hence* $P(S)$ *must be even.*

Proof: Suppose that $F \equiv 2k \bmod 5$, and let P_S be the Psuedo-Frobenius number such that $P_S \equiv k \bmod 5$. Similarly, let $P_P \equiv 4k \bmod 5, P_{P'} \equiv 3k \bmod 5$.

Now, let $T \subset B$ such that $A(T \cup S) = S$, $T^* = T$. Then T must include exactly one and exclude exactly one of each pair $\{b, F-b\} \subset B$. Since $P_S-(F-P_S) \equiv 0 \bmod 5$, and $A_S-(A_F-A_S) = P_S-(F-P_S)+5 > 0$, $P_S-(F-P_S) \in S$, and so $F-P_S \preccurlyeq P_S$, implying $P_S \in T, F-P_S \notin T$. Thus, there must exist a red triangle (P_S, a, b), and so either $P_P - P_S \in B$ or $P_{P'} - P_S \in B$. However, $F - P_{P'} + P_S \equiv 0 \bmod 5$, and so $P_{P'} - P_S \notin B$.

Thus, if (P_S, a, b), then $a \preccurlyeq P_P - P_S$. Note $a \in T \rightarrow F - a \notin T$ and similarly $F - b \notin T \rightarrow b \in T$; we must then have $P_P - P_S \in T$, and since $P_P - P_S \equiv P_{P'} \bmod 5$, $P_{P'} \in T, F - P_{P'} \notin T$. By the red antichain condition, if $F-P_P \preccurlyeq a$, then $P_P \in T, F-P_P \notin T$ As above, this would require $P_S-P_{P'} \in B$ or $P_P-P_{P'} \in B$, and $P_S-P_P \in B$ or $P_{P'}-P_P \in B$; however, since $P_P-P_S > 0$, we must have $P_{P'} - P_P, P_S - P_{P'} \in B$ to allow for red triangles; however, this implies $P_S - P_P > 0$, which is impossible. This, together with the fact that $F - P_S \not\preccurlyeq P_P - P_S$, implies $F - P'_P \preccurlyeq a, b$, and this is the unique minimal element below them; it must be true, then that $a \equiv F-P_{P'} \equiv P_P \bmod 5$, which would require $P_P \in T$, but this has already been shown to be impossible.

Lemma 9.81. *If $m(S) = 5$, $h(S) \geq 1$, $A(T \cup S) = S$ and $|T \cap PF(S)| = 1$, then T can only be one of the following:*

- *T_2 generated by $F - P_k$, it exists iff $GPF(S)$ has two connected components, it is self dual*
- *T_3 generated by P_k-P_{3k}, it exists iff $P_k-P_{3k} \in B(S)$ and $P_{4k}+P_{3k} \leq 2P_k$*
- *T_4 generated by $P_{3k} - P_{4k}$, it exists iff $P_{3k} - P_{4k} \in B(S)$ and P_k is not connected to P_{4k} in $GPF(S)$*

Proof:

- $P_k \in T$;
  - If $F - P_k \in T$ then T must be the order ideal generated by $F - P_k$, denote it by T_2. In this case the graph has 2 connected components, T_2 is one of the self dual order ideals. $T_2^* = B(S) \setminus T_2$ is another.
  - If $F-P_k \notin T$ then P_k must satisfy a triangle, but $P_{3k}-P_k \not\equiv k(mod\ 5)$ and $P_{4k} - P_k \not\equiv k(mod\ 5)$ so no such T is possible.
- $P_{3k} \in T$; So P_{3k} must satisfy a triangle. $P_{4k} - P_{3k} \not\equiv 3k(mod\ 5)$, but $P_k - P_{3k} \equiv 3k(mod\ 5)$. So if (P_{3k}, a, b) is satisfied then $a \preccurlyeq P_k - P_{3k}$. If $a \prec P_k - P_{3k}$ then by corollary 3.14.1 $a \preccurlyeq P_k$ which is impossible (in Case 1). So $a = P_k - P_{3k}$ and T is the order ideal generated by $P_k - P_{3k}$. Denote this order ideal by T_3, note that in this case $P_k - P_{3k} \in B(S)$

  Also $P_k - P_{3k} \not\preccurlyeq P_{4k}$ iff $P_{4k} - (P_k - P_{3k}) \leq P_k$

- $P_{4k} \in T$; it is similar to case when $P_{3k} \in T$, T must be the order ideal generated by $P_{3k} - P_{4k}$, we denote it by T_4, in this case $P_{3k} - P_{4k} \in B(S)$

  $P_{3k} - P_{4k} \not\preccurlyeq P_k$ iff $P_k - (P_{3k} - P_{4k}) \leq F$ iff $P_k + P_{4k} - F \leq P_{3k}$ iff $P_k + P_{4k} - F \notin S$

We are done. $\square$

Lemma 9.82. *If $m(S) = 5$, $h(S) \geq 1$, $A(T \cup S) = S$ and $|T \cap PF(S)| = 2$, then T can only be one of the following:*

- *T_5, generated by $P_{4k} - P_{3k}$ and $P_{4k} - P_k$*
- *If $P_{4k} - P_{3k} = F - P_k$ then there is a family of good numerical sets, the number of sets in the family increases by 1 when we go from S to $E(S, 1)$*
- *T_6, generated by $F - P_k, P_{3k} - P_{4k}$*
- *T_2^*, it exists iff $GPF(S)$ has two connected components, it is self dual*
- *The adjoin of one of the order ideal described above.*

  Proof:

- $P_k, P_{3k} \in T$; P_{3k} must satisfy a red triangle, say it satisfies (P_{3k}, a, b)
  - If $a \equiv 3k(mod\ m)$ then $P_k - P_{3k} \in B(S)$ and $a \preccurlyeq P_k - P_{3k}$. And $F - b = P_{3k} + a \equiv k(mod\ m)$. $F - b \notin T \implies F - P_k \notin T$. Therefore P_k satisfies a triangle. $P_{3k} - P_k < 0$ so its not in $B(S)$, so $P_{4k} - P_k \in B(S)$. Say (P_k, a_1, b_1) is satisfied then $a_1 \equiv 3(mod\ 5)$ and $a_1 \preccurlyeq P_{4k} - P_k$. Now if $a_1 \neq P_{4k} - P_k$ then by corollary 3.14.1 $a_1 \preccurlyeq P_{4k}$ which is a contradiction. Therefore $a_1 = P_{4k} - P_k$

    Also by corollary 3.19.1 $a_1 = a$, so $F - b = P_{3k} + a = P_{3k} + P_{4k} - P_k = A_{3k} + A_{4k} - A_k - m$. Max embedding dimension implies $F - b + m \in S$, $F - b \notin S$ shows $F - b = P_k \in T$ and we have a contradiction.
  - If $a \equiv k(mod\ m)$ then $P_{4k} - P_{3k} \in B(S)$ and $a \preccurlyeq P_{4k} - P_{3k}$. Now if $a \neq P_{4k} - P_{3k}$ then by corollary 3.14.1 $a \preccurlyeq P_{4k}$ which is a contradiction. Therefore $a = P_{4k} - P_{3k}$

    Now if $P_{4k} - P_{3k} \neq F - P_k$ then by corollary **??** $F - P_k \notin T$ and hence P_k satisfies a triangle. $P_{3k} - P_k \equiv F(mod\ 5)$ so $P_{3k} - P_k \notin B(S)$. Therefore $P_{4k} - P_k \in B(S)$ and $a_1 \preccurlyeq P_{4k} - P_k$. (here (P_k, a_1, b_1) is the triangle that is satisfied). Next if $a_1 \neq P_{4k} - P_k$ then $a_1 \preccurlyeq P_{4k}$ which is impossible. So $a_1 = P_{4k} - P_k$. It follows that T is generated by $P_{4k} - P_{3k}$ and $P_{4k} - P_k$. We denote this order ideal as T_5

    On the other hand if $P_{4k} - P_{3k} = F - P_k$ then $P_{4k} - (F - P_k) = P_{3k} \notin S$, in this case all order ideals not containing P_{4k}, and containing $F - P_k, P_{3k}$ work. Note that the number of such order ideals increases by one from S to $E(S, 1)$

- $P_k, P_{4k} \in T$; then P_{4k} does not have its conjugate in T, and hence must satisfy a red triangle (P_{4k}, a, b). $P_k - P_{4k} \equiv F(mod\ 5)$ so we must have $P_{3k} - P_{4k} \in B(S)$, $a \preccurlyeq P_{3k} - P_{4k}$. If $a \neq P_{3k} - P_{4k}$ then by corollary 3.14.1 $a \preccurlyeq P_{3k}$ which is a contradiction. Therefore $a = P_{3k} - P_{4k}$.

  $P_{3k} - P_k \equiv F(mod\ 5)$ and $P_{4k} - P_k \equiv P_{3k}(mod\ 5)$ so neither can be in T and hence P_k cannot satisfy a red triangle. So $F - P_k \in T$ and T is generated by $F - P_k, P_{3k} - P_{4k}$. Denote this order ideal by T_6

- $P_{3k}, P_{4k} \in T$;
  - If both $F - P_{3k}, F - P_{4k}$ are in T then T is self dual and $GPF(S)$ has two connected components.
  - If only $F - P_{3k}$ is in T then $T^* \cap PF(S) = \{P_k, P_{4k}\}$ and hence $T = T_6^*$
  - If only $F - P_{4k}$ is in T then $T^* \cap PF(S) = \{P_k, P_{3k}\}$ so either $T = T_5^*$ or T is the adjoin of a good numerical set of the family described in that case.
  - If neither of $F - P_{3k}, F - P_{4k}$ is in T then they both must satisfy a triangle. $P_k - P_{4k} \equiv F(mod\ 5)$ and $P_{3k} - P_{4k} \equiv 4k(mod\ 5)$, so $P_{3k} - P_{4k} \in B(S)$. Also $P_k - P_{3k} \equiv 3k(mod\ 5)$, $P_{4k} - P_{3k} \equiv k(mod\ 5)$, so $P_k - P_{3k} \in B(S)$. Say the triangles being satisfied are (P_{3k}, a_1, b_1) and (P_{4k}, a_2, b_2). $a_1 \preccurlyeq P_k - P_{3k}$, in fact $a_1 = P_k - P_{3k}$ (because $P_k \notin T$). $F - P_{3k} \preccurlyeq a_2 \preccurlyeq P_{3k} - P_{4k}$, $F - b_2 = P_{4k} + a_2 \equiv 3k(mod\ 5)$ which contradicts corollary 3.19.1

We are done □

Lemma 9.83. *If $m(S) = 5$, $h(S) \geq 1$, $A(T \cup S) = S$ and $|T \cap PF(S)| = 3$, then T can only be one of the following:*

- $T_1^* = B(S)$
- T_3^*
- T_4^*

Proof: If T has at least one minimal element then T^* has ≤ 2 Pseudo-Frobenius numbers and is covered by previous lemmas.

Otherwise P_k, P_{3k}, P_{4k} must all satisfy red triangles. But the largest one among them cannot, so no such T exists. □

Theorem 9.84. *If $m(S) = 5$, $h(S) \geq 1$ then either $P(E(S,1)) = P(S)$ or $P(E(S,1)) = P(S) + 2$*

Proof: Follows from last three lemmas.

Theorem 9.85. *If $m(S) = 5$, $h(S) \geq 1$ then $P(E(S,1)) = P(S)$ implies $P(S) \leq 8$*

Corollary 9.85.1. *$P \geq 10$ implies $\#\{S|m(S) = 5, P(S) = P, F(S) \leq F\} = \#\{S|m(S) = 5, P(S) = P + 2, F(S) \leq F + 5\}$*

Lemma 9.86. *If $m(S) = 5$, $h(S) \geq 1$ and S is not bad then*

- *$P_{3k} - P_k \notin B$*
- *$P_k - P_{4k} \notin B$*
- *P_k is connected to P_{4k} in $GPF(S)$ iff $P_k - P_{3k} \in B$ iff $P_{4k} - P_{3k} \in B$*
- *P_{3k} is connected to itself iff $P_{3k} - P_{4k} \in B$*
- *P_{4k} is connected to itself in $B(S)$ iff $P_{4k} - P_k \in B$*

Corollary 9.86.1. *Under the assumptions of the lemma an edge from P_k to P_{4k} and a loop on P_{3k} cannot simultaneously exist*

Corollary 9.86.2. *Under assumptions of the lemma*

- *T_1, T_1^* exist*
- *T_2, T_2^* exist iff P_k is not connected to P_{4k} in $GPF(S)$*
- *T_3, T_3^* exist iff P_k is connected to P_{4k} in $GPF(S)$ and $P_{4k} + P_{3k} \leq 2P_k$*
- *T_4, T_4^* exist iff P_{3k} is connected to itself in $GPF(S)$*
- *T_5, T_5^* exist iff P_k is connected to P_{4k} and there is a loop around P_{4k} in $GPF(S)$*
- *T_6, T_6^* exist iff there is a loop around P_{3k} in $GPF(S)$*

Theorem 9.87. *If $m(S) = 5$, $h(S) \geq 1$ and $F + P \neq Q + R$ for $\forall P, Q, R \in PF(S) \setminus \{F\}$. Then:*

Note the trivial edges of $GPF(S)$ are the edge between P_{3k}, P_{4k} and loop on P_k

- *If $GPF(S)$ only has the trivial edges then $P(S) = 4$*
- *If the only non trivial edge on $GPF(S)$ is a loop on P_{3k} then $P(S) = 8$*
- *If the only non trivial edge on $GPF(S)$ is a loop on P_{4k} then $P(S) = 4$*
- *If the only non trivial edges on $GPF(S)$ are loops on P_{3k} and P_{4k} then $P(S) = 8$*
- *If the only non trivial edge is P_k connected to P_{4k} then $P(S) = 4$*
- *If the only non trivial edges are the edge between P_k, P_{4k} and a loop around P_{4k} then $P(S) = 4$ or 6 according to $P_{4k} + P_{3k} > 2P_k$ or $P_{4k} + P_{3k} \leq 2P_k$*

Proof:

Claim1: If P_k is connected to P_{4k} but there is no loop around P_{4k} then $P_{4k} + P_{3k} \leq 2P_k$

This is because P_k connected to P_{4k} means $P_k - P_{4k} - F \in S$ which is iff $P_k + P_{4k} - F > P_{3k}$ And there is no loop around P_{4k} so $2P_{4k} - F \notin S$ i.e. $2P_{4k} - F \leq P_k$. Adding the two $(P_k + P_{4k} - F) + P_k > P_{3k} + (2P_{4k} - F)$ i.e. $2P_k > P_{3k} + P_{4k}$

Lemma 9.88. *$P_{4k} + P_{3k} \geq 2P_k$ (it follows in some manner from kunz inequalities)*

Corollary 9.88.1. *S is of max ED, $m(S) = 5$, S is not bad, $P(S) = 6$ imply $P_{4k} + P_{3k} = 2P_k$.*

In particular they lie on a hyperplane and have density 0

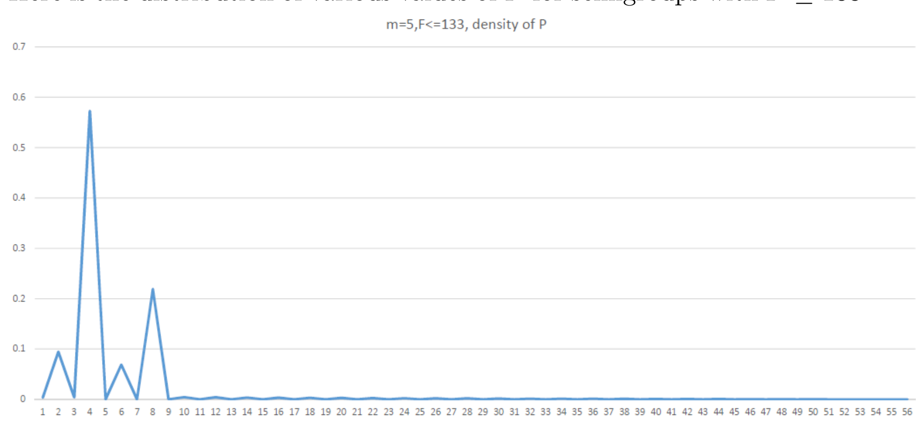
Theorem 9.89. *For multiplicity 5, $P(S) = 4, 8$ have positive densities, all other values of P combined have density 0.*

Moreover density of $P(S) = 4$ is approximately 0.29:

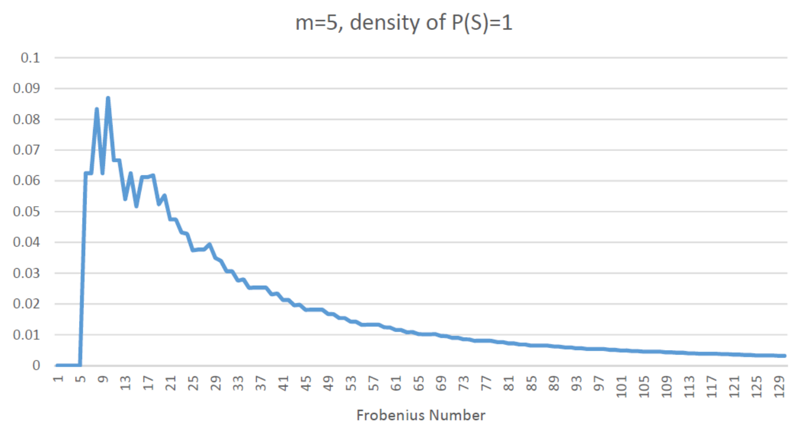
And density of $P(S) = 8$ is approximately 0.71 : (exact values can calculated by computing volumes)

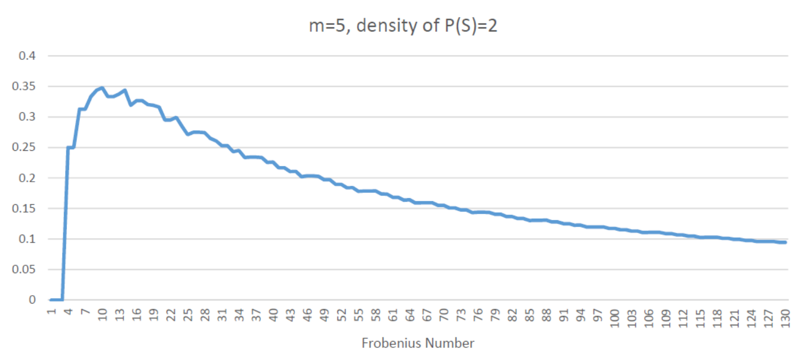
Remark 9.90. *We observe that finitely many hyperplanes divide the polyhedron into a number of regions, in each region the semigroups have the same $GPF(S)$ and it determines $P(S)$*

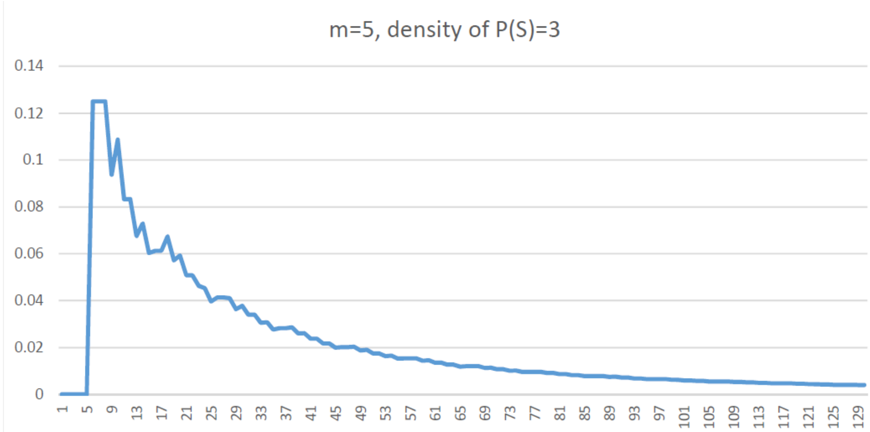
Here is the distribution of various values of P for semigroups with $F \leq 133$

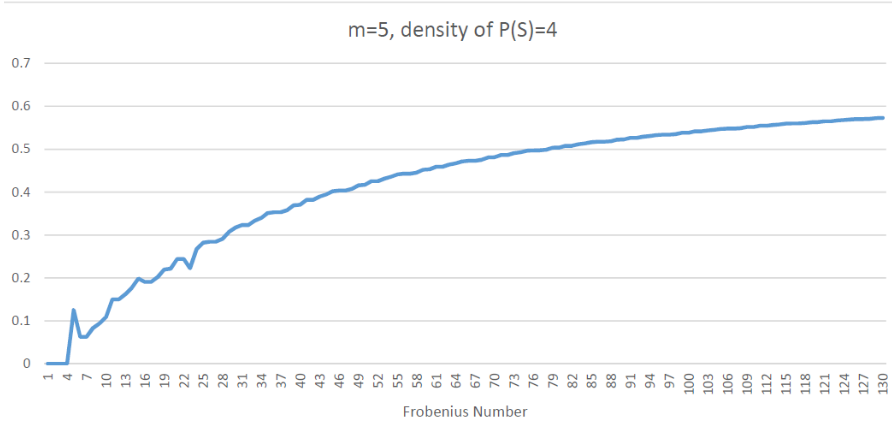


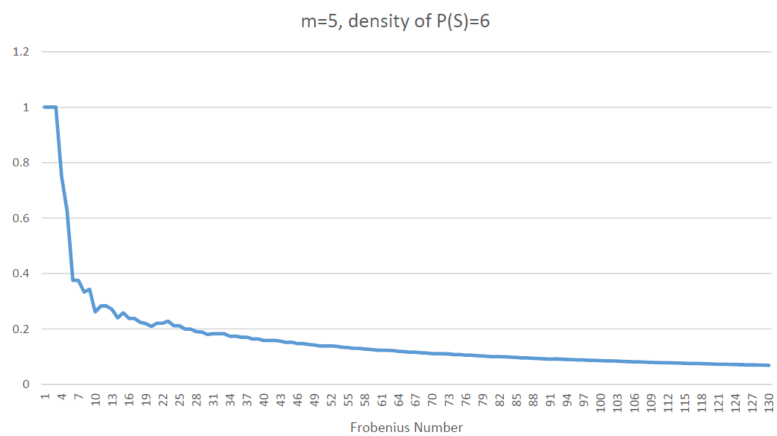
The following graphs show the density different values of P plotted against Frobenius number.

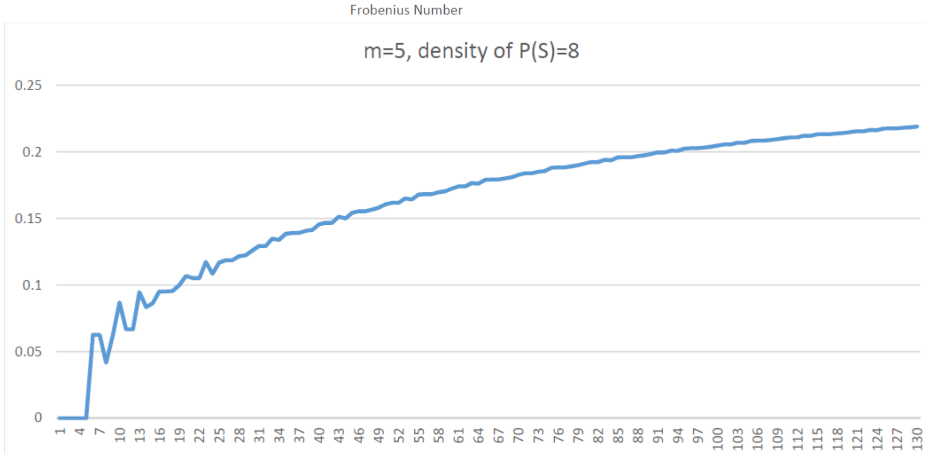

m=5, density of P(S)=1
0.1
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
Frobenius Number

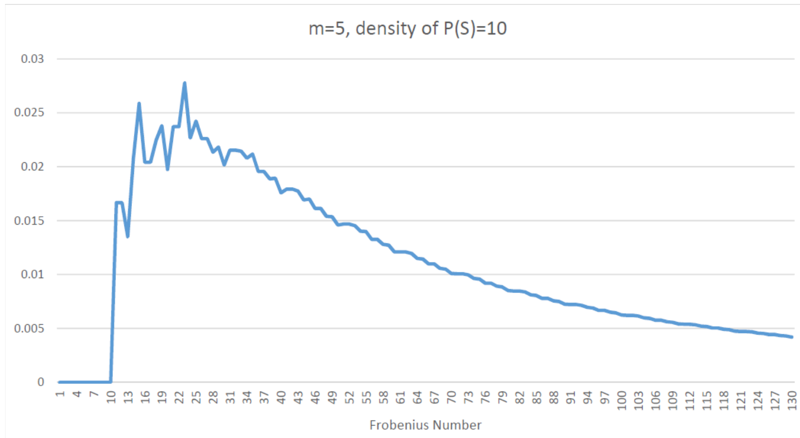

m=5, density of P(S)=2
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
Frobenius Number

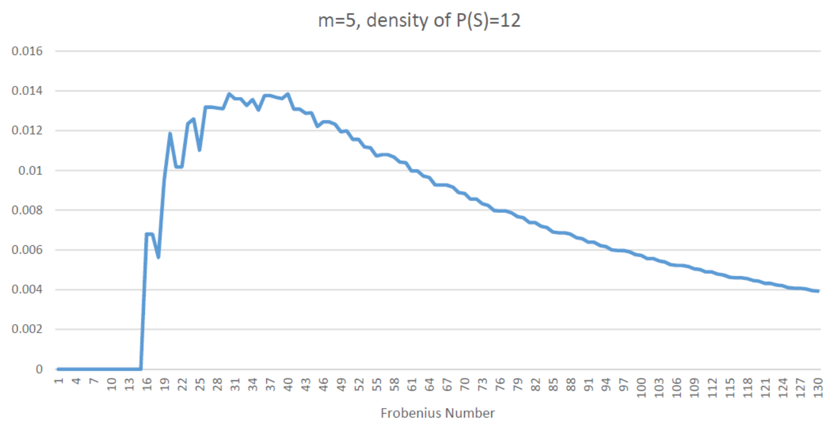

m=5, density of P(S)=3
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0

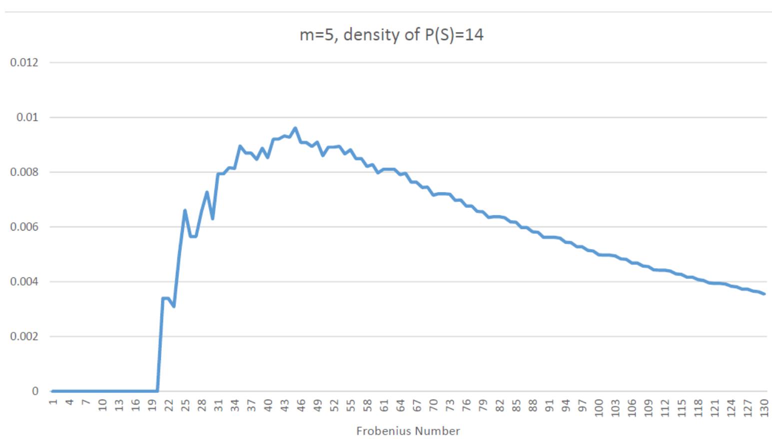

m=5, density of P(S)=4
Frobenius Number


m=5, density of P(S)=6
Frobenius Number


m=5, density of P(S)=8

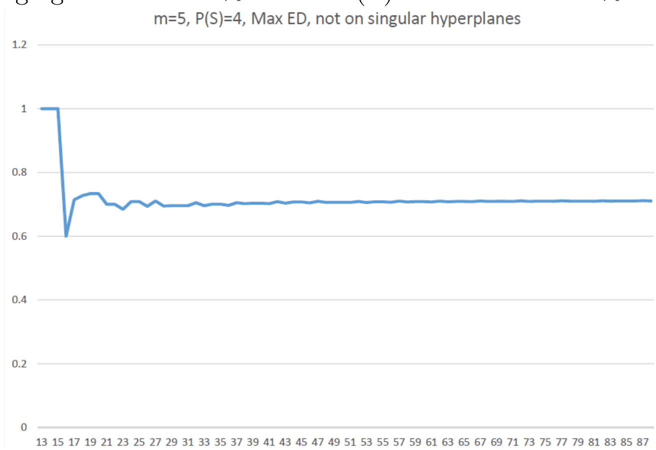


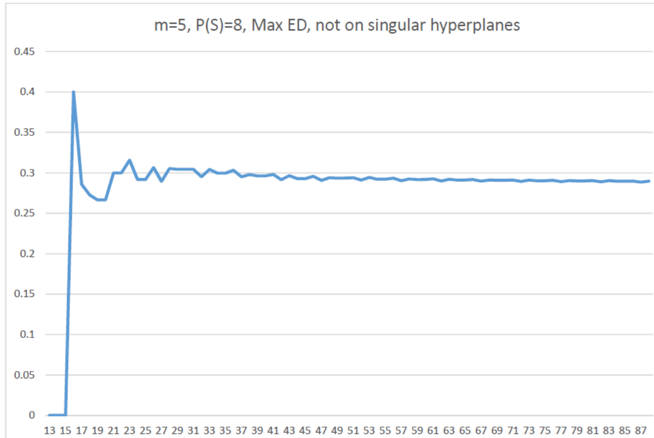




If we restrict ourselves to numerical semigroups that are of max embedding, not on bad hyperplanes and not on the hyperplane $P_{4k} + P_{3k} = 2P_k$ (we know that such semigroups have density 1). Then we get the following graphs (remember that $P(S)$ can only be 4 *or* 8) which show the density of $P(S) = 4$

converging to around 71% and of $P(S) = 8$ to around 29%





9.6 Multiplicity 6

Conjecture 9.91. *If $m(S) = 6$ (and no further restrictions) then $P(S)$ can take all values other than* 5

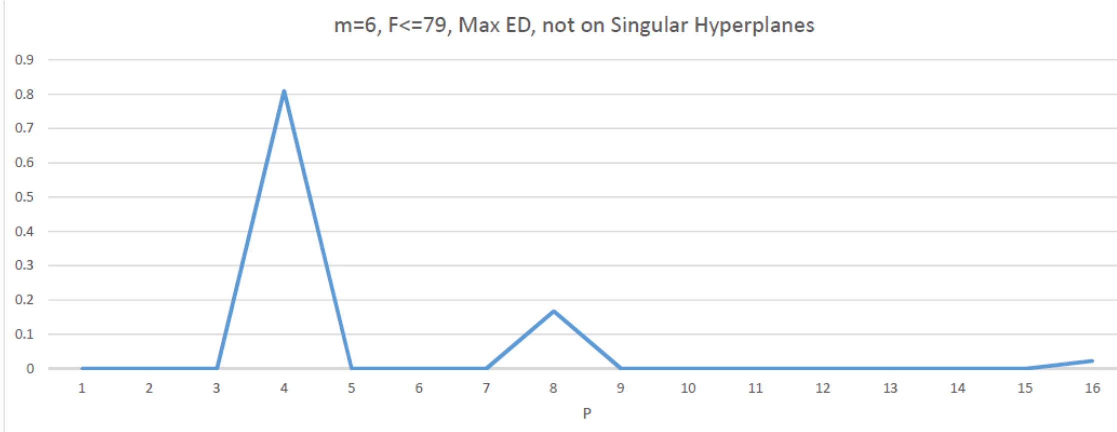
Conjecture 9.92. *Say S is of Multiplicity* 6 *and of max E.D. and not bad. Say the generators of S are $\langle 6, a_1, a_2, a_3, a_4, F\rangle$ s.t. $a_1 < a_2 < a_3 < a_4$ then:*

- *$P(S)$ is even as conjectured earlier*
- *If $P(E(E(S,1))) = P(S)$ then $P(S)$ is one of* $4, 6, 8, 12, 16$
- *If $P(S) \equiv 2(mod\ 4)$ and $P(S) \neq 6$ then $P(E(S,1)) = P(S) + 2$*
- *If $P(E(S,1)) \neq P(S)$ then $a_1 + a_4 = a_2 + a_3$*
- *$a_1 + a_4 \neq a_2 + a_3$ implies $P(S) = 4, 6, 8, 12, 16$*

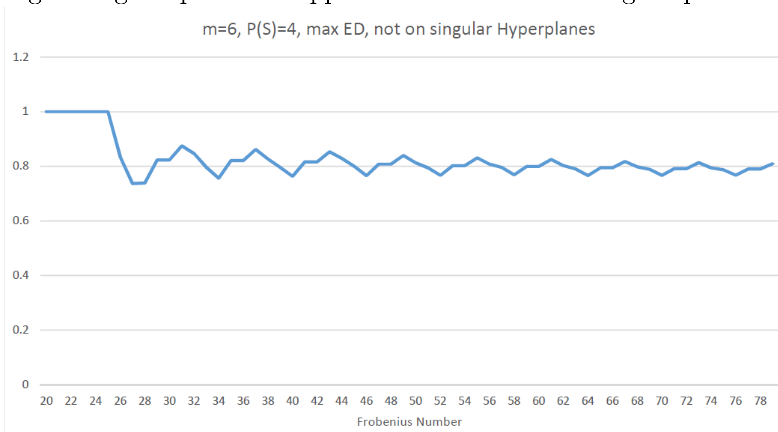
- *If whenever* $\{i_1, i_2\} \neq \{j_1, j_2\}$ *implies* $A_{i_1} + A_{i_2} \neq A_{j_1} + A_{j_2}$ *then* $P(S) = 4, 8, 16$ *and thus these are the only values that can have positive density*
- $P(S) = 4, 8, 16$ *are the only ones that have positive density*

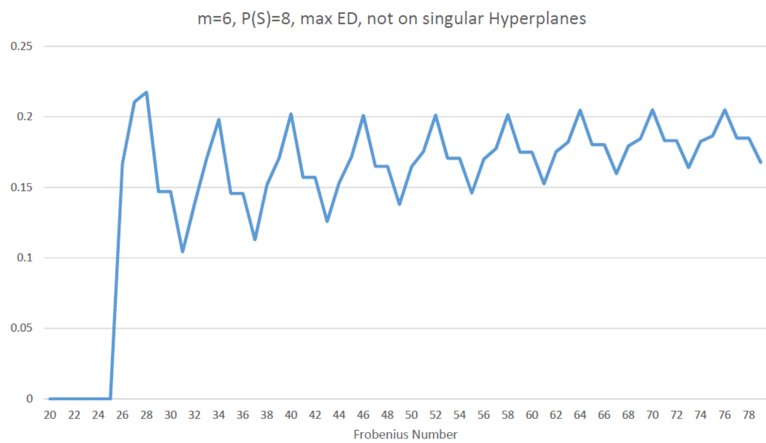
Definition 9.93 (Singular Hyperplanes)**.** *A hyperplane of the form* $A_{i_1} + A_{i_2} = A_{j_1} + A_{j_2}$, $\{i_1, i_2\} \neq \{j_1, j_2\}$ *is called a singular hyperplane.*

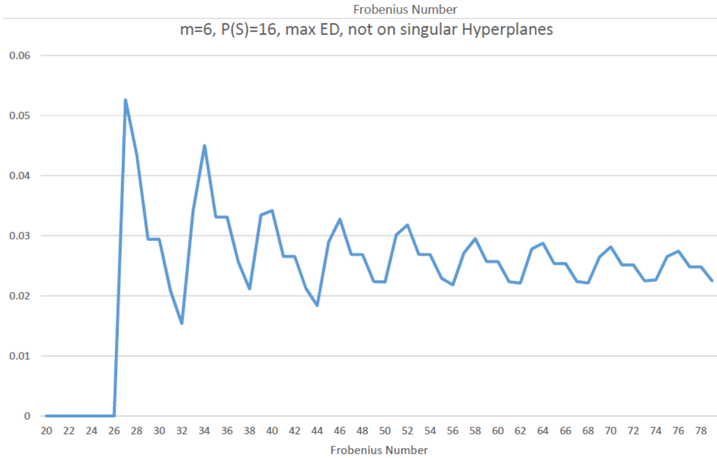
The following graph plots densities of of P for semigroups of $F \leq 79$ with max embedding dimension, not lying on a singular hyper plane (we know that singular hyperplanes behave differently and have density 0). The graph shows the only P values are $4, 8, 16$ and 4 appears around 80% times.



The next graphs show how densities of different values of P evolve with the frobenius number. We again restrict to semigroups of max embedding dimension not lying on singular planes. It appears that all 3 will converge to positive values.







9.7 Conjectures Regarding densities of $P(S)$ in kunz Polyhedron of fixed multiplicity

Conjecture 9.94. *S is of max embedding dimension*

- *If S is not bad then $P(S)$ is even*
- *If S is not bad and whenever i_1, i_2, j_1, j_2 are pairwise distinct $A_{i_1} + A_{i_2} \neq A_{j_1} + A_{j_2}$ then $P(S) = P(E(S,1))$*
- *If whenever $\{i_1, i_2\} \neq \{j_1, j_2\}$ implies $A_{i_1} + A_{i_2} \neq A_{j_1} + A_{j_2}$ then $P(S)$ takes the values (and only these values): $2^2, 2^3, \dots 2^{m-2}$ and thus these are the only values that can have positive density*
- *$2^2, 2^3, \dots 2^{m-2}$ all have positive densities*
- *density of $2^{\lceil \frac{m}{2} \rceil - 1}$ is the largest among them (probably > 0.5)*
- *density of $2^{\lceil \frac{m}{2} \rceil}$ is the second highest*

Conjecture 9.95. *The hyperplanes $A_{i_1} + A_{i_2} = A_{j_1} + A_{j_2}$ divide the kunz polyhedron into a number of regions. All points in the same region take the same value of $P(S)$*

Remark 9.96. *Things to investigate further:*

Can the degree of the polynomial of $P(E(S,n))$ be bounded in terms of the number of hyperplanes that S is on.

For $k < m - 2$ if $2^k \not| P(S)$ can we say which hyperplanes S must be on depending on k

Remark 9.97. *Possible approach:*

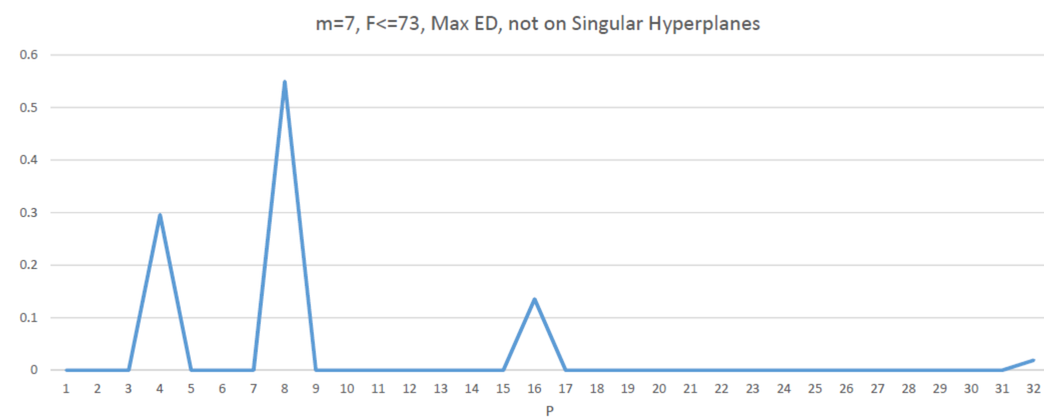
Prove that If S is of max embedding dimension and not on any of the hyperplanes described above then the only red triangles that can be satisfied are of the form $(Q, P - Q, F - P)$.

If this is true then all good numerical sets are generated by subsets of $DPF(S)$ and the DPF-Poset determines which subsets of $DPF(S)$ generate good numerical sets.

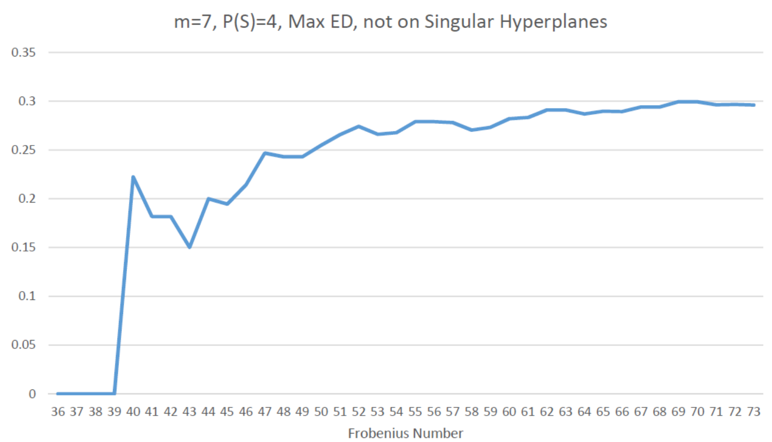
Numerical semigroups in the same region determined by those hyperplanes have the same DPF-Posets

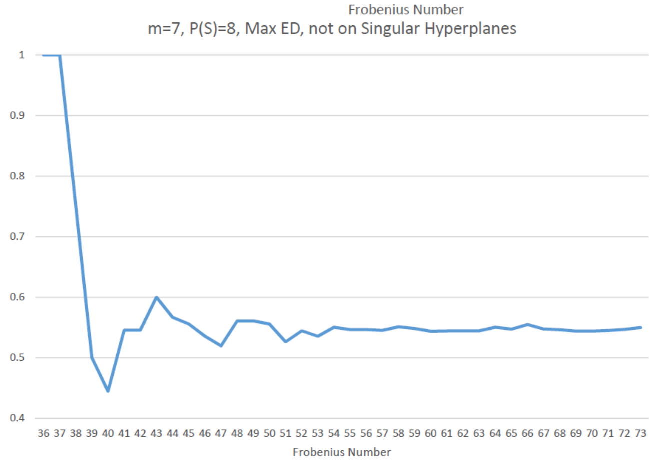
Some kind of combinatorial argument to show that $P(S)$ must be a power of 2

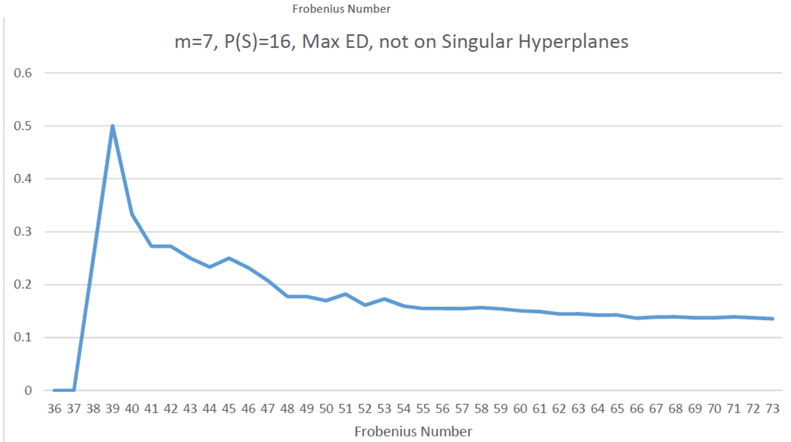
This graph shows the density of values of P for $m = 7$ when restricted to semigroups of max embedding dimension that are not on a singular hyperplane

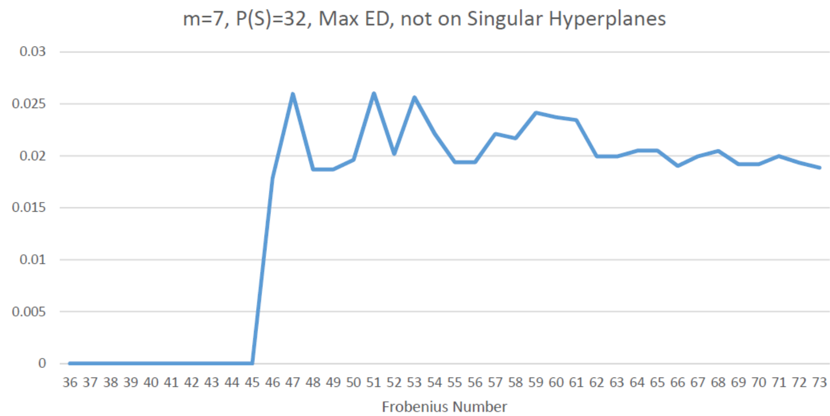


The next 4 graphs show that the densities of $P(S) = 4, 8, 16, 32$ actually seem to converge to positive values.


m=7, P(S)=4, Max ED, not on Singular Hyperplanes
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73
Frobenius Number

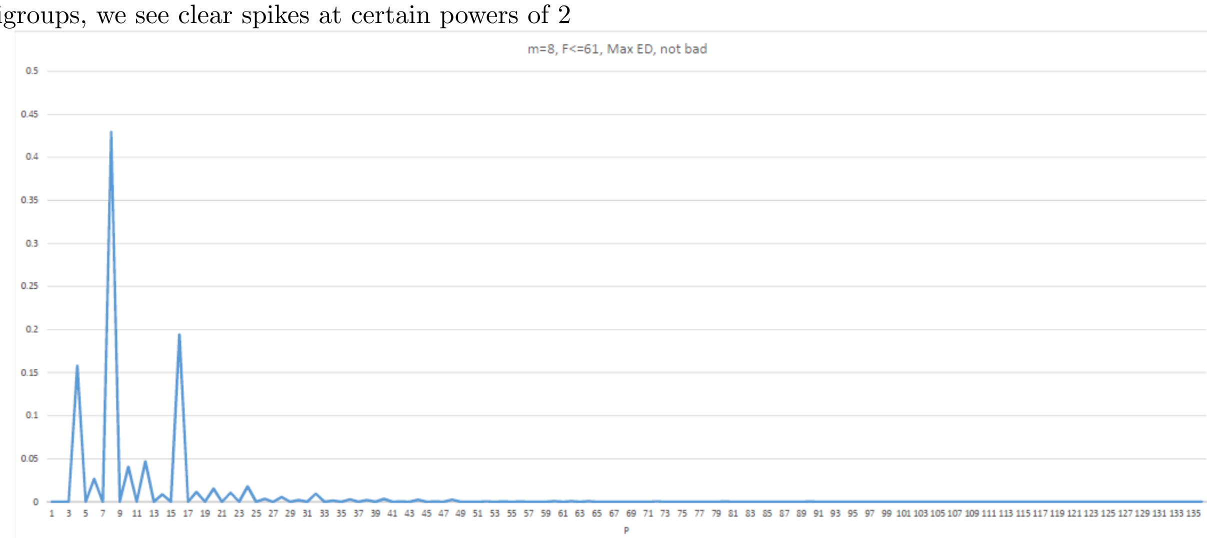

m=7, P(S)=8, Max ED, not on Singular Hyperplanes
1
0.9
0.8
0.7
0.6
0.5
0.4
36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73
Frobenius Number

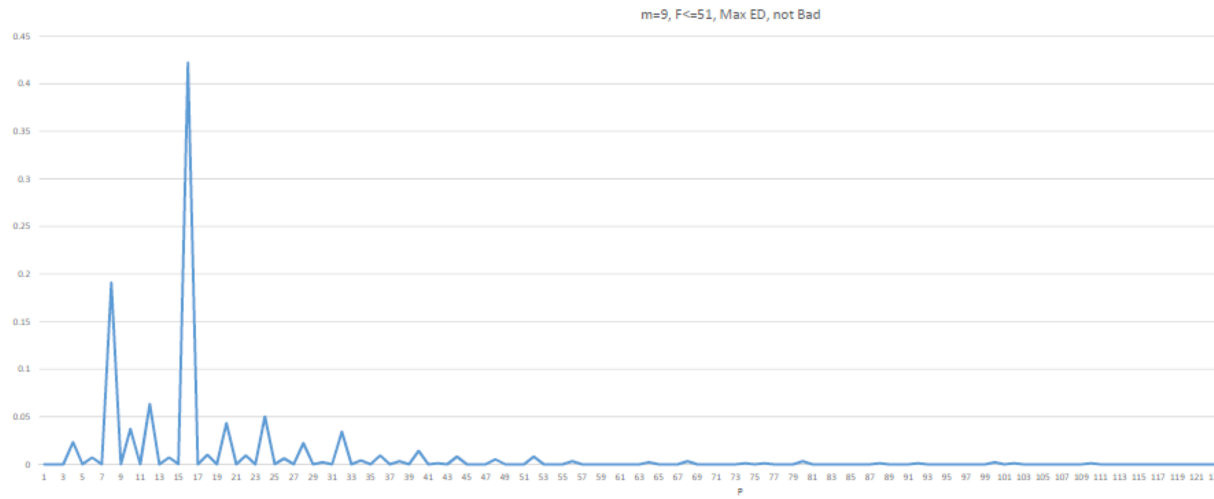

m=7, P(S)=16, Max ED, not on Singular Hyperplanes
0.6
0.5
0.4
0.3
0.2
0.1
0
36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73
Frobenius Number



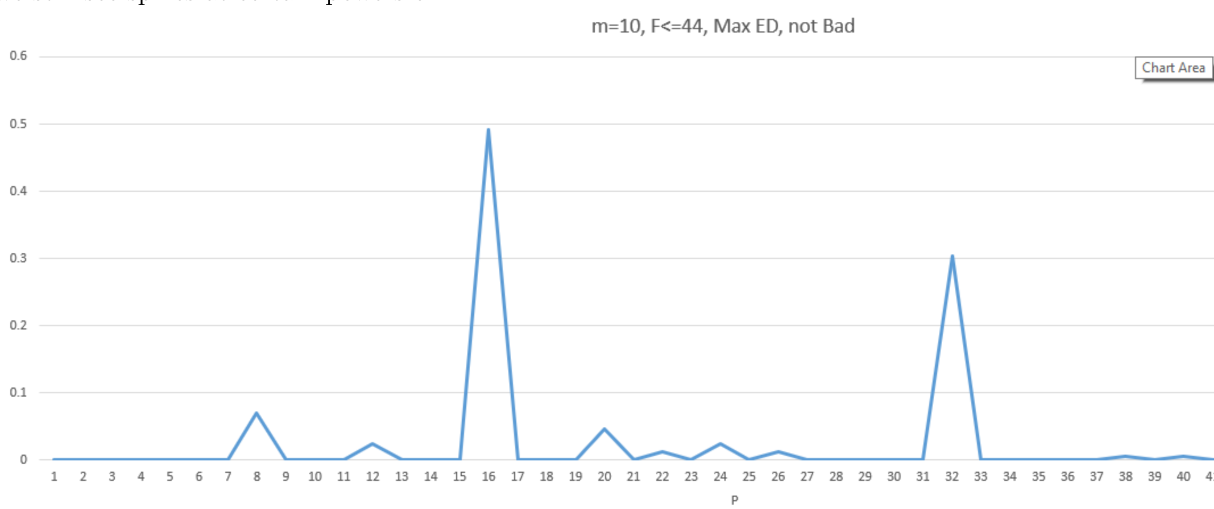
For higher multiplicities generating this data is a bit harder as even though semigroups outside of non-singular hyperplanes have density 1, they are quite sparse for smaller frobenius numbers

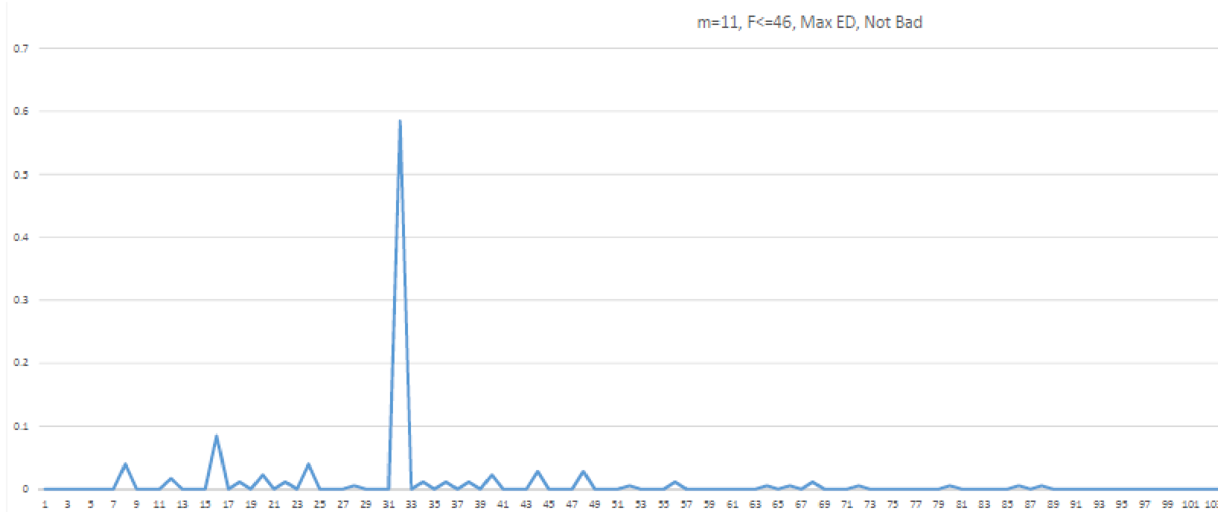
We therefore plot the densities of P values among Max ED, non bad semigroups, we see clear spikes at certain powers of 2





Note that for $m = 10, 11$ the total number of semigroups being considered is 171 and 176 respectively which is much smaller than previuos once. Nonetheless we still see spikes at certain powers of 2





10 Algorithmically Determining P(S)

We consider empty voids, noble semigroups, and ignoble semigroups separately.

```
if empty void then
  P(S)=1
end if
if noble semigroup then
  check Pseudo-Frobenius Graph for P(S)
end if
for all subsets of maximal void elements do
  put complement into "bad set", put subset into "good set"
  for all elts of subset do
    construct list of inclusion conditions
  end for
  construct all combinations of conditions
  for all combinations constructed do
    add described numbers to "good set", "bad set"
    take order ideal of "bad set", order filter of "good set"
    check that "good set" and "bad set" do not overlap
    for all antichains of remaining elts of void do
      add one to P(S)
    end for
  end for
end for
return P(S)
```