A Bunch of Tilde's and Squares

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ABSTRACT. In this article, we provide full characterizations of the associated semigroups of a numerical set and its complement when either one is a numerical semigroup. We also further develop a tool that arose during our investigation, eventually showing that they allow us to define a partial order on the set of all numerical semigroups that has thus far been unexplored.

Contents

1 Introduction

We call a subset of the nonnegative integers a *numerical set* provided it contains 0 and only has finitely many positive integers missing from the set. A *numerical semigroup* is a numerical set with additive closure. There are many numerical sets which are not numerical semigroups, but one can always find a natural numerical semigroup within the set which we will call the *atomic monoid* of the set or the *associated semigroup* of the set. We denote the associated semigroup of a numerical set S as $A(S)$, and it is given by

$$A(S) = \{s \in S \mid s + S \subseteq S\}.$$

It is straightforward to show that $A(S)$ is in fact a numerical semigroup. Also note that if S was a numerical semigroup, then the associated semigroup of S is itself since S is closed under addition.

Kaplan et al. [1] has shown that numerical sets have a bijective correspondence to Young Diagrams. A *Young Diagram* or sometimes *tableau* is an array given by stacking rows of squares of varying length, but with the property that the rows are decreasing in length as they progress down. An example is given below. If one wishes to understand the bijection between Young diagrams and numerical sets, please refer to [1].

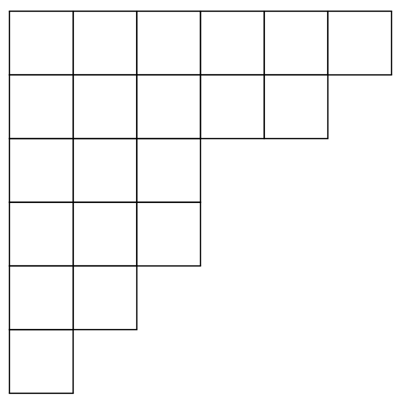


Figure 1: An example of a Young Diagram

Every box on the Young Diagram is associated to a *hook*. The hook is the set of boxes below and to the right of the particular box we are interested in. The number of boxes in the hook is called the *hook length* of that position or box of interest. This is seen in the following figure, where the number in each square is the hook length of that square, and a particular hook with hook length 5 is emphasized in green.

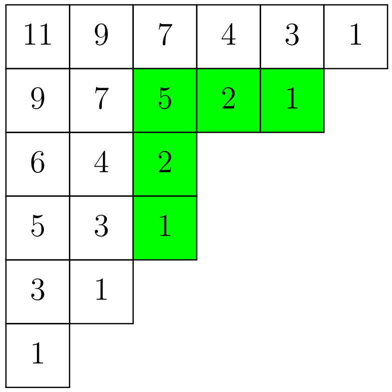


Figure 2: Since there are 2 squares to the right and 2 below, the hook length at this position is 5.

Kaplan et al. [1] have also shown that the hook lengths of a numerical set's Young diagram correspond precisely to the gaps of its associated semigroup. Herman and Chung [3] found that a given hook multi-set is not unique to a particular tableau, meaning there are often many numerical sets that correspond to a particular associated semigroup. They did however, find that a numerical set is fully characterized by its hook multi-set accompanied by the hook multi-set of the *complement* of the Young Diagram. The *complement* of a Young Diagram is found by completing the rectangular grid with length and width of the first row and first column respectively. A picture is shown below. A more precise definition of the complementary tableau can be found in [3].

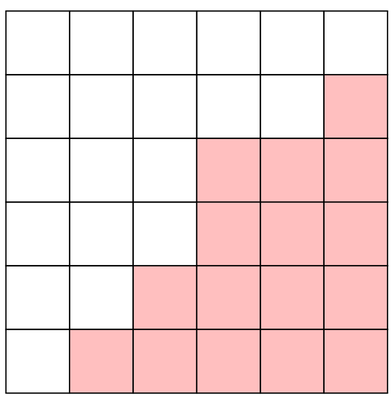


Figure 3: The complement of the Young diagram is in pink. Notice it is also a Young diagram when rotated by 180 degrees

Note that if you rotated the complement by 180 degrees, it is then also a tableau, and thus also corresponds to a unique numerical set. Therefore every numerical set has a unique complement numerical set. The converse, however, is not true: a numerical set can be the complementary numerical set of many numerical sets. If S is a numerical set, we will refer to this set as the *complementary numerical set* of S or S's *complement* for short, and denote it $\tilde{S}$. Combining the results of Herman and Chung with Kaplan's, we interpreted there to be underlying structure relating the associated semigroups of complementary numerical sets. In the succeeding sections, we characterize this correspondence for when either of the numerical sets are numerical semigroups.

2 Notation and Terminology

In this text, we will adopt the following notation for the objects of interest. We will typically use λ to denote the Young Diagram, with λ^* being λ's conjugate, and $\tilde{\lambda}$ being its complement. Similarly, S (and other capital, English letters) will primarily represent numerical sets, with S^* its conjugate, and $\tilde{S}$ its complement (note this is not the typical set complement that would be defined as $\mathbb{N}_0 \backslash S$). Since these objects are so fundamentally related, we may occasionally use $\lambda(S)$ to mean "the diagram associated with the numerical set S" when we wish to emphasize the difference, but otherwise S will refer to both the numerical set and its Young Diagram. We will call a numerical semigroup "nontrivial" if it is not all of the nonnegative integers (hereafter $\mathbb{N}_0$).

We will also commonly refer to the following quantities. Most of these are common terminology to anyone who has previously studied numerical semigroups, but we amend some of their definitions to extend to numerical sets. For the following, let S be a numerical set. Then

- The *Frobenius number* $F(S)$ of a numerical set S is the largest positive integer not in the set i.e. $F(S) = \max\{\mathbb{N}_0 \backslash S\}$
- $C(S) = F(S) + 1$ is *the conductor* of S. This is the first integer such that everything after it is in the set. To denote this, we will typically use a right arrow $\rightarrow$ after the conductor.
- *Small elements* of S are elements in S that are less that $F(S)$. *Large elements* are elements larger than $F(S)$.
- The *genus* of a numerical set is the number of positive integers not in S (these are frequently called the *gaps* of S). The genus of S is denoted $g(S)$. i.e. $g(S) = |\mathbb{N}_0 \backslash S|$. We will call $\mathbb{N}_0 \backslash S$ the *gap set* of S.
- The *multiplicity* of S is denoted $m(S)$ and is the first nonzero element that is larger than the first gap of S. That is, $m(S) = \min\{x \in S : x > n\}$ where $n = \min(\mathbb{N}_0 \backslash S)$.
- If S is a numerical semigroup, then the *atoms* of S are the elements that cannot be written as a sum of smaller elements. These are also often called the *generators* of S because every element of S can be expressed as a linear combination of them.
- The *embedding dimension* of S is the number of atoms, and it is denoted $e(S)$.
- The *Hook set* of S is the set of hooks in the Young Diagram of S. The *Hook multi-set* of S is the set of hooks in the Young Diagram of S counting repeats.

Example 2.0.1. *For the numerical semigroup* $S = \{0, 2, 4, 6 \rightarrow\}$
$C(S) = 6,$
$H(S) = \{1, 3, 5\},$
$F(S) = 5,$
$g(S) = 3,$
$m(S) = 2.$
The generators are $< 2, 7 >$, *and thus* $e(S) = 2.$

We will later introduce more terms as we need them.

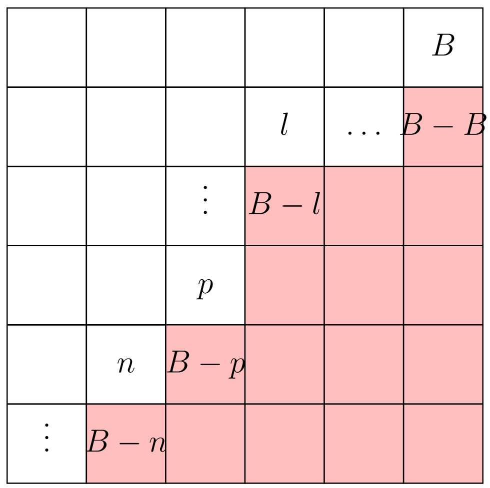
3 The Complementary Numerical Set

We begin our investigation by finding a better description of the complementary numerical set. We then describe some of the structural consequences the original set and the complement set impose on each other.

Definition 3.0.1. *Let S be a numerical set. We denote its Base $B(S)$ as its biggest small element, namely, $B(S) = \max\{s \in S \mid s < F(S)\}$*

Theorem 3.0.2. *Let S be a numerical set with Base B. If $n \leq B$, then $n \in S$ if and only if $B - n \in \tilde{S}$*

Proof. Consider the following Young tableau:



Since B is the largest element that remains in the set S, then we have that the horizontal line that corresponds to B in S is the same as the one that corresponds to 0 in $\tilde{S}$. From there, one can move to the adjacent line in S (the one that corresponds to $B - 1$ in S). Since we progressed by 1, it corresponds to 1 in $\tilde{S}$. This continues iteratively, and one can form couplets that sum to B (e.g. $(1, B-1), (2, B-2), ...(i, B-i)$). Since a horizontal line in S is a horizontal line in $\tilde{S}$, and the same is true for vertical lines, these pairs must both be in their respective sets or not at all. $\square$

Lemma 3.0.3. *Let S be a numerical set with $\tilde{S}$ its complement. Then $\{B(S) \rightarrow\} \subseteq \tilde{S}$.*

Proof. Since S is a numerical set, $0 \in S$ and due to Theorem 3.0.2, then $B(S) \in \tilde{S}$. In the Young diagram for $\tilde{S}$, since $B(S) \in \tilde{S}$ corresponded to $0 \in S$, then it is the last line of the Young Diagram, and therefore everything greater than $B(S)$ is also in $\tilde{S}$. $\square$

To emphasize, Lemma 3.0.3 only gives us $C(\tilde{S}) \leq B(S)$. We get equality only when $1 \notin S$.

Theorem 3.0.2 and Lemma 3.0.3 combine together to give a full characterization of $\tilde{S}$.

Theorem 3.0.4. *Let S be a numerical set with complement $\tilde{S}$. Then,*

$$\tilde{S} = \{B(S) - s \mid s \in S \text{ and } s \leq B(S)\} \cup \{B(S) \rightarrow\}.$$

Proof. Let $t \in \mathbb{N}_0$. Then there are two cases.
Case 1: If $t \leq B(S)$, then by Theorem 3.0.2, $t \in \tilde{S}$ if and only if $t \in \{B(S) - s \mid s \in S$ and $s \leq B(S)\}$.
Case 2: If $t > B(S)$, then $t \in \{B(S) \rightarrow\}$ and from our Lemma $t \in \tilde{S}$. Together these cases give us the desired equality. □

We now shift our attention to when at least one of the two sets is a numerical semigroup.

Theorem 3.0.5. *Consider a numerical semigroup S with Base B and multiplicity m. If $n \in \tilde{S}$ and $m \leq n \leq B$, then $n - m \in \tilde{S}$.*

Proof. Since $n \in \tilde{S}$ and $m \leq n \leq B$, then by Theorem 3.0.2 then $B - n \in S$. Since S is a semigroup, $B - n + m = B - (n - m) \in S$. Since $m \leq n, 0 \leq n - m$, then $B - (n - m) \leq B$ and again by 3.0.2, $n - m \in \tilde{S}$ as required. □

Corollary 3.0.6. *If S is a numerical semigroup with Base B and multiplicity m, then $km + r \in \tilde{S} \implies jm + r \in \tilde{S}$ for $0 \leq j \leq k$.*

Since we will use it later, we emphasize the contrapositive of Theorem 3.0.5:
"If $s \notin \tilde{S}$ and $0 < s < B(S) - m(S)$, then $s + m(S) \notin \tilde{S}$ ".

There is an analogue theorem to the previous for when $\tilde{S}$ is a semigroup.

Theorem 3.0.7. *Consider a numerical set S with Base B, and suppose $\tilde{S}$ is now a numerical semigroup. If $t \in S$ with $m(\tilde{S}) \leq t \leq B(S)$, then $t - m(\tilde{S}) \in S$.*

Proof. The proof is similar. From Theorem 3.0.2, we have that if $t \in S$, then $B - t \in \tilde{S}$. Since $\tilde{S}$ is a semigroup, then we must have $B - t + m(\tilde{S}) = B - (t - m(\tilde{S}) \in \tilde{S}$. Therefore we can again conlude we can conclude that $t - m(\tilde{S}) \in S$. □

Corollary 3.0.8. *If $\tilde{S}$ is a numerical semigroup, then $km(\tilde{S}) + r \in S \implies jm(\tilde{S}) + r \in S$ for $0 \leq j \leq k$.*

Again the contrapositive of 3.0.7 will be be useful:
"$y \notin S$ and $0 < y < B(S) - m(\tilde{S}) \Rightarrow y + m(\tilde{S}) \notin S$."

Fundamental to both of the previous theorems was the need for one of the sets to have additive closure. If neither of them do, you can still establish a similar statement with instead using the additive closure of the associated semigroup.

Theorem 3.0.9. *Consider a numerical set S and its complement $\tilde{S}$. If $n \in \tilde{S}$ and $m(A(S)) \leq n \leq B$, then $n - m(A(S)) \in \tilde{S}$. Similarly, if $n \in S$ and $m(A(\tilde{S})) \leq n \leq B$, then $n - m(A(\tilde{S})) \in S$*

The proofs are identical to those of the previous two.

Note all the theorems after 3.0.5 to this point don't actually require the subtraction to be by the multiplicity, merely just an element in the semigroup. For our purposes however, we will only need the statements as they are written.

To conclude this section, we will put a series of properties that we can determine of $\tilde{S}$ from knowing S. All but 2 use the step height and step width sequences of S $N(S)$ and $M(S)$ respectively. These will not be introduced until Section 8, so it is advisable to visit there first if needed.

Theorem 3.0.10. *Let S be a numerical set with $M(S) = (M_0, \ldots, M_k)$ and $N(S) = (N_0, \ldots, N_k)$. The following are properties of its complement $\tilde{S}$:*

1. $N(\tilde{S}) = (N_{k-1}, N_{k-2}, \ldots, N_2, N_1)$ *and* $M(\tilde{S}) = (M_k, M_{k-1}, \ldots, M_3, M_2)$
2. $F(\tilde{S}) = F(S) - (M_0 + N_k) = B(S) - M_0$
3. $B(\tilde{S}) = B(S) - (M_0 + N_0) = B(S) - m(S)$
4. $m(\tilde{S}) = B(S) - B_2(S)$
5. $\mathbb{N}_0 \backslash \tilde{S} = B(S) - \mathbb{N}_0 \backslash S$
6. $g(\tilde{S}) = g(S) - N_k$
7. $H(\tilde{S}) = \{a - b : a \in b, b \notin S, \ \& \ \ b < a \leq B(S)\}$
8. $\lambda(\tilde{S}) = \sum_{i=0}^{k} M_i - \lambda(S)$

Proof.

1. This will be shown in section 8
2. $F(\tilde{S}) = B(S) - \min\{\mathbb{N}_0 \backslash S\} = B(S) - M_0 = (F(S) - N_k) - M_0 = F(S) - (M_0 + N_k)$
3. $B(\tilde{S})$, by definition, is the last element in $\tilde{S}$ before the conductor. So by 3.0.2, it corresponds to the first element in S after the first set of gaps. This element is $M_0 + N_0$ in S. So, by 3.0.2, $M_0 + N_0 \in S \implies B(S) - (M_0 + N_0) = B(\tilde{S}) \in \tilde{S}$.
4. Since elements $s \in S$ map to $B - s \in \tilde{S}$, then its clear that to find the smallest element after the first gap of $\tilde{S}$, we should find the largest element in $\{M_{k-1}\}$. This is precisely the definition of $B_2(S)$, so $B_2(S)$ maps to $m\tilde{S}$ and $m(\tilde{S}) = B(S) - B_2(S)$.
5. In light of Theorem 3.0.2, an integer s is a gap of S if and only if $B - s$ is a gap of $\tilde{S}$. Thus, $\mathbb{N}_0 \backslash \tilde{S} = B(S) - \mathbb{N}_0 \backslash S$.
6. Since the N-sequence represents all strings of gaps, $\sum N_i$ is the genus of S. From property 1, it is clear that $g(\tilde{S}) = g(S) - N_k$.

7. Using the definition of the Hook set and part 5 of this theorem,

$$\begin{aligned} H(\tilde{S}) &= \{\tilde{b} - \tilde{a} | \tilde{b} \in \mathbb{N}_0 \backslash \tilde{S}, \tilde{a} \in \tilde{S}, \tilde{a} < \tilde{b} \\ &= \{(B(S) - b) - (B(S) - a) | b \in \mathbb{N}_0 \backslash S, b < B(S), a \in S, B(S) - b > B(S) - a \geq 0\} \\ &= \{a - b | b \in \mathbb{N}_0 \backslash S, a \in S, b < a \leq B(S)\} \end{aligned}$$

8. Consider an arbitrary r in the tuple $\lambda(S)$. There exists a corresponding $\tilde{r} = \sum_{i=0}^{k} M_i - r$ in $\lambda(\tilde{S})$. This map is bijective with itself as an inverse, so in fact $\lambda(\tilde{S}) = \sum_{i=0}^{k} M_i - \lambda(S)$.

□

4 Arithmetic Sequences

Here we begin to answer this paper's first essential question, "When are a numerical set and its complement both numerical semigroups?" This section will provide the answer for a particular family of semigroups, the family of generalized arithmetic semigroups. We will see that most of these semigroups have complements that are not numerical semigroups and discuss the few cases in which the complement is a numerical semigroup.

Definition 4.0.1. *We call a numerical semigroup S a* Generalized Arithmetic Semigroup *if it is generated by generators that are from a sequence of arithmetic progression. That is, for some $a, h, d, k \in \mathbb{N}$ with $1 \leq k \leq a - 1$ and $gcd(a, d) = 1$,*

$$S = \langle a, ha + d, ha + 2d, \ldots, ha + kd \rangle .$$

4.1 h=1

We will begin our investigation in the case of a standard arithmetic progression, in particular, when h=1.

Theorem 4.1.1. *If we have an arithmetic semigroup with maximum embedding dimension, $S = \langle a, a+d, \ldots, a+(a-1)d \rangle$ with $a \geq 3$ and $d \geq 2$ and we do not have $a = 3$ and $d = 2$ simultaneously, then we always have $(a-1)d - 1 \in S$ and at least one of $(a-1)d - 2$ or $(a-1)d - 3$ is in S.*

Proof. It is well known that it is possible to compute the Frobenius Number of a semigroup S from its Apery Set $\text{Ap}(S)$ and multiplicity $m(S)$, given by $F(S) = max(Ap(S)) - m(S)$. Additionally it is known that the Apery Set of an arithmetic semigroup with maximal generators is $\text{Ap}(S) = \{0, a+d, a+2d, \ldots, a+(a-1)d\}$. From this, it is clear that for our arithmetic progression semigroup $F(S) = (a-1)d$. Now consider $b = (a-1)d - 1 + a$. Since $a > 1$ by our hypothesis, then $b > F(S)$, so $b \in S$. Assume to the contrary that $B = (a-1)d - 1 \notin S$. Then $b = (a-1)d - 1 + a \in \text{Ap}(S)$ by definition. However, we know that elements of $\text{Ap}(S)$ have the form $a + kd$ for some $k = 1, 2, \ldots, a-1$. Hence, $b = (a-1)d - 1 + a = a + kd$ for some $k \in \mathbb{N}$ therefore $-1 = (k+1-a)d$, so $d \mid -1$, but this is contradiction since $d > 0$ and $d \neq 1$. So we must have $B = (a-1)d - 1 \in S$.

Suppose $d \neq 2$ and suppose towards contradiction that $(a-1)d-2 \notin S$. Then

$$(a-1)d-2+a > (a-1)d-2+2 = (a-1)d = F(S)$$

, so $(a-1)d-2+a \in S$ and $(a-1)d-2+a \in \mathrm{Ap}(S)$. Since

$$(a-1)d-2+a = (a-1)(d+1)-1 > (2-1)(1+1)-1 = 1$$

we must have $(a-1)d-2+a = a+kd$ for some $k = 1, \ldots, a-1$. Rearranging we see $-2 = (k-a+1)d$ so $d|-2$ and we must have $d = 1$ or $d = 2$. By assumptions we have $d \neq 1$ and $d \neq 2$, so this is a contradiction. Thus we have $(a-1)d-2 \in S$.

Now consider $d = 2$ and $a > 3$. Suppose towards contradiction that $(a-1)d-3 \notin S$. Then

$$(a-1)d-3+a > (a-1)d = F(S)$$

so $(a-1)d-3+a \in S$ and $(a-1)d-3+a \in \mathrm{Ap}(S)$. Since

$$(a-1)d-3+a = (a-1)(d+1)-2 > (3-1)(2+1)-2 = 2 > 0$$

we must have $(a-1)d-3+a = a+kd$ for some $k = 1, \ldots, a-1$, so $-3 = (k-a+1)d$ and $d|-3$. However, $2 \not| -3$, so this is a contradiction and we must have $(a-1)d-3 \in S$. □

Theorem 4.1.2. *For an arithmetic semigroup S with $h = 1$, $k = 1$, and $a \geq 3$, S is symmetric and $\tilde{S}$ is not a semigroup.*

Proof. Let $h = 1$, $k = 1$, and $a \geq 3$. So all of the semigroups are generated by 2 elements. From [5], we know any semigroup generated by 2 elements will be symmetric. Note, $a \geq 3$ so at least $1, 2 \notin S$, so by symmetry at least $F(S)-1, F(S)-2 \notin S$. If $F(S)-1, F(S)-2 \notin S$ then $0, 1 \in \tilde{S}$. Since $1 \in \tilde{S}$, then every element has to be in the complement for it to be a numerical semigroup. However, at least $0, 1 \in S$ so in our complement, there are at least 2 elements not $\tilde{S}$. Therefore $\tilde{S}$ is not a numerical semigroup. □

Theorem 4.1.3. *An arithmetic semigroup S with $h = 1$, $k = a-2$, and $a \geq 3$, $\tilde{S}$ is not a numerical semigroup.*

Proof. By Corollary 2.4 of [2] since $a \equiv 2 \mod k$, S is symmetric, so for the same reason as in the proof to Theorem 4.1.2, $\tilde{S}$ is not a numerical semigroup. □

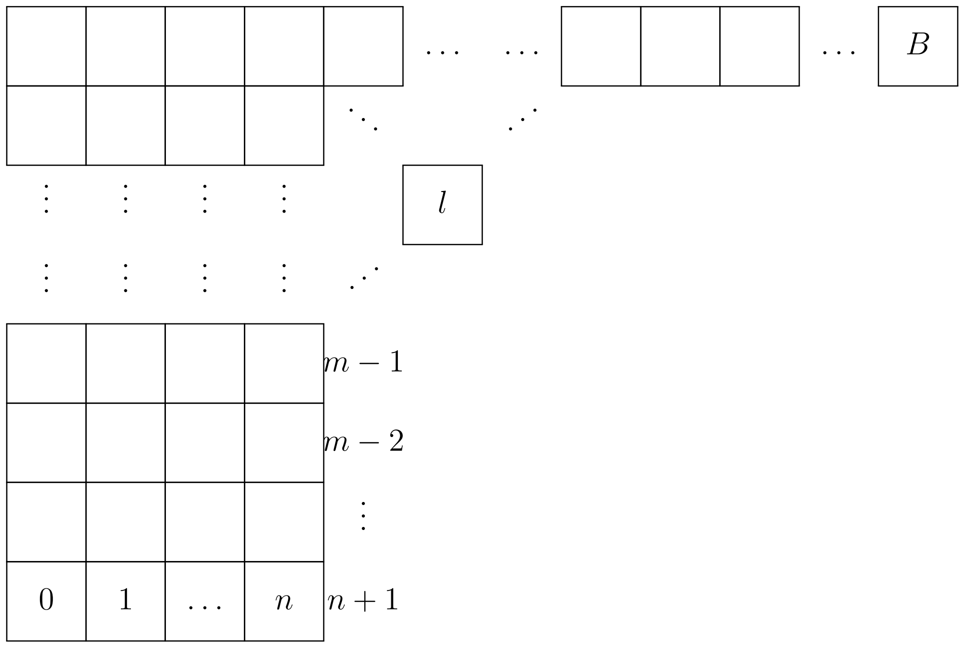
4.2 General $h \geq 1$

We will now extend our investigation by generalizing to $h \geq 1$.

In spirit of this text's first essential question on when S and $\tilde{S}$ are both semigroups, we will proceed by establishing a necessary conditions on S.

Lemma 4.2.1. *Let S be a numerical set with Base B and $n = \max\{s \in S : \{0, 1, \ldots, s\} \subseteq S\}$ and $m = \min\{s \in S : s > n\}$. If $S \cap \{n-1, \ldots, n-m+1\} \neq \emptyset$, then $\tilde{S}$ is not a numerical semigroup.*

Proof. Consider $B \in S$ and $l \in S$ with $l \in \{B-1, \dots B-m+1\}$. Then the young tableau will be:



So, $0 \in \tilde{S}$ since $B \in S$ and $B - l \in \tilde{S}$ since $l \in S$. Since $B - m + 1 \leq l \leq B - 1$ we have $1 \leq B - l \leq m - 1$. Since $n+1, \dots, m-1 \notin S$ we have $B-n-1, \dots, B-m+1 \notin \tilde{S}$ however $|\{B-n-1, \dots, B-m+1\}| = m-n-1$ and $1 \leq B-n-1 \leq m-n-1$ so some multiple of $B-n-l$ is in the set $\{B-n-1, \dots, B-m+1\}$ meaning $\tilde{S}$ is not closed under addition and thus is not a numerical semigroup. $\square$

Lemma 4.2.2. *The arithmentic semigroup $S = \langle a, ha+d, \dots, ha+kd \rangle$ with $h \geq 1$, $1 \leq k \leq a-1$, and a and d are relatively prime, has $B(S) = F(S) - 1$.*

Proof. By [2] $F(S) = \lceil \frac{a-1}{k} \rceil ha + (a-1)d - a$, so $F(S) - 1 = \lceil \frac{a-1}{k} \rceil ha + (a-1)d - a - 1$. Since $\gcd(a, d) = 1$ there exist $x, y \in \mathbb{Z}$ with $|x| < d$ and $|y| < a$ so that $xa + yd = 1$. If $y > 0$, $1 = xa + yd > xa$ so $x < \frac{1}{a}$ and since a is an integer $a \leq 0$. If $a = 0$, then $yd = 1$ so $d|1$ which is a contradiction so in fact we have $a < 0$. Then we can set $y' = y - a$ and $x' = x + d$ so we still have $x'a + y'd = (x+d)a + (y-a)d = xa + yd = 1$ and now $-a \leq y' \leq -1$ and $0 < x' < d$. Then we can rewrite

$$F(S) - 1 = \lceil \frac{a-1}{k} \rceil ha + (a-1)d - a - xa - yd = a\left(\lceil \frac{a-1}{k} \rceil h + d - x - 1\right) + d(-1-y) = aq + di$$

where $q = \lceil \frac{a-1}{k} \rceil h + d - x - 1 \geq 0$ since $x < d$ and $k \leq a-1$ so $\lceil \frac{a-1}{k} \rceil h \geq 1$. Also, $0 \leq i = -1-y < a-1$ since $-1 \geq y > -a$. Note, since $-1-y < a-1$ then $\frac{-1-y}{k} < \frac{a-1}{k}$ so $\lceil \frac{-1-y}{k} \rceil \leq \lceil \frac{a-1}{k} \rceil$ and $\lceil \frac{-1-y}{k} \rceil h \leq \lceil \frac{a-1}{k} \rceil h$. Also, since $x < d$, $d - x > 0$ so $\lceil \frac{a-1}{k} \rceil h \leq \lceil \frac{a-1}{k} \rceil h + d - x - 1$. Thus $\lceil \frac{i}{k} \rceil h = \lceil \frac{-1-y}{k} \rceil h \leq \lceil \frac{a-1}{k} \rceil h + d - x - 1 = q$, so by Proposition 2.1 in [2] $F(S) - 1 = aq + di \in S$. Then $B(S) = \max\{n \in S : n < F(S)\} = F(S) - 1$. $\square$

Theorem 4.2.3. *Let $S = \langle a, ha+d, \dots, ha+kd \rangle$ be an arbitrary arithmetic semigroup with $1 \leq k \leq a-1$, and $\gcd(a, d) = 1$, $a \geq 3$, and $d \geq 2$ but not simultaneously $a = 3$, $d = 2$, and $k = 2$. Then $B(S) - 1 \in S$ or $B(S) - 2 \in S$.*

Proof. Suppose $d > 2$. Suppose towards contradiction that $B(S) - 1 \notin S$, then $B(S) - 1 + a > B(S) + 1 = F(S)$ so $B(S) - 1 + a \in S$ and thus $B(S) - 1 + a \in \mathrm{Ap}(S)$. Notice $\mathrm{Ap}(S) = \{\lceil \frac{i}{k} \rceil ha + id : 0 \leq i \leq a-1\}$ is increasing in i since if $i > j$, then $\frac{i}{k} > \frac{j}{k}$ so $\lceil \frac{i}{k} \rceil \geq \lceil \frac{j}{k} \rceil$ and $id > jd$ so $\lceil \frac{i}{k} \rceil ha + id > \lceil \frac{j}{k} \rceil ha + jd$. Since $F(S) + a = B(S) + 1 + a$ is the largest element of the Apery Set of S, $B(S) + 1 + a = \lceil \frac{a-1}{k} \rceil ha + (a-1)d$. Then since $B(S) \in S$, $B(S) - 1$ is the second largest gap of S, so $B(S) - 1 + a$ is the second largest element of the Apery Set: $B(S) - 1 + a = \lceil \frac{a-2}{k} \rceil ha + (a-2)d$. Note $(B(S) + 1 + a) - (B(S) - 1 + a) = 2$ and also

$$\begin{aligned}(B(S) + 1 + a) - (B(S) - 1 + a) &= \lceil \frac{a-1}{k} \rceil ha + (a-1)d - \left(\lceil \frac{a-2}{k} \rceil ha + (a-2)d \right) \\ &= ha \left(\lceil \frac{a-1}{k} \rceil - \lceil \frac{a-2}{k} \rceil \right) + d \geq d > 2\end{aligned}$$

which is a contradiction. Thus, $B(S) - 1 \in S$.

Now suppose $d = 2$ and $a > 3$. If $B(S) - 1 \in S$ we are done, so say $B(S) - 1 \notin S$ and we will show that $B(S) - 2 \in S$. Suppose towards contradiction that $B(S) - 2 \notin S$. Then $B(S) - 2 + a > B(S) + 1 = F(S)$ so $B(S) - 2 + a \in S$ and moreover $B(S) - 2 + a \in \mathrm{Ap}\, S$. As before $B(S) + 1 + a$ is the largest element of the Apery Set, $B(S) - 1 + a$ will be the second largest, so $B(S) - 2 + a$ must be the third largest element of the Apery Set. Thus, $B(S) - 2 + a = \lceil \frac{a-3}{k} \rceil ha + (a-3)2$. We also have

$$B(S) - 2 + a = \lceil \frac{a-1}{k} \rceil ha + 2a - a - 2 - 1 + a = \lceil \frac{a-1}{k} \rceil ha + 2a - 5.$$

Then

$$\begin{aligned}0 = (B(S) - 2 + a) - (B(S) - 2 + a) &= \lceil \frac{a-1}{k} \rceil ha + 2a - 5 - \left(\lceil \frac{a-3}{k} \rceil ha + (a-3)2 \right) \\ &= ha \left(\lceil \frac{a-1}{k} \rceil - \lceil \frac{a-3}{k} \rceil \right) + 1 > 0\end{aligned}$$

which is a contradiction, so $B(S) - 2 \in S$.

For the case $a = 3$, $d = 2$, and $k = 1$, S has two generators so S is symmetric, but $1, 2 \notin S$, so $B(S) - 1, B(S) - 2 \notin S$. ☐

Corollary 4.2.4. *The arithmetic semigroup $S = \langle a, ha + d, \ldots, ha + kd \rangle$ with $a \geq 3$ and $d \geq 2$ but not simultaneously $a = 3$, $d = 2$, and $k = 2$ has $\tilde{S}$ is not a semigroup.*

Proof. This is an immediate result of Theorem 4.2.3 and Lemma 4.2.1. ☐

We look briefly at the examples that do not fall under these characterizations (i.e. those that still could have a complementary semigroup).

Example 4.2.5. *$S = \langle 2, x \rangle$ with x relatively prime to 2*

We can rewrite $x = 2q + 1$ where $q \in \mathbb{N}_0$. If $q = 0$, then $x = 1$, so $S = \mathbb{N}_0$. If $q > 0$ then $S = \{0, 2, \ldots, 2q \rightarrow\}$, so S is a 2-staircase (defined in next section) with q steps.

Example 4.2.6. *$a = 3$, $d = 2$, and $k = 2$*

Let $S = \langle 3, 3h+2, 3h+4 \rangle$. Then $0, 3, \ldots, 3h \in S$, $3h+1$ is not a multiple of 3 so $3h+1 \notin S$, $3h+2, 3h+3, 3h+4 \in S$ so every integer greater than $3h+4$ will be in S and $S = \{0, 3, \ldots, 3h, 3h+2 \rightarrow\}$. Notice that S fits the definition of a truncated 3-staircase (next section) with $h+1$ steps.

Definition 4.2.7. *We call a numerical semigroup pseudo-arithmetic if $S = \langle a, ha+d, ha+2d, \ldots, ha+kd \rangle$ for some $a \in \mathbb{N}_0$, $h \in \mathbb{N}$, $1 \leq k \leq a-1$, and $d \in \mathbb{Z} \backslash \{0\}$. [special thanks to the Geogroup Union for showing us this]*

Proposition 4.2.8. *Let S be a pseudo-arithmetic semigroup, i.e. $S \langle a, ha+d, ha+2d, \ldots, ha+kd \rangle$ for some $a \in \mathbb{N}$, $h \in \mathbb{N}$, $1 \leq k \leq a-1$, and $d \in \mathbb{Z} \backslash \{0\}$. Then $\tilde{S}$ is a semigroup if and only if $k = a-1$ and $d = 1$ or $a = 3$, $d = 2$, and $k = a-1$.*

Proof. Suppose S is a pseudo-arithmetic semigroup and $\tilde{S}$ is a semigroup. Then by 6.0.11 (this theorem is proven and discussed in a later section, but we wanted to put this proposition here to provide a full characterization of when an arithmetic or pseudo-arithmetic semigroup has complement also a semigroup), S is a truncated n-staircase with l steps for some $n, l \in \mathbb{N}$. So either $S = \langle n, ln+1, ln+2, \ldots, ln+a-1 \rangle$ in which case $a = n$, $d = 1$, and $k = a-1$. Or

$$S = \langle n, ln-j, ln-j+1, \ldots, ln-1, ln+1, ln+2, \ldots, (l+1)n-j-1 \rangle$$

for $j \in \mathbb{N}$, $j \neq 0$. We would still have $a = n$ and $k = n-1$, but now the difference between $ln-1$ and $ln+1$ is $d = 2$ but the difference between $ln-j$ and $ln-j+1$ is $d = 1$ so this only works when $n = a = 3$, $d = 2$, $k = n-1$. Thus, if S is an arithmetic semigroup with $\tilde{S}$ a semigroup if and only if $k = a-1$ and $d = 1$ or $a = 3$, $d = 2$, and $k = a-1$. □

5 n-staircases

This section will define a special family of numerical sets and, unlike the family in the last section, we will later prove that for every S in this family, both S and $\tilde{S}$ are numerical semigroups.

Definition 5.0.1. *Let S be a numerical set. We call S an* n-staircase with k-steps *if $S = \{0, n, 2n, \ldots, kn \rightarrow\}$ for $n \in \mathbb{N} \backslash \{0, 1\}$.*

Definition 5.0.2. *If S is a numerical set. We call S a* truncated n-staircase with k steps *if and only if $n \in \mathbb{N} - \{1\}$, $j \in \mathbb{Z}$ with $0 \leq j \leq n-2$, and $S = \{0, n, 2n, \ldots, (k-1)n, kn-j \rightarrow\}$.*

In both cases above, we call k the *step number* of S, or alternatively, S is a *staircase with k steps.*

We could have easily instead defined a truncated staircase as a numerical semigroup S such that $x \in S$ and $x < F(S)$ implies x is a multiple of $m(S)$, or equivalently, all non-multiplicity generators are greater than the the Frobenius number. All these above defintions are equivalent.

Note under these definitions an n-staircase is a special case of a truncated n-staircase.

Theorem 5.0.3. *For all $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$, then S a truncated n-staircase with k steps is a numerical semigroup.*

Proof. Let S be a truncated n staircase with k steps. Then $S = \{0, n, 2n, \ldots, (k-1)n, kn - j \rightarrow\}$. Clearly S is cofinite and contains 0, all that remains is to show additive closure. Let $a, b \in S$. If $a \geq kn - j$ or $b \geq kn - j$, then $a + b \geq kn - j$ so $a + b \in S$. If both $a < kn - j$ and $b < kn - j$, then $a, b \in \{0, n, 2n, \ldots, (k-1)n\}$ so $a = in$ and $b = jn$ for some $i, j \in \{0, 1, \ldots, k-1\}$. Therefore, $a + b = (i + j)n$ with $i + j \in \mathbb{N}$, so $a + b \in S$ since $n \in S$. Thus, S is closed under addition. Therefore, S is a numerical semigroup. □

Theorem 5.0.4. *For a semigroup S, S is an n-staircase ($j = 0$) if and only if both $B(S) \equiv 0 \mod m(S)$ and $F(S) + 1 \equiv 0 \mod m(S)$.*

Proof. $\Rightarrow$
This is clear by Definition 5.0.1.
$\Leftarrow$
Let $m = m(S)$. Suppose S is a numerical semigroup and $B(S) = km(S)$ for $k \in \mathbb{N}_0$. Naturally, this gives us $F(S) + 1 = (k+1)m$. Then suppose towards contradiction that $\exists y \in S$ such that $y < F(S)$ and $y \not\equiv 0 \mod m(S)$. By the division algorithm, $y = jm + r$ for $r \in \{1, 2, \ldots, m-1\}$. Since B(S) is the largest small element by definition, then we must have $y < B(S)$, so $j < k$. Since S is a semigroup, we must have additive closure, so $\{(j+1)m + r, (j+2)m + r, \ldots, km + r\} \subset S$. In particular, $km + r <\in S$ is contradiction. Since $r \in \{1, 2, \ldots, m-1\}$, then $B(S) < km + r < F(S) + 1$, contradiction to the fact $B(S)$ was the largest small element. Since the assumption led to contradiction, clearly no such y can exist and S is an n-staircase. □

One of the reasons we look at this particular family of semigroups is that its complement is within the same family.

Theorem 5.0.5. *For all $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$, a truncated n-staircase with k-steps will have a complement that is an n-staircase with $(k-1)$-steps.*

Proof. Since S has k steps, $B(S) = (k-1)n$. By Theorem 3.0.4, then

$$\tilde{S} = \{B(S) - s \mid s \in S \text{ and } s \leq B(S)\} \cup \{B(S) \rightarrow\},$$

and since S is a truncated n-staircase, then

$$\{B(S) - s \mid s \in S \text{ and } s \leq B(S)\} = \{0, n, 2n, \ldots, (k-1)n\}.$$

Thus,

$$\tilde{S} = \{B(S) - s \mid s \in S \text{ and } s \leq B(S)\} \cup \{B(S) \rightarrow\}$$
$$= \{0, n, 2n, \ldots, (k-1)n\} \cup \{B(S) \rightarrow\} = \{0, n, 2n, \ldots, (k-1)n\} \cup \{(k-1)n \rightarrow\}.$$

□

Corollary 5.0.6. *If S is an n-staircase with k steps, then both S and $\tilde{S}$ are numerical semigroups.*

Proof. The result is an immediate consequence of Theorem 5.0.5 and Theorem 5.0.3. □

6 Characterizing When Both S and $\tilde{S}$ are Semigroups

This section will begin by giving an alternative characterization of a numerical set S. We will use this characterization as a tool to completely classify which numerical semigroups have their complement a semigroup.

Definition 6.0.1. *Let S be a numerical set and let*

$G_i = \{g_i, g_i+1, g_i+2, \ldots, g_i+n_{g_i} : g_i-1 \in S$ and $g_i+n_{g_i}+1 \in S$ but $g_i+l \in \mathbb{N}_0 \backslash S$ for all $l \in \{0, 1, \ldots, n_{g_i}\}\}$

so that $g_i < g_j$ if $i < j$ and define the sequence of step heights *as $N(S) = \{N_i = |G_i| = n_{g_i} + 1\}$.*

This N sequence counts the lengths of strings of consecutive gaps of a numerical set S. In terms of a young diagram it is the step heights, as previously mentioned.

Definition 6.0.2. *Let S be a numerical set and let*

$J_i = \{j_i, j_i+1, j_i+2, \ldots, j_i+n_{j_i} : j_i-1 \notin S$ and $j_i+n_{j_i}+1 \notin S$ but $j_i+l \in S$ for all $l \in \{0, 1, \ldots, n_{j_i}\}\}$

with again $j_i < j_k$ if $i < k$ and define the sequence of step widths *to be $M(S) = \{M_i = |J_i| = n_{j_i}+1\}$.*

The M sequence counts the lengths of strings of consecutive small elements of a numerical set S. This would be the widths of the steps in the Young diagram.

Much more will be developed on M(S) and N(S) in Section 8. For now we prove only what we need to for the concluding theorem at the end of this section.

Theorem 6.0.3. *If $S \neq \mathbb{N}_0$ is a numerical semigroup with $M(S) = (M_0, M_1, \ldots, M_k)$. Then $M_0 = 1$.*

Proof. The first string of elements must begin at 0 since it is the smallest integer in the set. Since $S \neq \mathbb{N}_0$, then $1 \notin S$. Hence this particular string of small elements only contains 0 and thus $M_0 = 1$. □

Theorem 6.0.4. *Let $S \neq \mathbb{N}_0$ be a numerical semigroup with $N(S) = (N_0, N_1, \ldots, N_k)$. Then $N_0 = \max\{N\}$.*

Proof. Let $S \neq \mathbb{N}_0$ be a numerical semigroup with multiplicity m. Then $1, 2, \ldots, m-1 \in H(S)$, so $N_0 = |\{1, 2, \ldots, m-1\}| = m-1 = n_{g_0} + 1$. Suppose towards contradiction that there exists some $N_i > N_0$, i.e. $|G_i| > m-1$ so $|G_i| \geq m$. Then we have at least m consecutive numbers missing from our semigroup S, so at least one of these gaps must be a multiple of m. However, this contradicts that S is closed under addition (every positive multiple of m must be in S), so for all i, $N_i \leq N_0$ and thus $N_0 = \max\{N_i\}$. □

Corollary 6.0.5. *If a nontrivial semigroup S has that $N(S) = (N_0, N_1, \ldots, N_k)$, then $N_0 + 1 = m(S)$.*

Proof. This is given in the second line of the proof of 6.0.4. $\square$

Observation 6.0.6. *It is noteworthy to see how the M and N sequences correspond for the complementary numerical sets. Due to 3.0.2, we have that for any* $(x, B(S)-x)$, $x \in S \iff B(S)-x \in \tilde{S}$. *Hence, while the order of the sequences changes, the elements of N do not. Even better, the order changes in a predictable way since if* $N(S) = (N_0, N_1, \ldots, N_k)$, *then* $N(\tilde{S}) = (N_{k-1}, N_{k-2}, \ldots, N_1)$, *and if* $M(S) = (M_0, M_1, M_2, \ldots, M_k)$, *then* $M(\tilde{S}) = (M_k, M_{k-1}, \ldots, M_2)$. *These properties have some useful consequences.*

Theorem 6.0.7. *Let S be a numerical set. If S and* $\tilde{S}$ *are both numerical semigroups, then* $N(S) = (N_0, \ldots, N_k)$ *has* $N_0 = N_{k-1}$.

Proof. In light of the last observation, note that $N(\tilde{S}) = (N_{k-1}, N_{k-2}, \ldots, N_0)$. If S is a numerical semigroup then $N_0 \geq N_{k-1}$ by Theorem 6.0.4. Also, by Theorem 6.0.4, if $\tilde{S}$ is a numerical semigroup then $N_{k-1} \geq N_1$, so $N_{k-1} = N_0$. $\square$

Theorem 6.0.8. *Suppose S is a numerical semigroup, with* $N(S) = (N_0, N_1, \ldots, N_k)$ *having that* $N_0 = N_{k-1}$. *Then* $\forall i$ *with* $0 < i < k-1, N_i = N_0$.

Proof. Let S be a numerical semigroup with $N(S) = \{N_0 \ldots, N_k\}$ and $N_0 = N_k$. Then $m = m(S) = N_0 + 1$, and every multiple of m must be in S. Now consider our second to last string of N_{k-1} gaps $\{nm+l+1, nm+l+2, \ldots, nm+l+m-1\}$ where $nm+l \in S$ and $n \in \mathbb{N}$ and $0 \leq l \leq m-1$. Suppose towards contradiction that $l > 0$. Then $nm+l$ is not a multiple of m. The set $\{nm+l, nm+l+1, \ldots, nm+l+m-1\}$ has size m, so some element in this set must be a multiple of m. We said $nm+l$ is not a multiple of m, so some element in the set $\{nm+l+1, \ldots, nm+l+m-1\} \subseteq \mathbb{N}_0 \backslash S$ must be a multiple of m. This contradicts that S contains every multiple of m, so we must have that $l = 0$ and our set of N_{k-1} gaps is $\{nm+1, \ldots, nm+m-1\}$ for some $n \in \mathbb{N}$.

Suppose towards a contradiction that there exists an $0 < i < n$ such that $im \in S$ and $im + j \in S$ for some $1 < j < m-1$. We will show that $nm + j \notin S$. Every multiple of m has to be in the set, by closure under addition. If $im + j \in S$ then $nm + j$ has to also be in S because you can add m to $im + j$ until you reach $nm + j$. $\square$

Theorem 6.0.9. *Let S be a numerical semigroup with multiplicity m and* $N(S) = \{N_0, \ldots, N_k\}$, *with* $N_0 = N_{k-1}$. *Then the step widths of S are* $M_i = 1$ *for* $i \in \{0, 1, 2, \ldots, k-1\}$ *and* $M_k \in \{1, 2, \ldots, m-1\}$.

Proof. For every $n \in \mathbb{N}$ we have $nm \in S$. Suppose there exists an n so that $nm+1 \in S$ and $nm+1$ occurs before the N_{k-1} gaps, then $(n+1)m$ is also in S so there can at most be

$$(n+1)m - (nm+1) - 2 = m - 2 = N_1 - 1$$

gaps between $nm+1$ and $(n+1)m$ so for some $i \in \{2, \ldots, k-1\}$, $N_i < N_1$. This contradicts Theorem 6.0.8 so for all n, $nm+1 \notin S$ if $nm+1$ occurs before the second to last set of gaps. Now suppose $nm - 1 \in S$ for some $n \in \mathbb{N}$. If $n = 1$, then we have $m - 1 \in S$ is strictly less than the multiplicity which is a contradiction. If $n > 1$, the $(n-1)m \in S$ so for some $i \in \{2, \ldots, k-1\}$ so that $N_i \leq m - 2 < N_1$ which again contradicts Theorem 6.0.8. Thus, for all $j \in \{1, \ldots, k-1\}$, the j^{th} step width $M_j = 1$.

Note that $M_k \leq m - 1$ since otherwise we would have m consecutive numbers in S, so we would be able to get all larger numbers and there would not even be a k^{th} step. □

Theorem 6.0.10. *Let S be a numerical set and $\tilde{S}$ its complement. If S and $\tilde{S}$ are numerical semigroups, then $M_i = 1$ for $0 \leq i \leq k$.*

Proof. 6.0.9 has given us that $M_i = 1$ for $0 \leq i \leq k - 1$. We show in this case $M_k = 1$ Recall that $M_k(S) = M_0(\tilde{S})$. Assume to the contrary that $M_0(\tilde{S}) > 1$. Then $\tilde{S}$ must contain 1, so $\tilde{S} = \mathbb{N}$. But this cannot be the case then that $\tilde{S}$ is the complement of S. □

Theorem 6.0.11. *Let S be a nontrivial numerical set and $\tilde{S}$ its complement. Then S and $\tilde{S}$ are numerical semigroups if and only if S is a truncated n-staircase.*

Proof. $\Leftarrow$ has already been proven in 5.0.6.

$\Rightarrow$ Recall the definition of a truncated n-staircase as a numerical set of the form $\{0, n, 2n, \ldots, (k-1)n, kn - j \rightarrow\}$ for $n \in \mathbb{N} \backslash \{0, 1\}$ and $j \in \mathbb{Z}$ with $0 \leq j \leq k - 1$. From 6.0.10 and 6.0.8, then we have that S contains strings of N_0 gaps up until the kth string, and between each gap, there is exactly one number in S. These gaps begin at 1 since S is nontrivial, so the included numbers are exactly the multiples of $n = N_0 + 1$. Hence all that remains is to find j in $\{0, n, 2n, \ldots, kn - j \rightarrow\}$. But since N_k describes the number of gaps in the last sequence, then $j = N_0 - N_k$ suffices. □

This gives a necessary and sufficient condition for determining when S and $\tilde{S}$ are numerical semigroups.

7 $A(\tilde{S})$ and $A(S)$

This section will concentrate on generalizing 6.0.11 to describe the relationship between $A(S)$ and $A(\tilde{S})$ when S or $\tilde{S}$ is not a semigroup. This section may rely on some results in Section 8 of this paper which focuses on modifying the M and N sequences.

7.1 S or $\tilde{S}$ is a Semigroup

Theorem 7.1.1. *If S is a nontrivial numerical semigroup, then no two integers that sum to $F(\tilde{S}) = B - 1$ are in $\tilde{S}$. That is, no two of the following pairs are in $\tilde{S}$.*

$$(0, B-1), (1, B-2), \ldots, (k, B-(k+1))$$

Proof. By definition we know $B+1 \notin S$. Since S is a semigroup, no two of

$$(B,1),(B-1,2),(B-2,3),\ldots,(B-(k-1),k)$$

are in S. Since all of these are less than or equal to B, by Theorem 3.0.2 then no two of

$$(0,B-1),(1,B-2),\ldots,(k,B-(k+1))$$

are in $\tilde{S}$. $\square$

The previous result gives an indication that even when $\tilde{S}$ is not a numerical semigroup, it will not fail its closure property at its Frobenius number if it came from a semigroup.

Corollary 7.1.2. *Let S be a semigroup. If $y \in S$, then either $y > B(S)$ or $y-1 \notin \tilde{S}$.*

Proof. Suppose $y \in S$ and $y \leq B(S)$. Then by Theorem 3.0.2 then $B-y \in \tilde{S}$. By Theorem 7.1.1, then since $(B-y)+(y-1) = B-1$, then $y-1 \notin \tilde{S}$. $\square$

Definition 7.1.3. *We call a numerical set S a* column *if it has the form $S = \{0, g(S)+1 \rightarrow\}$.*

It is well known that a column is a numerical semigroup. Also note it is a staircase of one step.

Theorem 7.1.4. *Let S be a numerical semigroup. If S is not a truncated staircase, then $A(\tilde{S})$ is a column.*

Proof. Recall from Theorem 3.0.10, that $F(A(\tilde{S})) = F(\tilde{S}) = B(S)-1$, and $B(S) = B(\tilde{S}) + m(S)$. It suffices to show no small elements (elements less than $B(S)-1$) are in $A(\tilde{S})$. We do this by finding corresponding elements that sum to gaps, in particular, the gaps between $B(\tilde{S})$ and $C(\tilde{S})$.

First we will show $B(\tilde{S}) \notin A(\tilde{S})$. It suffices to show $\exists r \in \tilde{S} \ni 0 < r < m$. Since S is not a truncated staircase gives us that $\exists y \in S \ni y \leq B(S)$ and $y \not\equiv 0 \mod m(S)$. If $y \equiv B(S) \mod m(S)$ then $B(S)-m(S) \in \tilde{S}$ and $B(S)-m(S) \not\equiv 0 \mod m(S)$. If not $y \not\equiv B(S) \mod m(S)$ then $B(S)-y \in \tilde{S}$ and $B(S)-y \not\equiv 0 \mod m(S)$. Regardless, $\exists x \in \tilde{S} \ni x = jm+r$ with $0 < r \leq m-1$. By Lemma 3.0.5 then $r \in \tilde{S}$ as required. Then $B(\tilde{S}) < B(\tilde{S})+r < B(\tilde{S})+m(S) = B(S)$, so $B(\tilde{S})+r$ is a gap of $\tilde{S}$ since everything between $B(\tilde{S})$ and the conductor of $\tilde{S}$ must be a gap.

Now consider the following small elements of $\tilde{S}$, namely $\{p \in \tilde{S} \mid 1 \leq p < B(\tilde{S})\}$. From Theorem 3.0.5, we know that $B(\tilde{S}) - qm(S) \in \tilde{S}$ for all q so that $qm(S) \leq B(\tilde{S})$ since $B(\tilde{S}) \in \tilde{S}$. Fix $p \in \{p \in \tilde{S} \mid 1 \leq p < B(\tilde{S})\}$ and write $p = qm(S)+r$ with $q \geq 0$ and $0 \leq r < m(S)$. If $r \neq 0$, then

$$(B(\tilde{S})-qm(S))+p = B(\tilde{S})-qm(S)+qm(S)+r = B(\tilde{S})+r \in \{B(\tilde{S})+1,\ldots,B(\tilde{S})+m(S)-1 = F(\tilde{S})\}$$

so $p \notin A(\tilde{S})$ for the same reason as before.

Last we handle elements of the form $p = km(S)$ for some k, $p < F(\tilde{S})$. There are two cases. Let $nm(S)$ be the maximal multiple of $m(S)$ in $\tilde{S}$ such that $nm(S) < F(\tilde{S})$

Case 1: Suppose $(n+1)m(S) \notin \tilde{S}$. Then by Theorem 3.0.5, all of

$$m(S), 2m(S), \ldots, nm(S) \in \tilde{S}.$$

Then pair them so that they sum to $(n+1)m(S)$, so clearly none can be in $A(\tilde{S})$.

Case 2: Now suppose $(n+1)m(S) \in S$. Since we chose $nm(S)$ to be the largest multiple of $m(S)$ less than $F(\tilde{S})$, this can only occur if $(n+1)m(S) > F(\tilde{S})$. Additionally, this means that we have all multiples of $m(S)$ in $\tilde{S}$ by 3.0.5 we have all multiples less than or equal to $nm(S)$ and since $(n+1)m(S) > F(\tilde{S})$ we have all multiples larger than or equal to $(n+1)m(S)$. Note that we then have $B(S) - 1 \notin \tilde{S}$, so

$$1)\ B(S) - 1 < (n+1)m(S).$$

Also, $nm(S) \leq B(\tilde{S}) = B(S) - m(S)$ so

$$2)\ (n+1)m(S) \leq B(S).$$

So from 2 we have $(n+1)m(S) \leq B(S)$ and from 1 we have $B(S) < (n+1)m(S) + 1$, so in fact $B(S) = (n+1)m(S)$ and $B(\tilde{S}) = B(S) - m(S) = nm(S)$.

Since S is not a truncated staircase, as before, there exists an $x \in \tilde{S}$ such that $x \not\equiv 0 \mod m(S)$. Choose x to be the maximum such x that is less than $B(\tilde{S})$ and write $x = jm(S) + r$ with $0 < r < m(S)$. Note $x + m(S) = (j+1)m(S) + r$ is not in $\tilde{S}$ since $x < B(\tilde{S}) = nm(S)$ and either $x + m(S) < B(\tilde{S})$ in which case $x + m(S)$ cannot be in $\tilde{S}$ since we said that x was the largest. If $x + m(S) > B(\tilde{S})$, then we have

$$B(\tilde{S}) < x + m(S) < B(\tilde{S}) + m(S) = B(S)$$

so $x + m(S)$ must be a gap. Additionally, for all $l > j$ with $lm(S) + r < B(S)$, then $lm(S) + r \notin \tilde{S}$. Also, by 3.0.5 $im(S) + r \in \tilde{S}$ for all $i \in \{0, \ldots, j\}$. For $k \in \{1, \ldots, n-j\}$,

$$x < km(S) + x = (k+j)m(S) + r \leq nm(S) + r < (n+1)m(S) + r$$

so $km(S) \notin A(\tilde{S})$. For $k \in \{n-j, \ldots, n\}$, $n-k \in \{0, \ldots, j\}$ so $(n-k)m(S) + r \in \tilde{S}$ and

$$km(S) + (n-k)m(S) + r = nm(S) + r \notin \tilde{S},$$

so $km(S) \notin A(\tilde{S})$. Together, this gives us $km(S) \notin A(\tilde{S})\ \forall\ k \in \{1, 2, \ldots, n\}$ as required. □

Corollary 7.1.5. *Let S be a numerical semigroup. Then either $A(\tilde{S}) = \tilde{S}$ so S and $\tilde{S}$ are staircases or S is not a staircase and $A(\tilde{S})$ is a column.*

Observation 7.1.6. *Note, if $A(\tilde{S})$ is a column, S is not necessarily a numerical semigroup.*

Consider $S = \{0, 3, 5, 6, 7, 8, 9, 11 \rightarrow\}$, $\tilde{S} = \{0, 1, 2, 3, 4, 6, 9 \rightarrow\}$ and $A(\tilde{S}) = \{0, 9 \rightarrow\}$.

Definition 7.1.7. *Let S be a numerical set with $N(S) = (N_0, N_1, \ldots, N_k)$ and $M(S) = (M_0, M_1, \ldots, M_k)$. Then denote the* second base of S *as $B_2(S)$ with $B_2(S) = -1 + \sum_{i=0}^{k-2} N_i + \sum_{i=0}^{k-1} M_i$*

Note under the previous definition, $B_2(S)$ is just the largest small element of S that is less than the gaps that are in N_k and N_{k-1}. This is analogous to $B(S)$, which is the largest small element of S that is less than the gaps of N_k.

Theorem 7.1.8. *Let S be a numerical set with $N(S) = (N_0, \ldots, N_k)$ and $M(S) = (M_0, \ldots, M_k)$ and complement $\tilde{S}$. If $\tilde{S}$ is a numerical semigroup, then either $A(S) = S$ and is a truncated staircase, or one of $1, \ldots, N_k \in S$ which implies $A(S) = \{0, B(S) + N_k + 1 \rightarrow\}$, or $1, \ldots, N_k \notin S$ which gives us $A(S) = \{0, B(S), B(S) + N_k + 1 \rightarrow\}$.*

Proof. Suppose S is not a truncated staircase. We show small elements of S are not in $A(S)$ by showing they sum to gaps.

First consider elements of $\{s \in S \mid 1 \leq s \leq M_0 - 1\}$ (these are the elements in $\{M_0\}$). By definition, $M_0 \notin S$, so pairing them as $(1, M_0 - 1)$, $(2, M_0 - 2)$,..., $(k, M_0 - k)$ gives a partner in S that sums to a gap. Hence none of $\{s \in S \mid 1 \leq s \leq M_0 - 1\}$ are in $A(S)$.

Now consider elements of $\{p \in S \mid M_0 + N_0 \leq p \leq B_2(S)\}$(these are the elements in $\{M_1\}$ through $\{M_{k-1}\}$). First note carefully that since $N_{k-1} = m(\tilde{S}) - 1$, none of $\{B_2(S)+1, B_2(S)+2, \ldots, B_2(S)+m(\tilde{S})-1\}$ are in S. For $p \in \{M_1\}$ through $\{M_{k-1}\}$, we can write $p = qm(\tilde{S}) + r$ where $0 \leq r < m(\tilde{S})$. By 3.0.7, $B_2(S) - qm(\tilde{S}) \in S$ and $B_2(S) - qm(\tilde{S}) + p = B_2(S) + r \in \{B_2(S)+1, \ldots, B_2(S)+m(\tilde{S})-1\}$ provided $r \neq 0$. So $B_2(S) - qm(\tilde{S}) + p \notin S$ since $N_k = m(\tilde{S}) - 1$ and $p \notin A(S)$.

If $r = 0$, then $p = qm(\tilde{S})$ for some q. Let $nm(\tilde{S})$ be the maximal multiple of $m(\tilde{S})$ in S such that $nm(\tilde{S}) \leq B_2(S)$. By 3.0.7, we know that $m(\tilde{S}), 2m(\tilde{S}), \ldots, nm(\tilde{S}) \in S$

If $(n+1)m(\tilde{S}) \notin S$, then we can pair up $m(\tilde{S}), 2m(\tilde{S}), \ldots, nm(\tilde{S}) \in S$ so that the pairs sum up to $(n+1)m(\tilde{S})$ and so $p = qm(\tilde{S}) \notin A(S)$.

If instead $(n+1)m(\tilde{S}) \in S$, then either $(n+1)m(\tilde{S}) = B(S)$ or $(n+1)m(\tilde{S}) > F(S)$. However, $(n+1)m(\tilde{S}) > F(S)$ cannot be the case. To see why, recall that $N_{k-1} = m(\tilde{S}) - 1$. So then $nm(\tilde{S}) \leq B_2(S)$ gives us $(n+1)m(\tilde{S}) \leq B_2(S) + m(\tilde{S}) = B(S)$. Hence $(n+1)m(\tilde{S}) = B(S)$. From 3.0.7, we have all multiples of $m(\tilde{S})$ less than or equal to $B(S)$ are also in S.

Since S is not a truncated staircase, $\exists y \in S$ such that y is not a multiple of $m(\tilde{S})$ or $N_k \geq m(S)$.

If $\exists y \in S$ such that y is not a multiple of $m(\tilde{S})$. Choose the maximal one so that from the division algorithm, $y = jm(\tilde{S}) + r$ with $0 < r < m(\tilde{S}) - 1$, and $(j+1)m(\tilde{S}) + r \notin S$. This maximal y can always be found since y is not a multiple of $m(\tilde{S})$, we must have $y < B_2(S)$. Since all integers between $B_2(S)$ and $B(S)$ are gaps, then at least one is congruent to $y \mod m(\tilde{S})$.

Hence, for $k \in \{1, 2, \ldots, n-j\}$, then $km(\tilde{S}) + y = km(\tilde{S}) + jm(\tilde{S}) + r = (j+k)m(\tilde{S}) + r$. Since $j < j + k < n + 1$ and since we chose y to be maximal, then $(j+k)m(\tilde{S}) + r$ is a gap, and $km(\tilde{S}) \notin A(S)$. For $k \in \{n-j, n-j+1, \ldots, n\}$ then note that $0 \leq n - k \leq j$. So by 3.0.7, $(n-k)m(\tilde{S}) + r \in S$, and $km(\tilde{S}) + (n-k)m + r = nm(\tilde{S}) + r \notin S$. Hence again, $km(\tilde{S}) \notin A(S)$.

If instead $N_k \geq m(\tilde{S})$, then $B(S) + m(\tilde{S})$ must be a gap. For each k, $B(S) - (k-1)m(\tilde{S}) \in S$ by 3.0.7, so $km(\tilde{S}) + (B(S) - (k-1)m(\tilde{S})) = B(S) + m(\tilde{S}) \notin S$ and $km(\tilde{S}) \notin A(S)$.

Thus none of the multiples of $m(\tilde{S})$ in $A(S)$ with the exception of possibly $B(S)$.

Lastly, consider the elements of $\{M_k\}$. Since $\tilde{S}$ is a nontrivial semigroup, then we must have $M_k = 1$. Hence the only element of $\{M_k\}$ is $B(S)$. If $1, \ldots, N_k \notin S$ (this is if and only if $M_0 = 1$ and $N_0 \geq N_k$) then $0 + B(S) = B(S) \in S$ and $x + B(S) > B(S) + N_k$ for all nonzero $x \in S$ and so $B(S) \in A(S)$ in which case $A(S) = \{0, B(S), B(S) + N_k + 1 \rightarrow\}$. Otherwise, $B(S) \notin A(S)$ since we have some $y \in S$ so that $B(S) + 1 \leq B(S) + y \leq B(S) + N_k$ and thus $B(S) + y \notin S$, in which case $A(S) = \{0, B(S) + N_k + 1 \rightarrow\}$. Both cases have been observed to be possible. □

7.2 Neither S nor $\tilde{S}$ are Semigroups

With full characterizations of $A(\tilde{S})$ when S is a semigroup and A(S) when $\tilde{S}$ is a semigroup, we now attempt to generalize to when both S and $\tilde{S}$ are numerical sets.

We start by offering a sufficient, but not necessary condition that might be useful to determine that S and $\tilde{S}$ are not numerical semigroups. It is related to Theorem 7.1.1.

Theorem 7.2.1. *If $\tilde{S}$ fails its additive closure property at $F(\tilde{S})$, then neither S or $\tilde{S}$ are numerical semigroups.*

Proof. Clearly $\tilde{S}$ is not a numerical semigroup since it doesn't have additive closure.
Suppose $\exists a, b \in \tilde{S}$ such that $a + b = F(\tilde{S}) = B(S) - M_0$. Then we have $B(S) - a$ and $B(S) - b \in S$ and $(B(S) - a) + (B(S) - b) = 2B(S) - (a + b) = 2B(S) - (B(S) - M_0) = B(S) + M_0$. If $M_0 = 1$, then we if we had additive closure, then $B(S) + 1 \in S$, which is contradiction. If $M_0 > 1$, then we cannot have S a semigroup since $1 \in S$ and S is nontrivial (we know its nontrivial since $\tilde{S}$ has at least one gap, namely $F(\tilde{S})$. Therefore if two elements in $\tilde{S}$ sum to $F(\tilde{S})$, then S is not a semigroup. □

Lemma 7.2.2. *Let S be a numerical set with $N(S) = (N_0, \ldots, N_k)$ and $M(S) = (M_0, \ldots, M_k)$ and complement $\tilde{S}$ and Base $B(S)$. Then*

$$S = \{0, \ldots, M_0 - 1\} \cup ((\tilde{\tilde{S}} + M_0 + N_0) \backslash \{B(S) \rightarrow\}) \cup \{M_0 + N_0 + \cdots + M_k + N_k \rightarrow\}.$$

Proof. Note, $M(\tilde{\tilde{S}}) = (M_1, \ldots, M_{k-1})$ and $N(\tilde{\tilde{S}}) = (N_1, \ldots, N_{k-1})$. So

$$\begin{aligned}\tilde{\tilde{S}} = \{0, \ldots, M_1 - 1\} \cup \{M_1 + N_1, \ldots, M_1 + N_1 + M_2 - 1\} \cup \cdots \cup \\ \{M_1 + N_1 + \cdots + M_{k-2} + N_{k-2}, \ldots, M_1 + N_1 + \cdots + M_{k-2} + N_{k-2} + M_{k-1} - 1\} \cup \\ \{M_1 + N_1 + \cdots + M_{k-1} + N_{k-1} \rightarrow\}\end{aligned}$$

Then

$$\begin{aligned}\tilde{\tilde{S}}+M_0+N_0 = &\{M_0+N_0,\dots,M_0+N_0+M_1-1\}\cup\\ &\{M_0+N_0+M_1+N_1,\dots,M_0+N_0+M_1+N_1+M_2-1\}\cup\dots\cup\\ &\{M_0+N_0+M_1+N_1+\dots+M_{k-2}+N_{k-2},\dots,M_0+N_0+M_1+N_1+\dots+M_{k-2}+N_{k-2}+M_{k-1}-1\}\cup\\ &\{M_0+N_0+M_1+N_1+\dots+M_{k-1}+N_{k-1}\rightarrow\}\end{aligned}$$

We also have

$$\begin{aligned}S &= \{0,1,\dots,M_0-1\}\cup\{M_0+N_0,\dots,M_0+N_0+M_1-1\}\cup\ldots\cup\\ &\qquad\{M_0+N_0+\ldots+M_{k-1}+N_{k-1},\dots,M_0+N_0+\ldots+M_{k-1}+N_{k-1}+M_k-1\}\cup\\ &\qquad\{M_0+N_0+\ldots+M_k+N_k\rightarrow\}\\ &= \{0,1,\dots,M_0-1\}\cup((\tilde{\tilde{S}}+M_0+N_0)\backslash\{B(S)\rightarrow\})\cup\{M_0+N_0+\dots+M_k+N_k\rightarrow\}\end{aligned}$$

□

FAKE!!!

Conjecture 7.2.3. *Let S be a numerical set. If $A(S)$ is not a truncated staircase, then $A(\tilde{S})$ is a column.*

Conjecture 7.2.4. *Let S be a numerical set with complement $\tilde{S}$. If $A(\tilde{S})$ is not a truncated staircase, then $A(S)=\{0,B(S)-M_{k-1},B(S)-M_{k-1}+1,\dots,B(S)-j,F(S)+1\rightarrow\}$ for some $0\le j\le M_{k-1}$ or $A(S)=\{0,F(S)+1\rightarrow\}$.*

Conjecture 7.2.5. *If $A(S)$ is a truncated staircase, then $A(\tilde{S})$ is a staircase*

This still appears to still be true. No it does not: $S=\{0,3,11,14,22,25\rightarrow\}$

Conjecture 7.2.6. *If $A(\tilde{S})$ is a staircase, then $A(S)$ is a staircase.*

This conjecture is false, but is usually true (it has to do with how large N_k is). Consider, $S=\{0,6,7,13,14,20,21,23\rightarrow\}$ as a counterexample.

Theorem 7.2.7. *If S is a numerical set with $N(S)=(N_0,\dots,N_k)$ and $M(S)=(M_0,\dots,M_k)$ such that $N_i=N_1$ for all $1\le i<k$ and $N_0\le N_1$ and $M_i=M_1$ for all $0<i\le k$ and $A(S)$ is a column, then $A(\tilde{S})$ is a truncated staircase.*

Proof. Note, $N(\tilde{S})=(N_{k-1},\dots,N_0)=(N_1,\dots,N_1,N_0)$ and $M(\tilde{S})=(M_k,\dots,M_1)=(M_1,\dots,M_1)$. So,

$$\begin{aligned}\tilde{S}=&\{0,\dots,M_1-1\}\cup\{M_1+N_1,\dots,2M_1+N_1-1\}\cup\dots\cup\\ &\{(k-1)M_1+(k-1)N_1,\dots,kM_1+(k-1)N_1-1\}\cup\{kM_1+(k-1)N_1+N_0\rightarrow\}\end{aligned}$$

Note, every small element in $\tilde{S}$ is either of the form lM_1+lN_1+r or lM_1+lN_1 for some $0<r\le M_1-1$ and $0\le l\le k$. Also note, the gaps are of the form $(l+1)M_1+lN_1+g$ for $0\le g\le N_1-1$. Consider $lM_1+lN_1+r+(M_1-r\in\tilde{S})=(l+1)M_1+lN_1\notin\tilde{S}$. So $lM_1+lN_1+r\notin A(\tilde{S})$. Now consider lM_1+lN_1. Note, $(lM_1+lN_1)+(nM_1+nN_1+r)=(l+n)M_1+(l+n)N_1+r\in S$ for all $0\le r\le M_1-1$, so $lM_1+lN_1\in A(\tilde{S})$. Thus, $A(\tilde{S})=\{0,M_1+N_1,2M_1+2N_1,\dots,(k-1)M_1+(k-1)N_1,kM_1+(k-1)N_1+N_0\rightarrow\}$ is a staircase. □

Example 7.2.8. *Even when you know what type of semigroup $A(S)$ is, it is hard to predict what $A(\tilde{S})$ will be (column, normal staircase, pseudo-staircase, truncated staircase, none of the above).*

Consider, $A(S) = \{0, 17 \rightarrow\}$ *(a column).*

$S = \{0, 2, 5, 9, 11, 17 \rightarrow\}$	$S = \{0, 5, 9, 11, 17 \rightarrow\}$	$S = \{0, 1, 5, 9, 17 \rightarrow\}$
$\tilde{S} = \{0, 2, 6, 9, 11 \rightarrow\}$	$\tilde{S} = \{0, 2, 6, 11 \rightarrow\}$	$\tilde{S} = \{0, 4, 8, 9 \rightarrow\}$
$A(\tilde{S}) = \{0, 9, 11 \rightarrow\}$	$A(\tilde{S}) = \{0, 11 \rightarrow\}$	$A(\tilde{S}) = \{0, 4, 8, \rightarrow\}$
truncated staircase	*column*	*regular staircase*

$S = \{0, 3, 6, 12, 17 \rightarrow\}$	$S = \{0, 2, 5, 7, 10, 17 \rightarrow\}$
$\tilde{S} = \{0, 6, 9, 12 \rightarrow\}$	$\tilde{S} = \{0, 3, 5, 8, 10 \rightarrow\}$
$A(\tilde{S}) = \{0, 6, 9, 12 \rightarrow\}$	$A(\tilde{S}) = \{0, 5, 8, 10 \rightarrow\}$
pseudo-staircase	*none*

Now consider when $A(S) = \{0, 11, 22, 33, 37 \rightarrow\}$ *(a truncated staircase).*

$S = \{0, 2, 11, 13, 22, 24, 33, 35, 37 \rightarrow\}$	$S = \{0, 1, 11, 12, 22, 23, 28, 33, 34, 37 \rightarrow\}$	$S = \{0, 1, 6, 11, 12, 17, 22, 23, 28, 33, 34, 37 \rightarrow\}$
$\tilde{S} = \{0, 2, 11, 13, 22, 24, 33, 35 \rightarrow\}$	$\tilde{S} = \{0, 1, 6, 11, 12, 22, 23, 33, 34 \rightarrow\}$	$\tilde{S} = \{0, 1, 6, 11, 12, 17, 22, 23, 28, 33, 34 \rightarrow\}$
$A(\tilde{S}) = \{0, 11, 22, 33, 35 \rightarrow\}$	$A(\tilde{S}) = \{0, 33 \rightarrow\}$	$A(\tilde{S}) = \{0, 11, 22, 33 \rightarrow\}$
truncated staircase	*column*	*regular staircase*

$$S = \{0, 4, 11, 15, 22, 26, 33, 37 \rightarrow\}$$
$$\tilde{S} = \{0, 7, 11, 18, 22, 29, 33 \rightarrow\}$$
$$A(\tilde{S}) = \{0, 11, 22, 29, 33 \rightarrow\}$$
none

But even when $A(S) = \{0, 5, 10, 15, 20, 25 \rightarrow\}$ *(a "regular" staircase).*

$S = \{0, 2, 5, 7, 10, 12, 15, 17, 20, 22, 25 \rightarrow\}$	$S = \{0, 2, 5, 6, 7, 10, 11, 12, 15, 16, 17, 20, 21, 22, 25 \rightarrow\}$	$S = \{0, 1, 5, 6, 10, 11, 15, 16, 20, 21, 25 \rightarrow\}$
$\tilde{S} = \{0, 2, 5, 7, 10, 12, 15, 17, 20, 22 \rightarrow\}$	$\tilde{S} = \{0, 1, 2, 5, 6, 7, 10, 11, 12, 15, 16, 17, 20, 22 \rightarrow\}$	$\tilde{S} = \{0, 1, 5, 6, 10, 11, 15, 16, 20, 21 \rightarrow\}$
$A(\tilde{S}) = \{0, 5, 10, 15, 20, 22 \rightarrow\}$	$A(\tilde{S}) = \{0, 22 \rightarrow\}$	$A(\tilde{S}) = \{0, 5, 10, 15, 20 \rightarrow\}$
truncated staircase	*column*	*regular staircase*

When $A(S) = \{0, 22, 33, 44, 55, 59 \rightarrow\}$ *(pseudo-staircase):*

$S = \{0, 4, 22, 26, 33, 37, 44, 48, 55, 59 \rightarrow\}$	$S = \{0, 1, 22, 23, 33, 34, 44, 45, 55, 56, 59 \rightarrow\}$	$S = \{0, 4, 11, 22, 26, 33, 37, 44, 48, 55, 59 \rightarrow\}$
$\tilde{S} = \{0, 7, 11, 18, 22, 29, 33, 51, 55 \rightarrow\}$	$\tilde{S} = \{0, 1, 11, 12, 22, 23, 33, 34, 55, 56 \rightarrow\}$	$\tilde{S} = \{0, 7, 11, 18, 22, 29, 33, 44, 51, 55 \rightarrow\}$
$A(\tilde{S}) = \{0, 51, 55 \rightarrow\}$	$A(\tilde{S}) = \{0, 55 \rightarrow\}$	$A(\tilde{S}) = \{0, 44, 51, 55 \rightarrow\}$
truncated staircase	*column*	*none*

The previous example and other data suggests

Conjecture 7.2.9. *When $A(S)$ is not any kind of truncated staircase or pseudo-truncated staircase (when $A(S)$ falls into the none category in the example), then $A(\tilde{S})$ is a column.*

Conjecture 7.2.10. *When $A(S)$ is a column we are sad.*

Conjecture 7.2.11. *When $A(S)$ is a truncated staircase, $A(\tilde{S})$ can be anything except a pseudo-staircase.*

Conjecture 7.2.12. *When $A(S)$ is a regular staircase, then $A(\tilde{S})$ is some type of truncated staircase.*

Conjecture 7.2.13. *When $A(S)$ is a pseudo-truncated staircase, $A(\tilde{S})$ is a truncated staircase, a column, or none.*

I think the previous example can be explained in terms of column/row extension. Neither column extending or row extending S will change $\tilde{S}$, and therefore $A(\tilde{S})$ is invariant under column and row extension. Clearly $A(S)$ is not invariant, so my hypothesis is row extending or column extending too far will yield columns, "regular" staircases and truncated staircases.

Theorem 7.2.14. *Let S be a numerical set with Base B. Then $F(A(\tilde{S})) = F(\tilde{S}) = B - s$ where $s = \min(\mathbb{N}_0 \backslash S)$.*

Proof. Since $s \in \mathbb{N}_0 \backslash S$, by 3.0.2 $B - s \in \mathbb{N}_0 \backslash \tilde{S}$. Also, for all $0 \leq x < s$, $x \in S$ so again by 3.0.2 $B - x \in \tilde{S}$. Thus for all $y > B - s$, $y \in \tilde{S}$, so $B - s = F(\tilde{S})$.

For all $y > B - s$, $y + x \in \tilde{S}$ for all $x \in \tilde{S}$ since $y + x > B - s = F(\tilde{S})$, so $y \in A(\tilde{S})$. Also, $B - s \notin A(\tilde{S})$ since $B - s + 0 = B - s \notin \tilde{S}$. Thus, $F(A(\tilde{S})) = B - s$. □

Theorem 7.2.15. *Let S be a numerical set with Base B. Then*

$$H(A(\tilde{S})) \supseteq \mathbb{N}_0 \backslash (\tilde{S}) = \{B - g : g \in \mathbb{N}_0 \backslash S, g < B\}.$$

Proof. By 3.0.2, $\mathbb{N}_0 \backslash (\tilde{S}) = \{B - g : g \in \mathbb{N}_0 \backslash S, g < B\}$. Let $x \in \mathbb{N}_0 \backslash (\tilde{S})$. Then $x \notin A(\tilde{S})$ since $x + 0 = x \notin \tilde{S}$, so $x \in H(A(\tilde{S}))$. □

Corollary 7.2.16. *Let S be a numerical set with $N(S) = (N_0, \ldots, N_k)$. Then $g(A(\tilde{S})) \geq g(\tilde{S}) = g(S) - N_k$.*

Lemma 7.2.17. *Let T be a numerical set. Then $(\tilde{T})^* = (\tilde{T^*})$.*

Proof. Let $M(T) = (M_0, \ldots, M_k)$ and $N(T) = (N_0, \ldots, N_k)$. Then $M(\tilde{T}) = (M_1, \ldots, M_k)$ and $N(\tilde{T}) = (N_0, \ldots, N_{k-1})$. So $M((\tilde{T})^*) = (N_{k-1}, \ldots, N_0)$ and $M((\tilde{T})^*) = (M_k, \ldots, M_1)$. Also, $M(T^*) = (N_k, \ldots, N_0)$ and $N(T^*) = (M_k, \ldots, M_0)$. So $M((\tilde{T^*})) = (N_{k-1}, \ldots, N_0) = M((\tilde{T})^*)$ and $N((\tilde{T^*})) = (M_k, \ldots, M_1) = N((\tilde{T})^*)$. □

Theorem 7.2.18. *Let S be a numerical semigroup. The number of numerical sets T such that $A(T) = S$ is $P(S)$. If $P(S) = 2$, then either $A(\tilde{T})$ is a staircase or $A(\tilde{T})$ is a column.*

Proof. Since $P(S) = 2$ and $A(S^*) = A(S) = S$, either $T = S$ or $T = S^*$.

If S is a truncated staircase, then $T = S \implies \tilde{T}$ is a regular staircase by 6.0.11, so $A(\tilde{T}) = \tilde{T}$ is a regular staircase. Also, $T = S^* \implies \tilde{T} = (\tilde{S^*}) = (\tilde{S})^*$ by 7.2.17, so $A(\tilde{T}) = A((\tilde{S})^*) = A(\tilde{S})$ is a regular staircase.

If S is not a truncated staircase, then $T = S \implies A(\tilde{T})$ is a column by 7.1.4. Also, $T = S^* \implies \tilde{T} = (\tilde{S^*}) = (\tilde{S})^*$ by 7.2.17, so $A(\tilde{T}) = A((\tilde{S})^*) = A(\tilde{S})$ is a column. □

8 Properties of $M(S)$ and $N(S)$

The focus of this section is to further explore $M(S)$ and $N(S)$ and their relationships to other invariants of S. This section will also construct other sequences from the M and N sequences that are sometimes more useful and intuitive to use than the M and N sequences themselves.

Observation 8.0.1. *Let S be a numerical set, then $|M(S)| = |N(S)|$ (that is, the sequences are of the same length).*

To see this observation more clearly start building the set by including 0 so we first count M_0, then at the end of M_0 we encounter the first element of $\mathbb{N}_0$ not in S so next we count N_0 and we keep repeating this process until we get that everything after the last gap of S is in S.

Observation 8.0.2. *Let S be a numerical set with $N(S) = (N_0, \ldots, N_k)$, then the genus of S is $g(S) = \sum_{i=0}^{k} N_i$.*

By referencing definition 6.0.1 one can see that $N(S)$ partitions $\mathbb{N}_0 \backslash S$ based on the lengths of consecutive gaps of the set. So the sum of the N_i's is the same size as $\mathbb{N}_0 \backslash S$. The size of $\mathbb{N}_0 \backslash S$ is the genus of the set. Therefore $g(s) = \sum_{i=0}^{k} N_i$.

Observation 8.0.3. *Let S be a numerical set with $M(S) = (M_0, \ldots, M_k)$, then the number of small elements of S is $\sum_{i=0}^{k} M_i$.*

This observation comes from the fact that the number of small elements form a sequence of finite length. This construction partitions this sequence into smaller strings (these strings are the J_i described in definition 6.0.2), where M_i the length of the i^{th} string, but of course these would sum up to the length of the original sequence.

Theorem 8.0.4. *Let S be a numerical set with $N(S) = (N_0, \ldots, N_k)$ and $M(S) = (M_0, \ldots, M_k)$, then $F(S) = -1 + \sum_{i=0}^{k}(M_i + N_i)$.*

Proof. Consider the set $\{n \in \mathbb{N}_0 \mid n \leq F(S)\}$. From this definition it is clear it contains F(S)+1 elements. We can partition these elements into two sets, those that are in S and those that are not. However, our observations tell us these quantities are given exactly by $\sum_{i=0}^{k} M_i$ and $\sum_{i=0}^{k} N_i$ respectively. This gives us the desired equality. $\square$

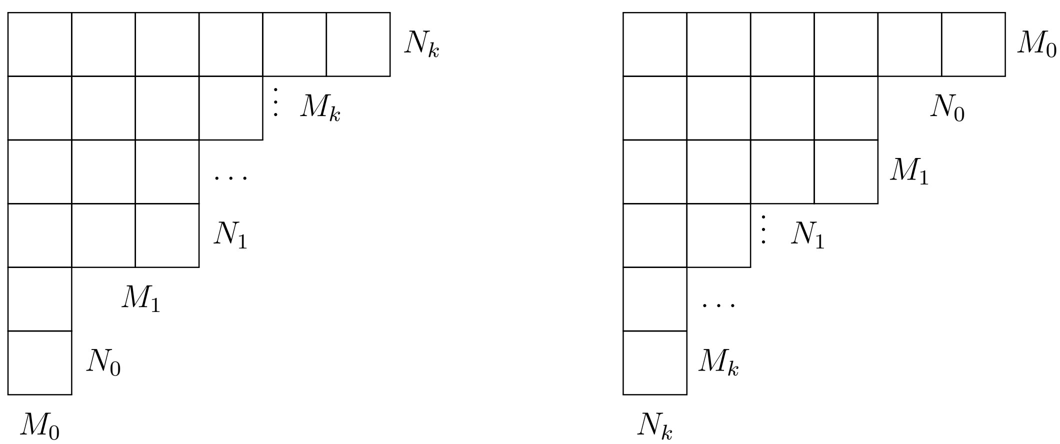
Observation 8.0.5. *If* $N(S) = (N_0, N_1, \ldots, N_{k-1}, N_k)$, *then* $N(\tilde{S}) = (N_{k-1}, \ldots, N_1, N_0)$

Observation 8.0.6. *If* $M(S) = (M_0, M_1, \ldots, M_{k-1}, M_k)$, *then* $M(\tilde{S}) = (M_k, M_{k-1}, \ldots, M_1)$

As aforementioned, the step height and widths transform nicely under complementation. They also do so under conjugation.

Theorem 8.0.7. *Let S be numerical set. Then* $N(S) = (N_0, N_1, \ldots, N_{k-1}, N_k)$ *and* $M(S) = (M_0, M_1, \ldots, M_{k-1}, M_k)$ *if and only if* $M(S^*) = (N_k, N_{k-1}, \ldots, N_0)$ *and* $N(S^*) = (M_k, M_{k-1}, \ldots, M_0)$.

Proof. Below is a proof by picture.



(a) $\lambda(S)$ (b) $\lambda(S^*)$

When we conjugate the partition, all of our rows become columns and vice versa, so our M's and N's will switch places, and then become reversed.

$\square$

Corollary 8.0.8. *Let S be a numerical semigroup with* $N(S) = (N_0, N_1, \ldots, N_k)$, *and* $M(S) = (M_0, M_1, \ldots, M_k)$. *Then* $M_i = N_{k-i}$ *for* $0 \leq i \leq k$ *if and only if S is symmetric.*

Proof. It is well known that a semigroup is symmetric if and only if it is self-conjugate. By Theorem 8.0.7, S is self-conjugate if and only if $M(S^*) = (N_k, N_{k-1}, \ldots, N_0) = M(S)$, so $(N_k, N_{k-1}, \ldots, N_0) = (M_0, M_1, \ldots, M_k)$ and thus $M_i = N_{k-i}$ for all $i \in \{0, 1, \ldots, k\}$. $\square$

Theorem 8.0.9. *Let S be a numerical set with $N(S) = (N_1, \ldots, N_k)$ and $M(S) = (M_1, \ldots, M_k)$. Then the hook length in the i^{th} row and j^{th} column of $\lambda(S)$ is*

$$h_{i,j} = M_1 + M_2 + \cdots + M_{m-1} + N_{n+1} + N_{n+2} + \cdots + N_k + 1 - i - j$$

where m and n are retrieved by $N_k + \cdots + N_m < j$ where $m \in \{1, 2, \ldots, k+1\}$ is the minimum such number that satisfies the condition and $M_1 + M_2 + \cdots + M_n < i$ where $n \in \{0, 1, \ldots, k\}$ is the maximum such number that satisfies the condition. By convention say $M_0 = N_{k+1} = 0$.

Proof. From [3], we have that $h_{i,j} = r_i - i + c_j - j + 1$. Under our notation, $r_i = \sum_{q=1}^{m-1} M_q$ where m is defined as above. Similarly, $c_j = \sum_{q=k}^{n+1} N_q$. Thus we have $h_{i,j} = \sum_{q=1}^{m-1} M_q + \sum_{q=n+1}^{k} N_q - i - j + 1$ $\square$

Proposition 8.0.10. *Suppose T is a numerical set with Frobenius number F. Then T^* is given by*

$$\{F - u : u \in \mathbb{Z} \backslash T\} = \{F - u : u \in \mathbb{N}_0 \backslash T\} \bigcup \{F + 1 \rightarrow\}.$$

Proposition 8.0.11. *Suppose T is a numerical set with $\lambda(T) = \lambda$, and the numerical set associated with λ^* is T^*. Then $A(T) = A(T^*)$.*

Proof. By Proposition 8.0.10, $T^* = \{F - u : u \in \mathbb{Z} \backslash T\} = \{F - u : u \in \mathbb{N}_0 \backslash T\} \bigcup \{F + 1 \rightarrow\}$, where $F = \max \mathbb{N}_0 \backslash T$. Note that since λ and λ^* are partitions of the same size, then $\max \mathbb{N}_0 \backslash T = \max \mathbb{N}_0 \backslash T^*$. This can also be seen by observing that $0 \in T$, so $F - 0 \notin \{F - u : u \in \mathbb{Z} \backslash T\}$, and $F < F + 1$, so $F \notin T^*$, and clearly it is the largest such integer. We first show that $A(T) \subseteq A(T^*)$, then since the conjugate of the conjugate of T is T, then it immediately follows that $A(T^*) \subseteq A(T)$, hence $A(T) = A(T^*)$. Let $n \in A(T)$. Suppose towards a contradiction that $F - n \in T$. Then $n + (F - n) = F \in T$, a contradiction. Thus $F - n \notin T$, and so $F - (F - n) = n \in T^*$. We now show that $\forall m \in T^*, n + m \in T^*$. Suppose towards a contradiction that $\exists m \in T^*$ such that $n + m \notin T^*$. Certainly $m \neq F$, and if $m > F$, then $n + m \geq F + 1$, so $n + m \in \{F + 1 \rightarrow\} \subset T^*$, a contradiction. Thus $m < F$, and so $0 < F - m < F + 1$. Now since $n + m \notin T^*$, then $F - (n + m) \in T$. Since $n \in A(T)$, then $n + F - (n + m) = F - m \in T = \{F - u : u \in \mathbb{N}_0 \backslash T^*\} \bigcup \{F + 1 \rightarrow\}$. But $F - m < F + 1$, so $F - m \in \{F - u : u \in \mathbb{N}_0 \backslash T'\}$. Hence $\exists u \in \mathbb{N}_0 \backslash T^*$ such that $F - m = F - u \iff m = u$. Thus $m \in \mathbb{N}_0 \backslash T^*$, i.e. $m \notin T^*$, a contradiction. Hence $\forall m \in T^*, n + m \in T^*$, i.e. $n \in A(T^*)$. Thus $A(T) \subseteq A(T^*)$, and similarly $A(T^*) \subseteq A(T)$. Hence $A(T) = A(T^*)$. $\square$

Proposition 8.0.12. *Let T be a numerical set with $N(T) = (N_0, \ldots, N_k)$ and $M(T) = (M_0, \ldots, M_k)$. Then $N(A(T)) = (M_k, \ldots, M_0)$ and $M(A(T)) = (N_k, \ldots, N_0)$ if and only if T^* is a semigroup.*

Proof. If T^* is a semigroup, the result follows immediately from 8.0.11 and 8.0.7.

If instead we have $N(A(T)) = (M_k, \ldots, M_0)$ and $M(A(T)) = (N_k, \ldots, N_0)$, then by 8.0.7 we have $N(T^*) = N(A(T))$ and $M(T^*) = M(A(T))$, so $A(T) = T^*$. By 8.0.11, $T^* = A(T) = A(T^*)$ so T^* is a numerical semigroup. $\square$

Theorem 8.0.13. *If T is a nontrivial numerical set with $N(T) = (N_0, \ldots, N_k)$ and $M(T) = (M_0, \ldots, M_k)$ and $N(A(T)) = (M_k, \ldots, M_0)$ and $M(A(T)) = (N_k, \ldots, N_0)$. Then $N_k = 1$.*

Proof. By the previous proposition, then we know T^* is a semigroup. Again, by proposition 8.0.11, then $A(T) = T^*$. Since $M(A(T)) = (N_k, \ldots, N_0)$ is of a semigroup, then N_k must be 1. $\square$

Conjecture 8.0.14. $N_i(T) \leq N_0(A(T))$

Proof. Not sure how to do this yet $\square$

Theorem 8.0.15. *Let S be a numerical set with $N(S) = (m-1, m-1, \ldots, m-1, 1, 1, \ldots, 1)$ where there are l ones and k $m-1$'s in the $N(S)$ sequence and $M(S) = (1, \ldots, 1)$. Then S is a semigroup if and only if $m \neq 1, 3, 5, \ldots, 2l-1$.*

Proof. We can write

$$\begin{aligned} S = &\{0, 1, \ldots, M_1 - 1\} \cup \{M_1 + N_1, \ldots, M_1 + N_1 + M_2 - 1\} \cup \ldots \cup \\ &\{M_1 + N_1 + \ldots + M_{k-1} + N_{k-1}, \ldots, M_1 + N_1 + \ldots + M_{k-1} + N_{k-1} + M_k - 1\} \cup \\ &\{M_1 + N_1 + \ldots + M_k + N_k \rightarrow\} \\ = &\{0, m, 2m, \ldots, km, km+2, km+4, \ldots, km + 2(l-1), km + 2l \rightarrow\} \end{aligned}$$

If S is a semigroup, then S must contain all the multiples of m, so $km+1, km+3, \ldots, km+2l-1$ cannot be multiples of m and thus, $m \neq 1, 3, \ldots, 2l-1$.

If $m \neq 1, 3, \ldots, 2l-1$, then either $m = 2n$ for some $n \in \{1, \ldots, l-1\}$ or $m \geq 2l$.

Suppose $m = 2n$ for some appropriate m, then $S = \{2n, 4n, 6n, \ldots, 2kn, 2kn+2, 2kn+4, \ldots, 2kn+2l \rightarrow\}$. Let $x, y \in S$ if at least one of x or y is greater than or equal to $2kn+2l$ then $x+y \geq 2kn+2l$ so $x+y \in S$. If both x and y are strictly less than $2kn+2l$, then both x and y are even. If at least one of x or y is greater than or equal to $2kn$, then $x+y \geq 2kn$ and $x+y$ is even so $x+y \in S$. If both x and y are strictly less than $2kn$, then $x = c_1 m$ and $y = c_2 m$ for some $c_1, c_2 \in \{0, \ldots, k-1\}$ so $x+y = (c_1+c_2)m$ is a multiple of m and thus $x+y \in S$. Therefore S is closed under addition and thus is a numerical semigroup.

Suppose $m \geq 2l$ and let $x, y \in S$. If $x \geq km+2l$ then $x+y \geq km+2l$ so $x+y \in S$. If $x, y < km+2l$, then $x, y \in \{0, m, 2m, \ldots, km, km+2, km+4, \ldots, km+2(l-1)\}$. If $x, y > km$, then $x+y > km+km \geq km+2l$ since $m \geq 2l$ and $k \in \mathbb{N}$ so $km \geq 2l$ thus $x+y \in S$. If $x > km$ and $0 < y \leq km$ then $y = cm$ for some $c \in \{1, 2, \ldots, k\}$ and $x = km+2q$ for some $q \in \{1, \ldots, l-1\}$, so $x+y = km+2q+cm \geq km+2q+2cl = km+2(q+cl) \geq km+2l$ since $q+cl \geq l$ since $c > 0$ and thus $x+y \in S$. If $x, y \leq km$, then $x = c_1 m$ and $y = c_2 m$ for some $c_1, c_2 \in \{0, 1, \ldots, k\}$ so $x+y = (c_1+c_2)m \in S$. If $y = 0$, then $x+y = x \in S$. So in all cases, S is closed under addition and thus S is a numerical semigroup. $\square$

Theorem 8.0.16. *Let S be a numerical semigroup. Then $N = (k, k-1, k-2, \ldots, 1)$ and $M = (1, 1, 2, 3, 4, \ldots, k-2, k-1)$ if and only if $S = \langle k+1, k+1+k, k+1+2k, \ldots, k+1+(k-1)k, k+1+(k)k\rangle$. This is an arithmetic semigroup where $a = k+1$ and $d = k$ and has maximum embedding dimension.*

Proof. Let $S' = \langle k+1, k+1+k, k+1+2k, \ldots, k+1+(k-1)k, k+1+(k)k\rangle = \langle k+1, 2k+1, 3k+1, \ldots, (k)k+1, (k+1)k+1\rangle$ and suppose S is a numerical semigroup with $N(S) = (k, k-1, k-$

$2, \ldots, 1)$ and $M(S) = (1, 1, 2, 3, 4, \ldots, k-2, k-1)$. We have

$$\begin{aligned}
S = &\{0, 1, \ldots, M_0 - 1\} \cup \{M_0 + N_0, \ldots, M_0 + N_0 + M_1 - 1\} \cup \ldots \cup \\
&\quad \{M_0 + N_0 + \ldots + M_{k-1} + N_{k-1}, \ldots, M_0 + N_0 + \ldots + M_{k-1} + N_{k-1} + M_k - 1\} \cup \\
&\quad \{M_0 + N_0 + \ldots + M_k + N_k \rightarrow\} \\
= &\{0\} \cup \{k+1\} \cup \{1 + k + 1 + k - 1, 1 + k + 1 + k - 1 + 1\} \cup \\
&\{1 + k + 1 + k - 1 + 2 + k - 2, 1 + k + 1 + k - 1 + 2 + k - 2 + 1, 1 + k + 1 + k - 1 + 2 + k - 2 + 2\} \\
&\cup \cdots \cup \{1 + k + 1 + k - 1 + \cdots + (i-1) + (k - i + 1), \ldots, \\
&1 + k + 1 + k - 1 + \cdots + (i-1) + (k - i + 1) + i - 1\} \\
&\cup \cdots \cup \{1 + k + 1 + k - 1 + \cdots + k - 2 + 2, \ldots, 1 + k + 1 + k - 1 + \cdots + k - 2 + 2 + k - 1 - 1\} \cup \\
&\{1 + k + 1 + k - 1 + 2 + k - 2 + \cdots + k - 2 + 2 + k - 1 + 1 \rightarrow\} \\
= &\{0\} \cup \{k+1\} \cup \{1 + k + 1 + k - 1, 1 + k + 1 + k - 1 + 1\} \cup \\
&\{1 + k + 1 + k - 1 + 2 + k - 2, 1 + k + 1 + k - 1 + 2 + k - 2 + 1, 1 + k + 1 + k - 1 + 2 + k - 2 + 2\} \\
&\cup \cdots \cup \{1 + k + (i-1)k, \ldots, 1 + k + (i-1)k + i - 1\} \cup \cdots \cup \\
&\{1 + k + (k-2)k, \ldots, 1 + k + (k-2)k + k - 1 - 1\} \cup \{1 + k + (k-1)k \rightarrow\}
\end{aligned}$$

Let $x \in S \backslash \{0\}$. If $x < 1 + k + (k-1)k = 1 + k^2$, then $x \in \{1 + k + (i-1)k, \ldots, 1 + k + (i-1)k + i - 1\}$ for some $i \in \{1, \ldots, k-1\}$. So $x = (k+1) + k(i-1) + j$ for some $j \in \{0, \ldots, i-1\}$. Then

$$x = (i)(k) + (j+1)(1) = (j)(k+1) + ((i-j)k + 1) \in S'$$

since $1 \leq i - j \leq k - 1$ because $0 \leq j \leq i - 1$ and $1 \leq i \leq k - 1$. If instead, $x \geq 1 + k^2$, then it suffices to show that $1 + k^2, 1 + k^2 + 1, \ldots, 1 + k^2 + k \in S'$. Note,

$$\begin{aligned}
1 + k^2 &= (k)k + 1 \in S' \\
1 + k^2 + 1 &= (k)k + (2)1 = (k+1) + ((k-1)k + 1) \in S' \\
&\vdots \\
1 + k^2 + k &= (k)k + (k)1 = (k-1)(k+1) + ((k - (k-1))k + 1) = (k-1)(k+1) + (k+1) \in S'
\end{aligned}$$

so for all $x \geq 1 + k^2$, $x \in S'$. Thus, $S \subseteq S'$.

We can prove that S is in fact a numerical semigroup. Let $x, y \in S$. If one of x or y is larger than or equal to $k^2 + 1$, then $x + y \geq k^2 + 1$ so $x + y \in S$. If instead $x, y < k^2 + 1$, then $x \in \{1 + k + (i-1)k, \ldots, 1 + k + (i-1)k + i - 1\}$ and $y \in \{1 + k + (j-1)k, \ldots, 1 + k + (j-1)k + j - 1\}$ for some $i, j \in \{1, \ldots, k-1\}$, so $x = 1 + ik + a$ and $y = 1 + jk + b$ for some $a \in \{0, \ldots, i-1\}$ and $b \in \{0, \ldots, j-1\}$. Then $x + y = 1 + ik + a + 1 + jk + b = 1 + (i+j)k + (a + b + 1)$ where $a + b + 1 \in \{1, \ldots, i + j - 1\}$ so $x + y \in \{1 + (i+j)k, \ldots, 1 + (i+j)k + (i + j - 1)\} \subseteq S$. Thus S is closed under addition and is a numerical semigroup.

Now let x be an arbitrary element of S'. Then $x = \sum_{i=1}^{k+1} c_i(ik + 1)$ for some $c_i \in \mathbb{N}_0$. Then since $ik + 1 \in S$ for all $i \in \{1, \ldots, k+1\}$ as can be seen above, $x \in S$ since S is a semigroup and is closed under addition. Thus $S = S'$. $\square$

Lemma 8.0.17. *For $S = \langle k, k+1 \rangle$, consecutive small elements, can be generated by finding all $p, q \in \mathbb{N}$ that satisfy $p + q = n$ and $x = p(k) + q(k+1)$ for all $x \in S$ and $n \in \mathbb{N}$. The amount of solutions can be counted with the formula $\binom{n+2-1}{n}$.*

Proof. Proven by induction on n. Consider the base case, such as $n = 0$ and $n = 1$. For the case where $n = 0$, we have to find how many solutions there are to the equation $p + q = 0$, and there is only one solution, when $p = 0$ and $q = 0$. This solution corresponds to the element $x = 0(k) + 0(k + 1) = 0 \in S$. Note, the number of solutions could also have been found with the formula $\binom{n+2-1}{n} = \binom{(0)+2-1}{(0)} = \binom{1}{0} = 1$. Now consider the case where $n = 1$. In this case we need to find how many solutions there are to $p + q = 1$. In this case we only have 2 options, $p = 1$ and $q = 0$ or $p = 0$ and $q = 1$. The first one corresponds to the elements $1(k) + 0(k + 1) = k$ and $0(k) + 1(k + 1) = k + 1$, which are clearly consecutive small elements of S. Note, there are 2 solutions, which can also be obtained from the formula: $\binom{n+2-1}{n} = \binom{(1)+2-1}{(1)} = \binom{2}{1} = 2$.

Now suppose this pattern holds for all n up to $n = h - 1$. Consider the $n = h$ case. We want to find the solutions to the equation $p + q = h$. For this case we have the following solutions: $p = h \;\&\; q = 0, p = h - 1 \;\&\; q = 1, p = h - 2 \;\&\; q = 2, \ldots, p = 2 \;\&\; q = h - 2, p = 1 \;\&\; q = h - 1, p = 0 \;\&\; q = h$. These correspond to the elements

$$
\begin{aligned}
h(k) + 0(k + 1) &= hk \\
h - 1(k) + 1(k + 1) &= hk - k + k + 1 = hk + 1 \\
h - 2(k) + 2(k + 1) &= hk - 2k + 2k + 2 = hk + 2 \\
&\vdots \\
2(k) + (h - 2)(k + 1) &= 2k + hk + h - 2k - 2 = hk + (h - 2) \\
1(k) + (h - 1)(k + 1) &= k + hk - k + h - 1 = hk + (h - 1) \\
0(k) + h(k + 1) &= hk + h.
\end{aligned}
$$

These are the distinct consecutive elements in S beginning with hk and going to $hk + h$, so there are h consecutive elements. We know these elements are distinct because suppose there were a p and q and p' and q' such that $p \neq p'$, $q \neq q'$, $p + q = p' + q'$, and $pk + q(k + 1) = p'k + q'(k + 1)$. Then $(p + q)k + q = (p' + q')k + q' = (p + q)k + q'$, so $q = q'$ which is a contradiction. With the formula we get $\binom{h+2-1}{h} = \binom{h+1}{h} = h$.
Hence, for all $n \in \mathbb{N}$, all small elements in the semigroup S of the form $S = \langle k, k + 1 \rangle$ will be generated by finding solutions to $p + q = n$ and all of the possible solutions are counted by the formula $\binom{n+2-1}{n}$. □

Theorem 8.0.18. *$N(S) = (k, k - 1, \ldots, 2, 1)$ and $M(S) = (1, 2, \ldots, k - 1, k)$ for some $k \in \mathbb{N}\backslash\{0\}$ if and only if S is symmetric and $S = \langle k + 1, k + 2 \rangle$.*

Proof. Suppose S is a numerical semigroup such that $N(S) = (k, k - 1, \ldots, 2, 1)$ and $M(S) =$

$(1, 2, \ldots, k-1, k)$. By 8.0.8, we know that S has to be a symmetric semigroup. We have

$$\begin{aligned}
S &= \{0, 1, \ldots, M_0 - 1\} \cup \{M_0 + N_0, \ldots, M_0 + N_0 + M_1 - 1\} \cup \ldots \cup \\
&\quad \{M_0 + N_0 + \ldots + M_{k-1} + N_{k-1}, \ldots, M_0 + N_0 + \ldots + M_{k-1} + N_{k-1} + M_k - 1\} \cup \\
&\quad \{M_0 + N_0 + \ldots + M_k + N_k \rightarrow\} \\
&= \{0\} \cup \{(1+k), (2+k)\} \cup \{[(1+k) + (k-1+2)], [(1+k) + (k-1+2) + 1)], \\
&\quad [(1+k) + (k-1+2) + 2)]\} \cup \cdots \cup \{[(1+k) + (2+k-1) + (3+k-2) + \ldots \\
&\quad + (k-1+2)] + \cdots + [(1+k) + (2+k-1) + (3+k-2) + \cdots + (k-1+2) + \\
&\quad (k-2)]\} \cup \{[(1+k) + (2+k-1) + (3+k-2) + \cdots + (k-1+2), (k+1)], \rightarrow\} \\
&= \{0\} \cup \{(k+1), (k+2)\} \cup \{(2k+2), (2k+3), (2k+4)\} \cup \{(3k+3), (3k+4), \\
&\quad (3k+5), (3k+6)\} \cup \cdots \cup \{[(k-1)(k+1)], [(k-1)(k+1) + 1], \ldots, \\
&\quad [(k-1)(k+1) + (k-1)]\} \cup \{(k)(k+1), \rightarrow\} \\
&= \{0\} \cup \{(k+1), (k+2)\} \cup \{2(k+1), (k+1) + (k+2), 2(k+2)\} \cup \{3(k+1), \\
&\quad 2(k+1) + (k+2), (k+1) + 2(k+2), 3(k+2)\} \cup \cdots \cup \{(k-1)(k+1), (k-2)(k+1) \\
&\quad + (k+2), (k-3)(k+1) + 2(k+2), \ldots, 2(k+1) + (k-3)(k+2), (k+1) + \\
&\quad (k-2)(k+2), (k-1)(k+2)\} \cup \{(k)(k+1), \rightarrow\}
\end{aligned}$$

Observe, the only unique elements of the set are $k+1$ and $k+2$, the generators, and every other element in the set is clearly a linear combination of $k+1$ and $k+2$. Therefore we know $S \subseteq \langle k+1, k+2 \rangle$.

Now suppose that S is a symmetric numerical semigroup such that $S = \langle k+1, k+2 \rangle$. By 8.0.17, we know that all of the consecutive small elements are generated by finding all $p, q \in \mathbb{N}$ that satisfy $p+q = n$ and $x = p(k) + q(k+1)$ for all $x \in S$ and $n \in \mathbb{N}$. This lemma also tells us that these small elements are counted by the formula $\binom{n+2-1}{n}$. M(S) is a sequence that keeps track of the consecutive small elements of S, which means, in this case, we have a formula for what any given M_n is. $M(S) = (\binom{0+2-1}{0}, \binom{1+2-1}{1}, \binom{2+2-1}{2}, \ldots, \binom{(k-3)+2-1}{k-3}, \binom{(k-2)+2-1}{k-2}, \binom{(k-1)+2-1}{k-1}, \binom{(k)+2-1}{k}) = (1, 2, \ldots, k-2, k-1, k)$. So now that we have our $M(S)$, by 8.0.8, we know that $N(S) = (k, k-1, k-2, \ldots, 2, 1)$ because S is a symmetric semigroup.
Hence $N(S) = (k, k-1, \ldots, 2, 1)$ and $M(S) = (1, 2, \ldots, k-1, k)$ for some $k \in \mathbb{N} \backslash \{0\}$ if and only if S is symmetric and $S = \langle k+1, k+2 \rangle$. ☐

Theorem 8.0.19. *$\forall$ M sequences beginning with $M_0 = 1$, $\exists$ an N sequence such that M and N correspond to a semigroup.*

Proof. Suppose $M(S) = (1, M_1, \ldots, M_k)$. Then construct $N(S)$ with $N_0 = (k-2) + \sum_{i=0}^{k} M_i$, and let $N_i = 1$ identically afterward. Since $M_i \geq 1 \forall i$, then $N_0 = (k-2) + \sum_{i=0}^{k} M_i \geq k - 2 + (k+1) = 2k - 1 \geq 0$ since $k \geq 1$. We show that the numerical set S with these M and N sequences is a semigroup.

From Theorem 8.0.4, we know

$$F(S) + 1 = \sum_{i=0}^{k} (M_i + N_i) = \sum_{i=0}^{k} (N_i) + N_0 + 2 - k = 2N_0 + 2 + k - k = 2(N_0 + 1)$$

. In other words, the conductor is twice the multiplicity. $M_0 = 1$ shows that $0 \in S$ and defining a numerical set with finite M and N sequences always gives cofiniteness. All that remains is additive closure. Let $x, y \in S$. Since $x, y \geq m(S)$, then $x + y \geq 2m(S) = F(S) + 1$. This gives us $x + y \in S$ and thus S is a semigroup. □

The previous theorem has no analogue for gap sequences since it is possible find N sequences with no corresponding M sequences to make it a semigroup. In a sense, the M sequences determine possible N sequences for semigroups, and not vice-versa.

8.1 Ideal M's

Definition 8.1.1. *We call $M(S)$* ideal *if it has that $M_k + M_j - 1 = M_{j+k}$ for j,k not simultaneously 0.*

This definition is useful because it pertains to some of the semigroups we have previously worked with.

Theorem 8.1.2. *If S is a truncated n-staircase, them M(S) is ideal.*

Proof. Recall from Theorem 6.0.10 that if S and $\tilde{S}$ are numerical semigroups, then $M(S) = (1, 1, \ldots, 1)$. It is easy to see that $M(S)$ is ideal. We also proved in 6.0.11 that S must be a truncated n-staircase, and therefore the statement follows. □

This was simple in the case of n-staircases, but to generalize, we will need more tools, as seen in the following proposition.

Proposition 8.1.3. *The following statements are equivalent.*
1. If $M_0 = 1$ and $M_1 = l$, then $M_{j+1} = M_j + M_1 - 1$ (recursive definition of ideal)
2. $M_n = n(l - 1) + 1$ (closed form)
3. $M_0 = 1$, $M_1 = l$, and $M_{j+k} = M_j + M_k - 1$. (generalized)

Proof. The proof will be split into three parts.
$1 \Rightarrow 2$

Given $M_0 = 1$ and $M_1 = l$, then note the following equivalences:

$$\begin{aligned} M_n &= M_{n-1} + M_1 - 1 \\ &= M_{n-1} + l - 1 \\ &= M_{n-2} + 2(l-1) \\ &= M_{n-3} + 3(l-1) \\ &\;\;\vdots \\ &= M_{n-(n-1)} + (n-1)(l-1) \\ &= l + nl + 1 - l - n \\ &= n(l-1) + 1 \end{aligned}$$

Thus statement 1 implies statement 2 of the proposition.

$2 \Rightarrow 3$
Given $M_n = n(l-1)+1$, note that $M_0 = 1$ and $M_1 = l$. Also,

$$\begin{aligned} M_{j+k} &= (j+k)(l-1) + 1 \\ &= j(l-1) + 1 + k(l-1) \\ &= j(l-1) + 1 + k(l-1) + 1 - 1 \\ &= M_j + M_k - 1 \end{aligned}$$

Thus statement 2 implies statement 3 of the proposition.

$3 \Rightarrow 1$ Given statement 3, it is simply let k=1, and statement 1 is obtained.

Thus all three statements are equivalent. □

The power of Proposition 8.1.3 is that to prove a numerical set is ideal we only need prove either the recursive definition or the closed form.

Theorem 8.1.4. *If S is a Arithmetic semigroup $\langle a, ha+d, ha+2d, \ldots, ha+kd\rangle$ with $d = h = 1$, then M(S) is ideal.*

Proof. To prove this result, we first prove the following Lemma.

Lemma 8.1.5. *If S is an Arithmetic semigroup with d=h=1, then let $\{g_0, g_1, \ldots, g_{k-1}\}$ be the set of generators. Then $\forall n \in \mathbb{N}$, $\{m \in \mathbb{N} \mid ng_0 \leq m \leq ng_{k-1}\} \subset S$.*

Proof. Proven with induction on n. Clearly the case when n=1 is true, but to understand the mechanism of the proof, we look at when n=2.

From $\{g_0, g_1, \ldots, g_{k-1}\}$, we have all multiples of g_i, but in particular, we have the smallest and largest multiples $2g_0$ and $2g_{k-1}$. To see how we can obtain every number in between, let the ordered pair (g_i, g_j) where $g_i, g_j \in \{g_0, g_1, \ldots, g_{k-1}\}$ correspond to g_i+g_j. Perform the following progression

$$2g_0 = (g_0, g_0) \to (g_1, g_0) \to (g_2, g_0) \to \ldots \to (g_{k-1}, g_0) \to (g_{k-1}, g_1) \to (g_{k-1}, g_2) \to \ldots \to (g_{k-1}, g_{k-1}) = 2g_{k-1}$$

Note that since the generators are in an arithmetic progression with common difference of 1, then so are these ordered pairs, and thus $\{2g_0, 2g_0 + 1, \ldots, 2g_{k-1}\} \subset S$.

Now for an inductive step. Suppose for $n \in \mathbb{N}$ that $\{m \in \mathbb{N} \mid ng_0 \leq m \leq ng_{k-1}\} \subset S$. In particular, note that ng_i and $ng_{i+1} \in S$ for $0 \leq i < k-1$, as well as every number in between. Hence, all but the last step of the following progression are justified by our inductive hypothesis.

$$(n+1)g_i = (ng_i, g_i) \to (ng_i + 1, g_i) \to (ng_i + 2, g_i) \to \ldots \to (ng_{i+1}, g_i) \to (ng_{i+1}, g_{i+1}) = (n+1)g_{i+1}$$

This proves that when $(n+1)g_i$ and $(n+1)g_{i+1}$ are in S, so is every number in between. Applying this for $0 \leq i < k-1$ completes the argument that $\{m \in \mathbb{N} \mid (n+1)g_0 \leq m \leq (n+1)g_{k-1}\} \subset S$, and mathematical induction proves the Lemma. □

Note that in our Lemma, since $g_{k-1} = g_0 + (k-1)$, then there are $n(k-1)+1$ integers in the string ng_0, ng_0+1,..., ng_{k-1}. This is statement 2 of Proposition 8.1.3, so all that remains to show is these strings correspond to M_n.

If $m \in S$, $ng_0 \leq m \leq ng_{k-1}$ for some $n \in \mathbb{N}$. If m is an atom, then clearly it holds for $n = 1$. If m is not an atom, we have $m = c_0g_0 + c_1g_1 + \ldots + c_{k-1}g_{k-1}$ for some coefficients, all positive. Then note that since $g_j = g_0 + j$, then $m = (c_0 + c_1 + \ldots + c_{k-1})g_0 + (c_1 + 2c_2 + \ldots + (k-1)c_{k-1})$ Since $\sum_{i=1}^{k-1} kc_k$ is positive by construction, then $m \geq (c_0 + c_1 + \ldots + c_{k-1})g_0$. In a similar manner, $g_j = g_{k-1} - j - k + 1$, so $m = (c_0 + c_1 + \ldots + c_{k-1})g_{k-1} - (c_0(k-1) + c_1(k-2) + \ldots + c_{k-2})$. Again, since $\sum_{i=0}^{k-1} c_i(k-1-i)$ is positive, then $m \leq (c_0 + c_1 + \ldots + c_{k-1})g_{k-1}$. This is the same coefficient as the previous, so $ng_0 \leq m \leq ng_{k-1}$. This means that counting the integers in these intervals also counts the number of consecutive elements in S. Thus giving M(S), and we conclude M(S) is ideal. □

Unfortunately, while idealism is a nice property, it in itself is not enough to justify a numerical set being a semigroup. Without some sort of *corresponding restriction on N*, then N can be made to break the set's additive closure.

While an N sequence can seemingly always be chosen to break a set's additive closure, it also can guarantee it.

8.2 M^l and N^l

Observation 8.2.1. *Every numerical semigroup has a maximal truncated n-staircase with k steps contained within it.*

Let S be a semigroup, let $m(S) = m$, and define $k = \min\{j \in \mathbb{N}_0 : jn > F(S)\}$. Then since S is a semigroup, S must contain every multiple of m, so $\{0, m, 2m, \ldots, (k-1)m, F+1 \rightarrow\} \subseteq S$. Therefore, every numerical semigroups contains a unique maximal truncated staircase.

Notation: From now on let $\{N_i\}$ denote the set of N_i consecutive gaps in S (or what was previously referred to as G_i). Similarly, let $\{M_i\}$ refer to the set of M_i consecutive elements of S.

Definition 8.2.2. *Let S be a numerical semigroup with $N(S) = (N_0, \ldots, N_{k-1})$ and multiplicity m. Define $N^l = \sum N_i$ where for all $x \in \{N_i\}$, $lm < x < (l+1)m$. Let $\{N^l\} = \bigcup_i \{N_i\}$.*

Definition 8.2.3. *Let S be a numerical semigroup with $M(S) = (M_0, \ldots, M_{k-1})$ and multiplicity m. Define $M^l = \sum M_i$ where for all $x \in \{M_i\}$, $lm \leq x \leq (l+1)m$. Let $\{M^l\} = \bigcup_i \{M_i\}$.*

Theorem 8.2.4. *Let S be a numerical semigroup. Then $M^l \leq M^{l+1}$.*

Proof. Let $k \in \{M^l\}$. Then $lm \leq k \leq (l+1)m$. This gives us $(l+1)m \leq k+m \leq (l+2)m$ gives us that $k+m \in M^{l+1}$. This is a one to one mapping, so we must have $M^l \leq M^{l+1}$ as required. □

Corollary 8.2.5. *Let S be a numerical semigroup. Then $N^l \geq N^{l+1}$.*

Theorem 8.2.6. *Let S be a numerical semigroup with multiplicity m and $N(S) = (m-1, N_1, \ldots, N_{k-1})$ and $M(S) = (1, M_1, \ldots, M_{k-1})$. Then for all $i \in \{1, \ldots, k-1\}$, $N_i + M_i \leq m$.*

Proof. Fix $i \in \{1, \ldots, k-1\}$ and let $\{l, l+1, \ldots, l+M_i-1\}$ be the set of the M_i consecutive elements in S. Notice that $M_i \leq N_0 = m-1$ since other wise would would have everything after l is in S, so there will not even be an M_i. Then $l+m \in S$, so there can be at most $(l+m)-(l+M_i-1)-1 = m-M_i$ gaps in the N_i string of consecutive gaps, i.e. $N_i \leq m-M_i$, so $N_i + M_i \leq m$. □

Observation 8.2.7. *Note that while the previous statement gives us a necessary condition on M(S) and N(S) for S to be a semigroup, it is not sufficient. For an example, let M(S)=(1,1,1) and N(S)=(5,3,3).*

Conjecture 8.2.8. *If S is a semigroup with at least one of it's generators being greater than the frobenius number and it's $M(S) = (M_0, M_1, \ldots, M_i, \ldots, M_{k-1})$ and $N(S) = (N_0, N_1, \ldots, N_i, \ldots, N_{k-1})$ with i being the maximal number so that $M_{k-1} = N_i$, $M_{k-2} = N_{i+1}$, $\ldots$, $M_{i+1} = M_{k-2}$, and $M_i \geq N_{k-1}$. Then S^{sym}, with $M(S^{sym}) = (M_0, M_1, \ldots, M_{k-1}, N_{i-1}, \ldots, N_0)$ and $N(S^{sym}) = (N_0, N_1, \ldots, N_{k-2}, M_i, M_{i-1}, \ldots, M_0)$, is a symmetric semigroup. If S has no generators larger than the frobenius number then there is no way to complete the N sequence of S to make a symmetric semigroup.*

Definition 8.2.9. *Let the* sequence of gap lengths *for a numerical set S be $MN = (MN_1, \ldots, MN_k, 0, 0, 0, \ldots)$ where MN_i counts the number of gaps between the i^{th} and $(i-1)^{\text{th}}$ elements of S and k the smallest nonegative integer so that $MN_k \neq 0$. By convention let $MN_0 = 0$.*

Equivalently, we can construct $MN(S)$ from $M(S) = (M_0, \ldots, M_k)$ and $N(S)$,

$$MN(S) = (0^{(M_0-1)}, N_0, 0^{(M_1-1)}, N_1, \ldots, 0^{(M_k-1)}, N_k, 0, 0, 0, \ldots)$$

where $0^{(l)}$ means that there are l zeroes in a row.

Theorem 8.2.10. *Let S be a numerical set with $MN(S) = (MN_1, \ldots, MN_k, 0, 0, 0, \ldots)$ then $S = \{n + \sum_{i=1}^{n} MN_i : n \in \mathbb{N}_0\}$.*

Proof. We have $0 \in S$, so we can partition the number line as follows: 0,1,2,$\ldots$,MN_1,MN_1+1,MN_1+2,MN_1+2+1,$\ldots$,MN_1+2+MN_2-1,MN_1+MN_2+2,$\ldots$,$MN_1+\cdots+MN_k+k \to$ □

Theorem 8.2.11. *Let S be a numerical semigroup with $MN(S) = (MN_1, \ldots, MN_k, 0, 0, 0, \ldots)$. Then the Hilbert Series of S has*

$$\mathcal{H}(S)(1-t) = 1 - t + \sum_{s \in \{1,\ldots,k\}: MN_s \neq 0} t^{(\sum_{i=1}^{s} MN_i)+s} - \sum_{s \in \{2,\ldots,k\}: MN_s \neq 0} t^{(\sum_{i=1}^{s-1} MN_i)+s}.$$

Proof. By 8.2.10, $S = \{n + \sum_{i+1}^{n} MN_i : n \in \mathbb{N}_0\}$, so $\mathcal{H}(S) = \sum_{n=0}^{\infty} t^{(\sum_{i=1}^{n} MN_i)+n}$. And

$$\begin{aligned}
\mathcal{H}(S)(1-t) &= (1-t)\sum_{n=0}^{\infty} t^{(\sum_{i=1}^{n} MN_i)+n} \\
&= \sum_{n=0}^{\infty} t^{(\sum_{i=1}^{n} MN_i)+n} - \sum_{n=0}^{\infty} t^{(\sum_{i=1}^{n} MN_i)+n+1} \\
&= t^0 + \sum_{n=1}^{\infty} t^{(\sum_{i=1}^{n} MN_i)+n} - t^1 - \sum_{n=1}^{\infty} t^{(\sum_{i=1}^{n} MN_i)+n+1} \\
&= 1 - t + \sum_{n=1}^{\infty} t^{(\sum_{i=1}^{n} MN_i)+n} - \sum_{n=2}^{\infty} t^{(\sum_{i=1}^{n-1} MN_i)+n} \\
&= 1 - t + t^{MN_1+1} + \sum_{n=2}^{\infty} \left(t^{(\sum_{i=1}^{n} MN_i)+n} - t^{(\sum_{i=1}^{n-1} MN_i)+n} \right) \\
&= 1 - t + t^{MN_1+1} + \sum_{s \in \{2,\ldots,k\}: MN_s \neq 0} \left(t^{(\sum_{i=1}^{n} MN_i)+n} - t^{(\sum_{i=1}^{n-1} MN_i)+n} \right) \\
&= 1 - t + \sum_{s \in \{1,\ldots,k\}: MN_s \neq 0} t^{(\sum_{i=1}^{s} MN_i)+s} - \sum_{s \in \{2,\ldots,k\}: MN_s \neq 0} t^{(\sum_{i=1}^{s-1} MN_i)+s}
\end{aligned}$$

□

Corollary 8.2.12. *Let S be a numerical semigroup with $M(S) = (M_0, \ldots, M_k)$ and $N(S) = (N_0, \ldots, N_k)$. Then the Hilbert Series of S has*

$$\mathcal{H}(S)(1-t) = (1-t^{M_0}) + (t^{M_0+N_0} - t^{M_0+N_0+M_1}) + \cdots + (t^{\sum_{i=0}^{k-1}(M_i+N_i)} - t^{\sum_{i=0}^{k-1}(M_i+N_i)+M_k}) + t^{\sum_{i=0}^{k}(M_i+N_i)}$$

Proof. We have

$$\begin{aligned}
MN(S) &= (0^{(M_0-1)}, N_0, 0^{(M_1-1)}, N_1, \ldots, 0^{(M_k-1)}, N_k, 0, 0, 0, \ldots) \\
&= (N_0, 0^{(M_1-1)}, N_1, \ldots, 0^{(M_k-1)}, N_k, 0, 0, 0, \ldots) \\
&= (MN_1, \ldots, MN_z, 0, 0, 0, \ldots)
\end{aligned}$$

so by 8.2.11 we have

$$\begin{aligned}
\mathcal{H}(S)(1-t) &= 1 - t + \sum_{s \in \{1,\ldots,z\}: MN_s \neq 0} t^{(\sum_{i=1}^{s} MN_i)+s} - \sum_{s \in \{2,\ldots,z\}: MN_s \neq 0} t^{(\sum_{i=1}^{s-1} MN_i)+s} \\
&= (1-t^{M_0}) + \sum_{s=0}^{k} t^{(\sum_{i=0}^{s} N_i)+(s+1)+(\sum_{i=0}^{s}(M_i-1))} - \sum_{s=1}^{k} t^{(\sum_{i=0}^{s-1} N_i)+(s+1)+(\sum_{i=0}^{s}(M_i-1))} \\
&= (1-t^{M_0}) + \sum_{s=0}^{k} t^{(\sum_{i=0}^{s} N_i)+(\sum_{i=0}^{s} M_i)} - \sum_{s=1}^{k} t^{(\sum_{i=0}^{s-1} N_i)+(\sum_{i=0}^{s} M_i)} \\
&= (1-t^{M_0}) + \sum_{s=0}^{k} t^{(\sum_{i=0}^{s} N_i+M_i)} - \sum_{s=1}^{k} t^{(\sum_{i=0}^{s-1} N_i+M_i)+M_s} \\
&= (1-t^{M_0}) + (t^{M_0+N_0} - t^{M_0+N_0+M_1}) + \cdots + (t^{\sum_{i=0}^{k-1}(M_i+N_i)} - t^{\sum_{i=0}^{k-1}(M_i+N_i)+M_k}) + t^{\sum_{i=0}^{k}(M_i+N_i)}
\end{aligned}$$

□

Algorithm 8.2.13. Jackson Autry's Algorithm *for testing whether a numerical set is a numerical semigroup and finding the generators of the numerical semigroup given the M and N sequences.*

Input: $M(S) = (M_0, \dots, M_k)$ *and* $N(S) = (N_0, \dots, N_k)$

1. *Find the potentional Hilbert Series for S using 8.2.12, $\mathcal{H}(S)$*
2. *Multiply $\mathcal{H}(S)(1-t)$ by $(1 + t + t^2 + \cdots + t^{M_0+N_0-1})$*
3. *If the coefficient of $P = \mathcal{H}(S)(1-t)(1 + t + t^2 + \cdots + t^{M_0+N_0-1})$ form a valid Apery set (i.e. the corresponding Kunz coordinates satisfy the Kunz inequalities), then S is a numerical semigroup generated by the small $\mathbb{N}_0$-linearly independent elements of this Apery set*

Output: True or False. If True, generators of S are also outputted.

Finding the generators from the M and N sequences can be very useful, particularly for constructing the Bras-Amoros Tree and right extension.

Theorem 8.2.14. *If S is a leaf in the Bras-Amoros Tree, then S does not have maximal embedding dimension.*

Proof. Note, S is a leaf if and only if it has no generators larger than the Frobenius Number. Notice, $F(S) + m(S) \in \operatorname{Ap} S$, but since $F(S) + m(S) > F(S)$, $F(S) + m(S)$ is not a generator of S. Recall that all non-multiplicity generators of S must be in the Apery Set of S and $|\operatorname{Ap} S \backslash \{0\}| = m(S) - 1$, so at most S has $m(S) - 2$ non-multiplicity generators and thus $e(S) \leq m(S) - 1$ and S is not maximal embedding dimension. □

9 Operations on M and N

In this section, we explore how modifying the M and N sequence of a numerical set change the set.

Definition 9.0.1. *Let S be a numerical set given by $N(S) = (N_0, N_1, \dots, N_k)$ and $M(S) = M_0, M_1, \dots, M_k)$, then S_* is the right truncation of S such that $N(S_*) = (N_0, N_1, \dots, N_{k-1})$ and $M(S_*) = M_0, M_1, \dots, M_{k-1})$.*

Theorem 9.0.2. *Let S be a numerical semigroup with $N(S) = (N_0, \dots, N_k)$ and $M(S) = (M_0, \dots, M_k)$. Let S_* be the right truncation of S with $N(S_*) = (N_0, \dots, N_{k-1})$ and $M(S_*) = (M_0, \dots, M_{k-1})$. Then $\tilde{S} = \{0, \dots, M_k - 1\} \cup (\tilde{S}_* + M_k + N_{k-1})$.*

Proof. We have $S = \{0, \dots, M_0 - 1\} \cup \{M_0 + N_0, \dots, M_0 + N_0 + M_1 - 1\} \cup \cdots \cup \{M_0 + N_0 + \cdots + M_{k-1} + N_{k-1}, \dots, M_0 + N_0 + \cdots + M_{k-1} + N_{k-1} + M_k - 1\} \cup \{M_0 + N_0 + \cdots + M_{k-1} + N_{k-1} + M_k + N_k \rightarrow\}$ and $S_* = \{0, \dots, M_0 - 1\} \cup \{M_0 + N_0, \dots, M_0 + N_0 + M_1 - 1\} \cup \cdots \cup \{M_0 + N_0 + \cdots + M_{k-1} + N_{k-1}, \dots, M_0 + N_0 + \cdots + M_{k-2} + N_{k-2} + M_{k-1} - 1\} \cup \{M_0 + N_0 + \cdots + M_{k-1} + N_{k-1} \rightarrow\}$, so

$S \subseteq S_*$ and S has Base $B(S) = M_0 + N_0 + \cdots + M_{k-1} + N_{k-1} + M_k - 1$ while S_* has Base $B(S_*) = M_0 + N_0 + \cdots + M_{k-2} + N_{k-2} + M_{k-1} - 1$.

Also,

$$\begin{aligned}\tilde{S} &= \{B(S) - s : s \in S, s \leq B(S)\} \cup \{B(S) \rightarrow\} \\ &= \{M_0 + N_0 + \cdots + M_{k-1} + N_{k-1} + M_k - 1 - s : s \in S, s \leq M_0 + N_0 + \cdots + M_{k-1} + N_{k-1} + M_k - 1\} \cup \\ &\qquad \{M_0 + N_0 + \cdots + M_{k-1} + N_{k-1} + M_k - 1 \rightarrow\}\end{aligned}$$

and

$$\begin{aligned}\tilde{S}_* &= \{B(S_*) - s : s \in S_*, s \leq B(S_*)\} \cup \{B(S_*) \rightarrow\} \\ &= \{M_0 + N_0 + \cdots + M_{k-2} + N_{k-2} + M_{k-1} - 1 - s : s \in S_*, s \leq M_0 + N_0 + \cdots + M_{k-2} + N_{k-2} + M_{k-1} - 1\} \\ &\qquad \cup \{M_0 + N_0 + \cdots + M_{k-2} + N_{k-2} + M_{k-1} - 1 \rightarrow\}\end{aligned}$$

so

$$\begin{aligned}&\tilde{S}_* + M_k + N_{k-1} \\ &\quad = \{M_0 + N_0 + \cdots + M_{k-1} + N_{k-1} + M_k - 1 - s : s \in S_*, s \leq M_0 + N_0 + \cdots + M_{k-2} + N_{k-2} + M_{k-1} - 1\} \cup \\ &\qquad\quad \{M_0 + N_0 + \cdots + M_{k-1} + N_{k-1} + M_k - 1 \rightarrow\} \\ &\quad = \{M_0 + N_0 + \cdots + M_{k-1} + N_{k-1} + M_k - 1 - s : s \in S, s \leq M_0 + N_0 + \cdots + M_{k-2} + N_{k-2} + M_{k-1} - 1\} \cup \\ &\qquad\quad \{M_0 + N_0 + \cdots + M_{k-1} + N_{k-1} + M_k - 1 \rightarrow\}\end{aligned}$$

Note, $B(S) - x \in \{0, \ldots, M_k - 1\}$ where $x \in \{M_0 + N_0 + \cdots + M_{k-1} + N_{k-1}, \ldots, M_0 + N_0 + \cdots + M_{k-1} + N_{k-1} + M_k - 1\}$, so $\tilde{S} = \{0, \ldots, M_k - 1\} \cup (\tilde{S}_* + M_k + N_{k-1})$. □

Theorem 9.0.3. *Let S be a numerical set with $N(S) = (N_0, N_1, \ldots, N_k)$. Let T be a numerical set with $N(T) = (N_0, N_1, \ldots, N_k - j)$ for some $0 \leq j \leq N_k$. Then if S is a numerical semigroup, then T is also a numerical semigroup.*

Proof. Note, $T = S \cup \{F(S) + 1 - j \rightarrow\}$. Let $x, y \in T$. If $x \geq F(S) + 1 - j$, then $x + y \geq F(S) + 1 - j$ and $x + y \in T$. If $x, y < F(S) + 1 - j$ then $x, y \in S$ so $x + y \in S$ since S is a numerical semigroup and $S \subseteq T$ so $x + y \in T$. □

Definition 9.0.4. *Let S be a numerical set given by $N(S) = (N_0, N_1, \ldots, N_k)$ and $M(S) = (M_0, M_1, \ldots, M_k)$. Then call S^L the* left extension *of S, such that S^L is determined by adding an element to the left of the N(S) and M(S) sequences. That is, $N(S^L) = (N_{-1}, N_0, N_1, \ldots, N_k)$ and $M(S^L) = (M_{-1}, M_0, M_1, \ldots, M_k)$.*

Theorem 9.0.5. *The following are equivalent:*

1. *S^L is a left extension of the numerical set S with $M(S) = (M_0, \ldots, M_k)$ and $N(S) = (N_0, \ldots, N_k)$, $M(S^L) = (M_{-1}, M_0, \ldots, M_k)$ and $N(S^L) = (N_{-1}, N_0, \ldots, N_k)$*
2. *$S^L = \{0, 1, \ldots, M_{-1} - 1\} \cup (S + N_{-1} + M_{-1})$*

Proof. Note,

$$\begin{aligned}S =&\{0, 1, \ldots, M_0 - 1\} \cup \{M_0 + N_0, \ldots, M_0 + N_0 + M_1 - 1\} \cup \ldots \cup \\ &\{M_0 + N_0 + \ldots + M_{k-1} + N_{k-1}, \ldots, M_0 + N_0 + \ldots + M_{k-1} + N_{k-1} + M_k - 1\} \cup \\ &\{M_0 + N_0 + \ldots + M_k + N_k \rightarrow\}\end{aligned}$$

And

$$\begin{aligned}&S + M_{-1} + N_{-1} = \{M_{-1} + N_{-1}, M_{-1} + N_{-1} + 1, \ldots, M_{-1} + N_{-1} + M_0 - 1\} \cup \\ &\{M_{-1} + N_{-1} + M_0 + N_0, \ldots, M_{-1} + N_{-1} + M_0 + N_0 + M_1 - 1\} \cup \ldots \cup \\ &\{M_{-1} + N_{-1} + N_0 + \ldots + M_{k-1} + N_{k-1}, \ldots, M_{-1} + N_{-1} + M_0 + N_0 + \ldots + M_{k-1} + N_{k-1} + M_k - 1\} \cup \\ &\{M_{-1} + N_{-1} + M_0 + N_0 + \ldots + M_k + N_k \rightarrow\}\end{aligned}$$

So,

$$\begin{aligned}&\{0, \ldots, M_{-1} - 1\} \cup (S + M_{-1} + N_{-1}) = \{0, \ldots, M_{-1} - 1\} \cup \\ &\quad \{M_{-1} + N_{-1}, M_{-1} + N_{-1} + 1, \ldots, M_{-1} + N_{-1} + M_0 - 1\} \cup \\ &\quad \{M_{-1} + N_{-1} + M_0 + N_0, \ldots, M_{-1} + N_{-1} + M_0 + N_0 + M_1 - 1\} \cup \ldots \cup \\ &\quad \{M_{-1} + N_{-1} + N_0 + \ldots + M_{k-1} + N_{k-1}, \ldots, M_{-1} + N_{-1} + M_0 + N_0 + \ldots + M_{k-1} + N_{k-1} + M_k - 1\} \cup \\ &\quad \{M_{-1} + N_{-1} + M_0 + N_0 + \ldots + M_k + N_k \rightarrow\} \\ &\quad = S^L\end{aligned}$$

□

Corollary 9.0.6. *We also can write S in terms of S^L: $S = (S^L - m(S^L)) \backslash \{-m(S^L)\}$.*

Theorem 9.0.7. *Let S be a numerical set with $N(S) = (N_0, N_1, \ldots, N_k)$ and $M(S) = (M_0, M_1, \ldots, M_k)$ and let S^L be a numerical set with $N(S^L) = (N_0, N_0, N_1, \ldots, N_k)$ and $M(S^L) = (1, M_0, M_1, \ldots, M_k)$. If S is a numerical semigroup then S^L is a numerical semigroup.*

Proof. Note, $S^L = \{0\} \cup (N_0 + 1 + S)$.

Suppose S is a numerical semigroup and let $x^L, y^L \in S^L$. Without loss of generality, if $x^L = 0$ then $x^L + y^L = y^L \in S^L$, so suppose that $x^L, y^L \neq 0$. Then $x^L = N_0 + 1 + x$ and $y^L = N_0 + 1 + y$ for some $x, y \in S$. Since $x, y, N_0 + 1 = m(S) \in S$ and S is a semigroup, then $x + y + N_0 + 1 \in S$ so $(x + y + N_0 + 1) + N_0 + 1 = x^L + y^L \in S^L$ by construction. So S^L is closed under addition and thus is a numerical semigroup. □

Note that $N_0 + 1 \in S$, and left-extending N_0 was enough to justify S^L being a semigroup. It turns out that this generalizes.

Theorem 9.0.8. *Let S be a numerical set with $N(S) = (N_0, N_1, \ldots, N_k)$ and $M(S) = (M_0, M_1, \ldots, M_k)$ and let S^L be a numerical set with $N(S^L) = (N_{-1}, N_0, N_1, \ldots, N_k)$ and $M(S^L) = (1, M_0, M_1, \ldots, M_k)$. If S is a numerical semigroup and $m = N_{-1} + 1 \in S$ then S^L is a numerical semigroup. Also, if S^L is a numerical semigroup then $m \in S$.*

Proof. Note, $S^L = \{0\} \cup (N_{-1} + 1 + S)$.

Suppose S is a numerical semigroup and let $x^L, y^L \in S^L$. If $x^L = 0$ then $x^L + y^L = y^L \in S^L$, so suppose that $x^L, y^L \neq 0$. Then $x^L = m + x$ and $y^L = m + y$ for some $x, y \in S$. Since $x, y, m \in S$ and S is a semigroup, then $x + y + m \in S$ so $(x + y + m) + m = x^L + y^L \in S^L$ by construction. So S^L is closed under addition and thus is a numerical semigroup.

If instead we suppose that S^L is a numerical semigroup, then notice $S = (S^L \backslash \{0\}) - m$ and as before since $2m \in S^L$, $2m - m = m \in S$. □

Theorem 9.0.8 is very nearly an if and only if statement, but counterexamples have been found to show it is not enough for S^L to be semigroup to guarantee S is. This is exemplified in the next example.

Example 9.0.9. *Let S be a numerical set with $N(S) = (N_0, N_1, \ldots, N_k)$ and $M(S) = (M_0, M_1, \ldots, M_k)$ and let S' be a numerical set with $N(S') = (N_1, \ldots, N_k)$ and $M(S') = (M_1, \ldots, M_k)$. It is* not *always true that if S is a numerical semigroup, then S' is also a numerical semigroup.*

Consider $N(S) = (5, 1, 2)$ with $M(S) = (1, 1, 1)$ and $N(S'') = (1, 2)$ with $M(S'') = (1, 1)$. Obviously S'' is not a numerical semigroup, but $S = \{0, 6, 8, 11 \rightarrow\}$ is a numerical semigroup.

Theorem 9.0.10. *Let P^L be a semigroup that is a left extension of the numerical semigroup P, then $H(P^L) = \{1, \ldots, N_{-1}\} \cup \{H(P) + N_{-1} + 1\}$.*

Proof. Let P^L be a left extension of the numerical semigroup P with $N(P^L) = (N_{-1}, N_0, \ldots, N_k)$. Let the hook set of P be $H(P)$. Recall that by def 9.0.4, when we do a left extension, we are adding N_{-1} consecutive gaps into our original numerical semigroup, at the begining of the semigroup. So $H(P^L)$ is composed of the new gaps we just added $\{1, \ldots, N_{-1}\}$, plus all of our old gaps, $H(P)$, except that all of our old gaps have to be shifted up by $N_{-1} + 1$. Hence $H(P^L) = \{1, \ldots, N_{-1}\} \cup (H(P) + N_{-1} + 1)$. □

Theorem 9.0.11. *Let S be a numerical set with $N(S) = (N_0, \ldots, N_k)$ and $M(S) = (M_0, \ldots, M_k)$. Let S^L be a left extension of S with $N(S^L) = (N_{-1}, N_0, \ldots, N_k)$ and $M(S^L) = (1, M_0, \ldots, M_k)$. Then the hook set of S^L can be written in terms of the hook set of S as follows*

$$H(S^L) = H(S) \cup (\mathbb{N}_0 \backslash S + N_{-1} + 1) \cup \{1, 2, \ldots, N_{-1}\}.$$

Proof. Notice, $\{1, 2, \ldots, N_{-1}\} \subseteq H(S^L)$ and $H(S) \subseteq H(S^L)$ by construction. Let $x \in (\mathbb{N}_0 \backslash S + N_{-1} + 1)$, then $x = b + N_{-1} + 1$ where b is a gap of S. Since $S^L = \{0\} \cup (S + N_{-1} + 1)$, notice that if b is a gap of S then $b + N_{-1} + 1$ is a gap of S^L. So $x = b + N_{-1} + 1 = (b + N_{-1} + 1) - 0 \in H(S^L)$. Thus $H(S^L) \supseteq H(S) \cup (\mathbb{N}_0 \backslash S + N_{-1} + 1) \cup \{1, 2, \ldots, N_{-1}\}$.

Now consider $x \in H(S^L)$, so $x = b' - a'$ for some $b' \notin S^L$ and some $a' \in S^L$ with $b' > a'$. Either $b' \in \{1, \ldots, N_{-1}\}$ or $N_{-1} + 1 \leq b' \notin (S + N_{-1} + 1)$ so $b' = b + N_{-1} + 1$ for some $0 < b \notin S$. Additionally, either $a' = 0$ or $a' \in S + N_{-1} + 1$ so $a' = a + N_{-1} + 1$ for some $a \in S$. If $a' = 0$, then $b' - a' = b'$ so $\{1, \ldots, N_{-1}\} \subseteq H(S^L)$ and $(\mathbb{N}_0 \backslash S) + N_{-1} + 1 \subseteq H(S^L)$. If $a' = a + N_{-1} + 1$ we must have $b' = b + N_{-1} + 1$ with $b > a$. Then $b' - a' = b - a \in H(S)$. Thus, $H(S^L) \subseteq H(S) \cup (\mathbb{N}_0 \backslash S + N_{-1} + 1) \cup \{1, 2, \ldots, N_{-1}\}$. □

Theorem 9.0.12. *Let S be a numerical set with $N(S) = (N_0, \ldots, N_k)$ and $M(S) = (M_0, \ldots, M_k)$ and let $r_1(S) = \{F(S) - r : r \in S \ \ \& \ \ r < F(S)\}$ (these are the hooks of S located in the top row of the Young diagram). Let S^U be an upper extension of S with $N(S^U) = (N_0, \ldots, N_k, 1)$ and $M(S^U) = (M_0, \ldots, M_k, M_{k+1})$. Then $H(S^U) = H(S) \cup (r_1(S) + M_{k+1} + 1) \cup \{1, 2, \ldots, M_{k+1}\}$.*

Proof. Let S be a numerical set with $N(S) = (N_0, \ldots, N_k)$ and $M(S) = (M_0, \ldots, M_k)$ and let $r_1(S) = \{F(S) - r : r \in S \ \ \& \ \ r < F(S)\}$. Consider S^U such that $N(S^U) = (N_0, \ldots, N_k, 1)$ and $M(S^U) = (M_0, \ldots, M_k, M_{k+1})$. We will now show $H(S) \cup (r_1(S) + M_{k+1} + 1) \cup \{1, 2, \ldots, M_{k+1}\} \subseteq H(S^U)$.

Note, by construction, that $H(S) \subseteq H(S^U)$ and $\{1, 2, \ldots, M_{k+1}\} \subseteq H(S^U)$. Let $x \in (r_1(S) + M_{k+1} + 1)$. So $x = (F(S) - r) + M_{k+1} + 1$ for some $r \in S$ such that $r < F(S)$. Note, that $S^U = S \backslash \{F(S) + M_{k+1} + 1\}$ because, when you upper extend with $N_{k+1} = 1$, you are just creating one more gap namely a new Frobenius Number for the set. This means $F(S^U) = F(S) + M_{k+1} + 1$. So $x = (F(S) - r) + M_{k+1} + 1 = (F(S) + M_{k+1} + 1) - r = F(S^U) - r \in H(S^U)$. Hence $\{r_1 + M_{k+1} + 1\} \subseteq H(S^U)$, therefore $H(S) \cup (r_1(S) + M_{k+1} + 1) \cup \{1, 2, \ldots, M_{k+1}\} \subseteq H(S^U)$.

Now we will show $H(S^U) \subseteq H(S) \cup (r_1(S) + M_{k+1} + 1) \cup \{1, 2, \ldots, M_{k+1}\}$.

Note, $\mathbb{N}_0 \backslash S^U = \mathbb{N}_0 \backslash S \cup (F(S) + M_{k+1} + 1)$ and $H(S^U) = \{b - a : b \in \mathbb{N}_0 \backslash S^U, a \in S^U, \ \& \ \ b > a\}$. Let $y \in H(S^U)$.So either $b \in \mathbb{N}_0 \backslash S$ and $a \in S^U$, $b = F(S) + M_{k+1} + 1$ and $a \in S^U$ such that $a \in S$, or $b = F(S) + M_{k+1} + 1$ and $a \in S^U$ such that $a \notin S$. If $b \in \mathbb{N}_0 \backslash S$ and $a \in S^U, then y \in H(S)$. If $b = F(S) + M_{k+1} + 1$ and $a \in S^U$ such that $a \in S$, then $y \in (r_1(S) + M_{k+1} + 1)$. And finally, if $b = F(S) + M_{k+1} + 1$ and $a \in S^U$ such that $a \notin S$, then $y \in \{1, \ldots, M_{k+1}\}$. Therefore $H(S^U) \subseteq H(S) \cup (r_1(S) + M_{k+1} + 1) \cup \{1, 2, \ldots, M_{k+1}\}$. Hence, $H(S^U) = H(S) \cup (r_1(S) + M_{k+1} + 1) \cup \{1, 2, \ldots, M_{k+1}\}$. □

Definition 9.0.13. *Let S be a numerical set with $N(S) = (N_0, N_1, \ldots, N_k)$ and $M(S) = (M_0, M_1, \ldots, M_k)$ then S_C is a column extension of S if $N(S_C) = N(S)$ and $M(S_C) = (M_0 + 1, M_1, \ldots, M_k)$. Alternatively, we say $S_C = \{0\} \cup (S + 1)$.*

Theorem 9.0.14. *Let S be a numerical set with $N(S) = (N_0, \ldots, N_k)$ and $M(S) = (M_0, \ldots, M_k)$ and let S_C be the column extension of S numerical set with $N(S_C) = N(S)$ and $M(S_C) = (M_0 + 1, M_1, \ldots, M_k)$. Then the hook set of $H(S_C)$ is $H(S_C) = H(S) \cup (\mathbb{N}_0 \backslash S + 1)$.*

Proof. Note, $S_C = \{0\} \cup (S + 1)$ so $\mathbb{N}_0 \backslash S_C = \mathbb{N}_0 \backslash S + 1$. If $x \in H(S_C)$ then $x = b_C - a_C$ where $b_C \in \mathbb{N}_0 \backslash S_C$, $a_c \in S_C$, and $b_C > a_C$. So $b \in \mathbb{N}_0 \backslash S$. Also, either $a_C = 0$ or $a_C = a + 1$ for some $a \in S$ with $a < b$. If $a_C = 0$, then $x = b_C - a_C = b_C = b + 1 \in \mathbb{N}_0 \backslash S + 1$. If $a_C = a + 1$, then $x = b_C - a_C = b + 1 - (a + 1) = b - a \in H(S)$. Thus, $H(S_C) \subseteq H(S) \cup (\mathbb{N}_0 \backslash S + 1)$.

Let $x \in H(S) \cup (\mathbb{N}_0 \backslash S + 1)$. Then either $x \in H(S)$ or $x \in \mathbb{N}_0 \backslash S + 1$. If $x \in H(S)$, then $x = b - a$ for some $b \in \mathbb{N}_0 \backslash S$ and $a \in S$ with $a < b$. Note, $x = (b + 1) - (a + 1)$ where $b + 1 \in \mathbb{N}_0 \backslash S_C$ and $a + 1 \in S_C$, so $x \in H(S_C)$. If $x \in \mathbb{N}_0 \backslash S + 1$, then $x = b + 1 = b + 1 - 0$ for some $b \in \mathbb{N}_0 \backslash S$ so $x \in H(S_C)$. Thus, $H(S_C) \supseteq H(S) \cup (\mathbb{N}_0 \backslash S + 1)$. □

Definition 9.0.15. *Let S be the numerical set with $N(S) = (N_0, \ldots, N_k)$ and $M(S) = (M_0, \ldots, M_k)$, the* row extension *of S is the numerical set S^R with $N(S^R) = (N_0, \ldots, N_{k-1}, N_k + 1)$ and $M(S^R) = M(S)$. Equivalently, $S^R = S \backslash \{F(S) + 1\}$.*

Theorem 9.0.16. *Let S be a numerical set with $N(S) = (N_0, \ldots, N_k)$ and $M(S) = (M_0, \ldots, M_k)$ and let S^R be the numerical set with $N(S^R) = (N_0, \ldots, N_{k-1}, N_k + 1)$ and $M(S^R) = M(S)$. Then the hook set of $H(S^R)$ is $H(S^R) = H(S) \cup (r_1(S) + 1)$ where $r_1(S)$ is defined as in 9.0.12.*

Proof. Notice, $\mathbb{N}_0 \backslash S^R = (\mathbb{N}_0 \backslash S) \cup \{F(S) + 1\}$. Let $x \in H(S^R)$. Then $x = b - a$ where $b \in \mathbb{N}_0 \backslash S^R$ and $a \in S^R$ with $a < b$. Either $b = F(S) + 1$ or $b \in \mathbb{N}_0 \backslash S$ and $a \in S^R \subseteq S$. If $b = F(S) + 1$, then $x = b - a = F(S) + 1 - a = (F(S) - a) + 1 \in r_1(S) + 1$. If $b \in \mathbb{N}_0 \backslash S$, then $x = b - a \in H(S)$. So $H(S^R) \subseteq H(S) \cup (r_1(S) + 1)$.

If $x \in H(S) \cup (r_1(S) + 1)$, then either $x \in H(S)$ so $x = b - a$ for some $b \in \mathbb{N}_0 \backslash S$ and $a \in S$ with $a < b$, or $x \in r_1(S) + 1$ so $x = (F(S) - a) + 1$ for some $a \in S$ with $a < F(S)$. If $x \in H(S)$, then $b \in \mathbb{N}_0 \backslash S \subseteq \mathbb{N}_0 \backslash S^R$ and $a \in S^R$ since $a < b$ so $a < F(S)$. Thus $x \in H(S^R)$. If $x \in r_1(S) + 1$, then $x = (F(S) - a) + 1 = (F(S) + 1) - a \in H(S^R)$. So $H(S^R) \supseteq H(S) \cup (r_1(S) + 1)$. □

Theorem 9.0.17. *Let S be a numerical set and let S^R be the row extension of S as defined above. Then $A(S^R) \subseteq A(S)$.*

Proof. Note, $S^R = S \backslash \{F(S) + 1\} \subset S$. By definition,

$$\begin{aligned} A(S^R) &= \{a \in S^R : a + x \in S^R \, \forall x \in S^R\} \\ &= \{a \in S : a + x \in S \, \forall x \in S \text{ with } a, x, a + x \neq F(S) + 1\} \\ &\subseteq \{a \in S : a + x \in S \, \forall x \in S\} \\ &= A(S) \end{aligned}$$

□

Theorem 9.0.18. *Let S be a numerical set and let S_C be the column extension of S as defined above. Then $A(S_C) \subseteq A(S)$.*

Proof. Note, $S_C = \{0\} \cup (S + 1)$. By definition,

$$\begin{aligned} A(S_C) &= \{a \in S_C : a + x \in S_C \, \forall x \in S_C\} \\ &= \{a \in S_C : a + s + 1 \in S_C \, \forall s \in S\} \\ &= \{a \in S_C : a + s \in S \, \forall s \in S\} \\ &\subseteq \{a \in \mathbb{N}_0 : a + x \in S \, \forall x \in S\} \\ &= A(S) \end{aligned}$$

□

Theorem 9.0.19. *Let S be a numerical set. Then for $t \in A(S)$, $t \in A(S_C)$ if and only if $t \in S_C$.*

Proof. $\Rightarrow$ Since $A(S_C) \subseteq S_C$, then clearly this is true.

$\Leftarrow$ Let $t \in S_C$ such that $t \in A(S)$. Then for nonzero $x \in S_C$, $x - 1 \in S$. Since $t \in A(S)$, $t + (x - 1) \in S$. This gives us $t + x \in S_C$. Since x was arbitrary, we can conclude $t \in A(S_c)$. □

In particular, the last theorem gives us that if multiples of m(A(S)) are in S_C, then they must be in $A(S_C)$.

Definition 9.0.20. *Let S be a numerical semigroup. We call S an n-staircase with k steps and l missing steps or a* pseudo-n-staircase *if and only if $S = \{0, (l+1)n, (l+2)n, \dots, kn \to\}$. Similarly, S is a truncated n-staircase with k steps and l missing steps or a* pseudo-truncated-n-staircase *if and only if $S = \{0, (l+1)n, (l+2)n, \dots, kn - j \to\}$ for some $0 \leq j < n$.*

Theorem 9.0.21. *Let S be a numerical set. If $A(S)$ is a truncated staircase, then $A(S_C)$ and $A(S^R)$ are truncated staircases or pseudo-truncated staircases. In particular, if $A(S)$ is a regular staircase, then $A(S_C)$ and $A(S^R)$ are columns.*

Proof. Suppose $A(S)$ is a regular staircase, i.e. $A(S) = \{0, m, 2m, \dots, km \to\}$.

By 9.0.17, since $A(S^R) \subseteq A(S)$ we only need to check if $m, 2m, \dots, (k-1)m$ are in $A(S^R)$. Note, $S^R = S \backslash \{F(S)+1\} = S \backslash \{km\}$, so $km \notin A(S^R)$. Since, $m, 2m, \dots, (k-1)m < km$ and are in S, they are also in S^R, so we can pair them to sum to km like $km = m + (k-1)m = 2m + (k-2)m = \cdots$. Thus, $m, 2m, \dots, (k-1)m \notin A(S^R)$. Then $A(S^R) = \{0, km+1 \to\}$.

By 9.0.18, since $A(S_C) \subseteq A(S)$ we again only need to check if $m, 2m, \dots, (k-1)m \in A(S_C)$. Note, $S_C = \{0\} \cup (S+1)$ so $F(S_C) = F(S) + 1 = km$. Suppose $nm \in S_C$ for some $n \in \{1, 2, \dots, k-1\}$, then $nm - 1 \in S$, so since $(k-n)m \in A(S)$, $km - 1 \in S$. But then we would have $km - 1 + x \geq km - 1$ for all $x \in S$ so $km - 1 + x \in S$ and $km - 1 \in A(S)$ which is a contradiction (Im not sure what author intended to be contradiction). Thus, $m, 2m, \dots, (k-1)m \notin S_C$, so $m, 2m, \dots, km \notin A(S_C)$, so $A(S_C)$ is a column.

Now, suppose $A(S)$ is a strictly truncated staircase, i.e. $A(S) = \{0, m, 2m, \dots, km, km+j \to\}$ for some $1 < j < m$.

Here we prove if everything in the equivalence class of j modulo m is in S, then $A(S^R)$ is a column. Again we have $A(S^R) \subseteq A(S)$ so we only need to check if $m, 2m, \dots, km$ are in $A(S^R)$. Notice, since $m, 2m, \dots, (k-1)m \in A(S)$ then $j, j+m, j+2m, \dots, j+(k-1)m \in S$ and also $j, j+m, j+2m, \dots, j+(k-1)m \in S^R$. Then $j + km = (j+m) + (k-1)m = (j+2m) + (k-2)m = \cdots = (j + (k-1)m) + m \notin S^R$ so $m, 2m, \dots, km \notin A(S^R)$ and $A(S^R) = \{0, km+j+1 \to\}$.

If not everything in the equivalence class of j modulo m is in S, but we have at least the $(k-1)m + j \in S$ (say we have n elements of the equivalence class j modulo m in S) and perhaps some other equivalence classes in S, then $A(S^R)$ is like a truncated staircase with the same step size as $A(S)$ but missing the first n steps. By (Sir) Deepriliam, the n elements of $[j]_m$ in S are $(k-1)m + j, (k-2)m + j, \dots, (k-n)m + j$. So

$$m + (k-1)m + j = 2m + (k-2)m + j = \cdots = nm + (k-n)m + j = km + j \notin S^R$$

and $m, 2m, \dots, nm \notin A(S^R)$. We still need to show that $(n+1)m, (n+2)m, \dots, km \in A(S^R)$. Let $i \in \{n+1, \dots, k\}$. Since $im \in A(S)$, $im + x \in S$ for all $x \in S$, so the only way $im \notin A(S^R)$ is if $im + x = km + j$ for some $x \in S^R \subseteq S$. So then $x = (k-i)m + j$, but we have $(k-i)m + j \notin S$ so no such x exists and $im \in A(S^R)$. Thus, $A(S^R) = \{0, (n+1)m, (n+2)m, \dots, km, km+j+1 \to\}$.

Now suppose nothing from $[j]_m$ is in S. Again $A(S^R) \subseteq A(S)$ so we only need to check $m, 2m, \ldots, km$. Since $m, 2m, \ldots, km \in A(S)$, $im + x \in S$ for alll $x \in S$. And since $S^R \subseteq S$, $im + x \in S$ for all $x \in S^R$. Then the only way $im + x \notin S^R$ is if $im + x = km + j$ so $x = (k-i)m + j$. But we have $(k-i)m + j \notin S$ so $(k-i)m + j \notin S^R$ and thus, nothing pushes im out of $A(S^R)$. So $A(S^R) = \{0, m, 2m, \ldots, km, km + j + 1 \rightarrow\}$. Note, if in the above case, $j = m - 1$, then nothing at all changes, except that now S^R is a regular staircase, not a strictly truncated one. This is because if $j = m - 1$, then $km + j + 1 = km + m - 1 + 1 = km + m = (k+1)m$.

If anything from $[m-1]_m$ is in S (say n elements of $[m-1]_m$ are in S), then $A(S_C) = \{0, (k-n+1)m, (k-n+2)m, \ldots, km, km + j + 1 \rightarrow\}$. By (Sir) Deepriliam, the n elements of $[m-1]_m$ that are in S are the largest n such elements, i.e. $km - 1, (k-1)m - 1, \ldots, (k-n+1)m - 1$. Then $S_C = \{0\} \cup (S+1)$ so $km, (k-1)m, \ldots, (k-n+1)m \in S_C$ and no other multiples of m (other than 0) are in S_C. Since $A(S_C) \subseteq A(S)$, we only need to check whether $km, (k-1)m, \ldots, (k-n+1)m \in A(S_C)$. Note, if $im \in S_C$, then for every non-zero $x \in S_C$, $x = c + 1$ for some $c \in S$ and $im + x = im + (c+1) = (im + c) + 1 \in S_C$ since $im \in A(S)$ so $im + c \in S$. Thus, $im \in A(S_C)$ and $A(S_C) = \{0, (k-n+1)m, (k-n+2)m, \ldots, km, km + j + 1 \rightarrow\}$. □

Conjecture 9.0.22. *Let S be a semigroup with $S = \langle p, q \rangle$ such that $p, q \in \mathbb{N} \backslash \{0, 1, 2\}$, then, under left extension, S will generate the arithmetic semigroup $P = \langle p, p+q, \ldots, p + (p-1)q \rangle$ when you left extend with the element $N_{-1} = p - 1$ and $Q = \langle q, q+p, \ldots, q + (q-1)p \rangle$ when you left extend with the element $N_{-1} = q - 1$. These are the arithmetic sequences of full embedding dimension for $a = p$, $d = q$, & $h = 1$ and $a = q$, $d = p$, & $h = 1$. If you left extend these further with the same element they were first extended with, you get the arithmetic semi groups that correspond to the same a and d values but you're h increases by one with each extension.*

So every arithmetic semigroup with full embedding dimension is the product of a left extension of the semigroup generated by it's a value and d value.

9.1 The Numerical Semigroup Poset

We now use this left-extension operation to establish a partially-ordered set on the set of all semigroups. It is first helpful to see some relations between S and one of its left-extensions S'. We will also be looking in more detail at Arf numerical semigroups and the poset they make.

Theorem 9.1.1. *With S and S^L defined as before (S^L is a left extension of S), S is Arf if and only if S^L is Arf*

Proof. $\Rightarrow$
Suppose x_1, y_1, and z_1 map to x_2, y_2, and z_2 from S to S' respectively with $x_1 \geq y_1 \geq z_1$. Then since the action under this mapping is adding $N_0 + 1$, then we also have $x_2 \geq y_2 \geq z_2$, and $x_2 + y_2 - z_2 = x_1 + y_1 - z_1 + 2(N_1 + 1) - (N_1 + 1) = x_1 + y_1 - z_1 + (N_1 + 1)$. Since S is Arf, we know $x_1 + y_1 - z_1 \in S$, and since adding $N_1 + 1$ is the same mapping, then $x_1 + y_1 - z_1 + (N_1 + 1) \in S'$. Thus S' is Arf provided S is.

$\Leftarrow$
The proof is similar, but with x_2, y_2, and z_2 from S' mapping to x_1, y_1, and z_1 in S by subtracting $N_0 + 1$. We still have $x_2 \geq y_2 \geq z_2 \quad \rightarrow \qquad x_1 \geq y_1 \geq z_1$, and

$$x_1 + y_1 - z_1 = x_2 + y_2 - z_2 - 2(N_1 + 1) + (N_1 + 1) = x_2 + y_2 - z_2 - (N_1 + 1).$$

S' is Arf, so $x_2 + y_2 - z_2 \in S'$, and subtracting $N_1 + 1$ gives an element in S by construction. Thus S is Arf provided S' is. $\square$

This statement has a powerful corollary regarding when you can left-truncate from $N(S)$.

Corollary 9.1.2. *If S is an Arf semigroup, then left-truncating will result in a numerical semigroup.*

Proof. Define T such that $T' = S$. Since S is Arf, from Theorem 9.1.1 then T is Arf. It is well known that this means T is a semigroup. T is defined so that $0 \in T$, but it remains to show that T is co-finite. However, since S had a finite N-sequence, the N-sequence of T is one shorter, and thus the number of gaps is finite. Hence, T is a semigroup. $\square$

To ease notation, and since at this point we will primarily focus on left extension, we will denote S' as a left extension of S unless otherwise stated.

Definition 9.1.3. *Let S be a numerical set with $N(S) = (N_0, N_1, \ldots, N_{k-1})$ then define $S^{(k)}$ recursively by $S^{(0)} = S$, $S^{(1)} = S^L$, and $S^{(n+1)} = (S^{(n)})^L$.*

Theorem 9.1.4. *Let S be a numerical semigroup with $N(S) = (N_0, \ldots, N_{k-1})$. Then S is Arf if and only if $S = \mathbb{N}_0^{(k)}$.*

Proof. The following will be a proof by induction on k.

If $k = 0$, then $N(S) = \emptyset$ so $S = \mathbb{N}_0 = \mathbb{N}_0^{(0)}$.

Suppose for some $n \in \mathbb{N}_0$, if S is Arf and has $N(S) = (N_0, \ldots, N_{n-1})$, then $S = \mathbb{N}_0^{(n)}$.

Now suppose T is Arf with $N(T) = (N_0, \ldots, N_n)$, then by 9.1.1 $T = S'$ for some Arf numerical semigroup S with $N(S) = (N_1, \ldots, N_n)$. Note that S fits the conditions of our induction hypothesis, so $S = \mathbb{N}_0^{(n)}$, then $T = S' = (\mathbb{N}_0^{(n)})' = \mathbb{N}_0^{(n+1)}$.

So by induction, if S is Arf, then $S = \mathbb{N}_0^{(k)}$ for some $k \in \mathbb{N}_0$.

Indeed, for all k, $\mathbb{N}_0^{(k)}$ is an Arf numerical semigroup by 9.1.1. $\square$

Corollary 9.1.5. *Let S be a numerical set. Then S is an Arf numerical semigroup if and only if $N(S) = (N_0, \ldots, N_k)$ and $M(S) = (1, \ldots, 1)$ and for each $i \in \{0, \ldots, k\}$, $N_i + 1$ is in the semigroup T with $N(T) = (N_{i+1}, \ldots, N_k)$ and $M(T) = (1, \ldots, 1)$.*

The power of Corollary 9.1.5 is that it gives us a way of constructing Arf semigroups sequentially, without referring to the actual semigroup, only the N sequence. To see why, if we wanted an Arf numerical semigroup, we start from $\mathbb{N}_0$ and add any gap length to its N sequence. We underline this mapping of N sequences below. For all of the following, remember that M_i is identically 1.

$$\begin{aligned}
\mathbb{N}_0 &\to (x) \\
(x) &\to (y, x) \text{ where } y \geq x \\
(y, x) &\to (z, y, x) \text{ where } z \geq x + y + 1 \text{ or } z = y \\
(z, y, x) &\to (w, z, y, x) \text{ where } w \geq z + y + x + 2 \text{ or } w = z + y + 1 \text{ or } w = z \\
&\dots
\end{aligned}$$

Any of the above sequences is an N-sequence for an Arf semigroup.

We will now show left extension is a partial ordering on semigroups (providing we are left-extending with elements in the respective semigroups).

Theorem 9.1.6. *The left extension is a partial order on numerical semigroups with $S \preceq T$ if $S = T^{(k)}$ for some $k \in \mathbb{N}_0$.*

Proof. We will show that left-extension is reflexive, anti-symmetric, and transitive.
First, we must show that $\preceq$ is reflexive. So suppose $S \preceq S$, then $S = S^{(k)}$ for some $k \in \mathbb{N}_0$. So $S = S^{(0)} = S$. Therefore $\preceq$ is reflexive.

We will now show left-extension is antisymmetric. We will first prove a Lemma.

Lemma 9.1.7. *If $S \preceq T$, then $S \subseteq T$.*

Proof. We first consider when $S = T'$ so that $S = \{0\} \cup m + T$. Let $s \in S$. If $s = 0$, then we know $0 \in T$ since it is a semigroup, and since $m \in T$ and T is closed under addition, then $s \in m + T$ implies that $s \in T$. This gives us $S \subseteq T$. Now suppose $S = T^k$ for $k \geq 2$. By the previous result, we know $S = (T^{k-1})'$, implies $S \subseteq T^{k-1}$. Since $(T^{k-1}) = (T^{k-2})'$, then again by the last result $T^{k-1} \subseteq T^{k-2}$. Together these give us $S \subseteq T^{k-2}$. This continues inductively, so that we have $S \subseteq T$. $\square$

Now for the proof of antisymmetry. Suppose $S \preceq T$ and $T \preceq S$. Then by our Lemma, $S \subseteq T$ and $T \subseteq S$. This gives us $S = T$ as required.

Suppose S, T, and V are numerical semigroups with $S \preceq T$ and $T \preceq V$. Let $N(V) = (N_0, N_1, \dots, N_j)$. Then $T = V^{(k)}$ for some $k \in \mathbb{N}_0$ so $N(T) = (N_{-k}, N_{-k+1}, \dots, N_{-1}, N_0, N_1, \dots, N_j)$ for some appropriate $N_{-k}, \dots, N_{-1}$. Also, $S = T^{(l)}$ for some $l \in \mathbb{N}_0$ so

$$N(S) = (N_{-k-l}, N_{-k-l+1}, \dots, N_{-k-1}, N_{-k}, N_{-k+1}, \dots, N_{-1}, N_0, N_1, \dots, N_j)$$

and in fact $S = (V^{(k)})^{(l)} = V^{(k+l)}$ and $S \preceq V$. $\square$

Theorem 9.1.8. *The left extension is a graded partial order on numerical semigroups and each numerical semigroup has a unique minimal element that it spawns from.*

Proof. Let T and V be minimal semigroups in the left extension partial order. Let $N(T) = (T_0^N, T_1^N, \ldots, T_x^N)$, $M(T) = (T_0^M, T_1^M, \ldots, T_x^M)$, $N(V) = (V_0^N, V_1^N, \ldots, V_y^N)$, and $M(V) = (V_0^M, V_1^M, \ldots, V_y^M)$. Let S be a semigroup such that $S \preceq T$ and $S \preceq V$. So $S = T^{(k)}$ with $N(S) = (T_{-k}^N, \ldots, T_{-1}^N, T_0^N, \ldots, T_x^N)$ and $S = V^{(l)}$ with $N(S) = (V_{-l}^N, \ldots, V_{-1}^N, V_0^N, \ldots, V_y^N)$. We show $(T_{-k}^N, \ldots, T_{-1}^N, T_0^N, \ldots, T_x^N)$ must equal $(V_{-l}^N, \ldots, V_{-1}^N, V_0^N, \ldots, V_y^N)$.

Here we eliminate the case that V and T have different lengths. Assume to the contrary they don't and without loss of generality, suppose $y > x$. This gives us that $N(V)$ contains $N(T)$, that is, $N(V) = (V_0^N, V_1, V_2^N, \ldots, T_0^N, T_1^N, \ldots, T_x^N)$. However, since V is minimal, then $(V_1, V_2, \ldots, T_0, T_1, \ldots, T_x)$ is not a semigroup. If we tried to left-extend from T to match S, we get $(V_1^N, V_2^N, \ldots, T_0^N, T_1^N, \ldots, T_x^N)$ before reaching S, but since this isn't a semigroup then there is no way to construct S from T. Therefore our assumption that $y > x$ was wrong and $x = y$. Furthermore, this also gives us $l = k$ since $N(S)$ and $M(S)$ must have the same length as $T^{(k)}$ and $V^{(l)}$.

We have $(T_{-k}^N, \ldots, T_{-1}^N, T_0^N, T_1^N, \ldots, T_x^N) = (V_{-k}^N, \ldots, V_{-1}^N, V_0^N, V_1^N, \ldots, V_x^N)$ so $V_i^N = T_i^N$ for all $i \in \{0, \ldots, x\}$ and $N(V) = N(T)$. Similarly, $M(S) = (1, \ldots, 1, T_0^M, \ldots, T_x^M) = (1, \ldots, 1, V_0^M, \ldots, V_x^M)$ so $V_i^M = T_i^M$ for all $i \in \{0, \ldots, x\}$ and $M(V) = M(T)$, so $V = T$.

So the height function h defined by $h(S) = |N(S)| - |N(T)|$ where T is the unique minimal element that S spawns from is well-defined. $\square$

The natural question is then whether it is possible to determine the minimal elements in this partially ordered set of semigroups. We know of one being $\mathbb{N}_0$. The others are given in the following theorem (credit to Maria Amoros for showing us this).

Theorem 9.1.9. *A numerical semigroup S is a non-minimal element of the left-extension poset if and only if it has maximal embedding dimension.*

Proof.
First recall that a necessary and sufficent condition for a semigroup S to have maximal embedding dimension is that for any $x, y \in S$ with neither being 0, then $x + y - m(S) \in S$. The proof hinges on this property.
$\Rightarrow$
Suppose S is a non-minimal element. Then this means $S = T'$ for some T, with T being a semigroup. Let $x, y \in S$ with neither being 0. Since S spawned from T, then $\exists a, b \in T$ such that $x = a + m(S)$ and $y = b + m(S)$. Since T is a semigroup, then $a + b = x + y - 2m(S) \in T$. Again, since $S = T'$, $a + b \in T$ gives us $a + b + m(S) = x + y - m(S) \in S$. This gives the desired implication, so S has maximal embedding dimension.
$\Leftarrow$
Now suppose S is a numerical semigroup with maximal embedding dimension. This means that the set $T = (S - m(S)) \backslash \{-m(S)\}$ is a semigroup. To see why, note that $0 \in T$ since $m(S) \in S$ and T is cofinite is guaranteed because S was cofinite. For $a, b \in T$, then $\exists$ nonzero $f, g \in S$ with $a = f - m(S)$ and $b = g - m(S)$. S having maximal embedding dimension guarantees

$f+g-m(S) \in S$, and therefore $a+b = f+g-2m(S) = (f+g-m(S)) - m(S) \in T$ as long as $f+g-m(S) \neq 0$. This is guaranteed though since $f, g \geq m(S)$, so $f+g-m(S) \geq m(S)$. Recall from 9.0.5, that T is actually a parent of S. Since we now know T is a semigroup, then S is non-minimal. □

Corollary 9.1.10. *The minimal elements of the poset are the semigroups that do not have maximal embedding dimension.*

As an addendum since it may be unclear, we prove the essential result used in Theorem 9.1.9

Theorem 9.1.11. *S is a numerical semigroup with maximal embedding dimension if and only if S is a numerical set with* $\forall$ non-zero $x, y \in S, x+y-m(S) \in S$

Proof. le □

Theorem 9.1.12. *For each $m-1 \in \mathbb{N}$, then the numerical set S given by $N(S) = (m-1, 1, k)$ and $M(S) = (1,1,1)$ is a semigroup for $k \leq m-3$.*

Proof. Let $M(S) = (1,1,1)$ and let $N(S) = (m-1, 1, k)$. Then $S = \{0, m, m+2, m+k+1 \rightarrow\}$. Note, $m+k+1 \leq 2m$ since $k \leq m-3$ and $m+m+2 > 2m$ so S is closed under addition and thus is a semigroup. □

Observation 9.1.13. *The semigroups given in Theorem 9.1.12 are minimal with respect to the left extension partial order provided $k > 1$.*

Theorem 9.1.14. *Let S be a numerical semigroup. Then S lives in a tree of numerical semigroups and has a parent P so that $S = P^{(n)}$ for some $n \in \mathbb{N}_0$. Say $N(P) = (S_0, \ldots, S_k)$ and $N(S) = (S_{-n}, \ldots, S_k)$. The gap set of S is*

$$H(S) = \bigcup_{i=0}^{n} \left\{ \sum_{j=-n}^{-n+i-1} S_j + i + 1, \sum_{j=-n}^{-n+i-1} S_j + i + 2, \ldots, \sum_{j=-n}^{-n+i} S_j + i \right\} \bigcup \left(H(P) + \sum_{j=-n}^{-1} S_j + n \right)$$

Proof. To avoid confusion, we make a note here that a sum in the form of $\sum_m^n = 0$ when $n < m$.

Proven with induction on n. First consider base cases where n=0 and n=1. When n=0,

$$H(S) = \bigcup_{i=0}^{n} \left\{ \sum_{j=-n}^{-n+i-1} S_j + i + 1, \sum_{j=-n}^{-n+i-1} S_j + i + 2, \ldots, \sum_{j=-n}^{-n+i} S_j + i \right\} \bigcup \left(H(P) + \sum_{j=-n}^{-1} S_j + n \right)$$

reduces to

$$\bigcup_{i=0}^{0} \left\{ \sum_{j=-0}^{-0+i-1} S_j + i + 1, \sum_{j=-0}^{-0+i-1} S_j + i + 2, \ldots, \sum_{j=-0}^{-0+i} S_j + i \right\} \bigcup \left(H(P) + \sum_{j=-0}^{-1} S_j + 0 \right)$$

which becomes

$$\{1,2,\ldots,S_0\}\bigcup\left(H(P)\right).$$

Hence the statement holds since the elements of $\{1,2,\ldots,S_0\}$ are contained in the H(P), and clearly $H(P)=H(S)$ since $S=P$. The case of n=1 is handled in Theorem 9.0.10.

Now for an inductive step. Suppose

$$H(P^{(m)})=\bigcup_{i=0}^{m}\left\{\sum_{j=-m}^{-m+i-1}S_j+i+1,\sum_{j=-m}^{-m+i-1}S_j+i+2,\ldots,\sum_{j=-m}^{-m+i}S_j+i\right\}\bigcup\left(H(P)+\sum_{j=-m}^{-1}S_j+m\right)$$

for some $m\in\mathbb{N}_0$ for all valid $P^{(m)}$. Then for all valid $P^{(m+1)}$ we have $P^{(m+1)}=(P^{(m)})'$ for some valid $P^{(m)}$, so by 9.0.10

$$\begin{aligned}
H(P^{(m+1)}) &= \{1,\ldots,S_{-m-1}\}\cup\left(H(P^{(m)})+S_{-m-1}+1\right)\\
&= \{1,\ldots,S_{-m-1}\}\cup\\
&\left(\bigcup_{i=0}^{m}\left\{\sum_{j=-m}^{-m+i-1}S_j+i+1+S_{-m-1}+1,\sum_{j=-m}^{-m+i-1}S_j+i+2+S_{-m-1}+2,\ldots,\sum_{j=-m}^{-m+i}S_j+i+S_{-m-1}+2\right\}\right.\\
&\quad\left.\bigcup\left(H(P)+\sum_{j=-m}^{-1}S_j+m+S_{-m-1}+1\right)\right.\\
&= \{1,\ldots,S_{-m-1}\}\cup\\
&\quad\left(\bigcup_{i=0}^{m}\left\{\sum_{j=-m-1}^{-m+i-1}S_j+i+2,\sum_{j=-m-1}^{-m+i-1}S_j+i+3,\ldots,\sum_{j=-m-1}^{-m+i}S_j+i+1\right\}\right)\\
&\quad\bigcup\left(H(P)+\sum_{j=-m-1}^{-1}S_j+m+1\right)\\
&= \{1,\ldots,S_{-m-1}\}\cup\\
&\quad\left(\bigcup_{i=1}^{m+1}\left\{\sum_{j=-m-1}^{-m+i-2}S_j+i+1,\sum_{j=-m-1}^{-m+i-2}S_j+i+2,\ldots,\sum_{j=-m-1}^{-m+i-1}S_j+i\right\}\right)\\
&\quad\bigcup\left(H(P)+\sum_{j=-m-1}^{-1}S_j+m+1\right)\\
&= \bigcup_{i=0}^{m+1}\left\{\sum_{j=-m-1}^{-m+i-2}S_j+i+1,\sum_{j=-m-1}^{-m+i-2}S_j+i+2,\ldots,\sum_{j=-m-1}^{-m+i-1}S_j+i\right\}\\
&\quad\bigcup\left(H(P)+\sum_{j=-m-1}^{-1}S_j+m+1\right)\\
&= \bigcup_{i=0}^{m+1}\left\{\sum_{j=-(m+1)}^{-(m+1)+i-1}S_j+i+1,\sum_{j=-(m+1)}^{-(m+1)+i-1}S_j+i+2,\ldots,\sum_{j=-(m+1)}^{-(m+1)+i}S_j+i\right\}\\
&\quad\bigcup\left(H(P)+\sum_{j=-(m+1)}^{-1}S_j+(m+1)\right)
\end{aligned}$$

Therefore, for all $n \in \mathbb{N}_0$,

$$H(P^{(n)}) = \bigcup_{i=0}^{n} \left\{ \sum_{j=-n}^{-n+i-1} S_j + i + 1, \sum_{j=-n}^{-n+i-1} S_j + i + 2, \ldots, \sum_{j=-n}^{-n+i} S_j + i \right\} \bigcup \left(H(P) + \sum_{j=-n}^{-1} S_j + n \right).$$

□

Corollary 9.1.15. *Let S be a numerical semigroup with parent P, i.e. $S = P^{(k)}$ where $N(P) = (N_0, N_1, \ldots, N_l)$ and $N(S) = (N'_k, N'_{k-1}, \ldots, N'_1, N_0, \ldots, N_l)$, then S is a n-core if and only if*

$$n \in \bigcup_{j=1}^{k} \left\{ \sum_{i=0}^{j-1} N'_{k-i} + j \right\} \bigcup \left(Core(P) + \sum_{i=1}^{k} N'_i + k \right)$$

where $Core(P) = \{p \in \mathbb{N}_0 \mid P \text{ is a } p-core\}$.

Proof. Note, $Core(S) = \{y \in S : y < F(S)\} = S \cap \{1, \ldots, F(S)\}$. We have $F(S) = \sum_{i=1}^{k} N'_i + k + F(P)$, so

$$S = \{0, N'_k + 1, N'_k + 1 + N'_{k-1} + 1, \ldots, N'_k + 1 + \cdots + N'_1 + 1\} \cup \left(P + \sum_{i=1}^{k} N'_i + k \right)$$

so

$$\begin{aligned} S \cap \{1, \ldots, F(P) + \sum_{i=1}^{k} N'_i + k\} &= \bigcup_{j=1}^{k} \left\{ \sum_{i=0}^{j-1} N'_{k-i} + j \right\} \cup \left(\left(P + \sum_{i=1}^{k} N'_i + k \right) \cap \left\{ 1, \ldots, F(P) + \sum_{i=1}^{k} N'_i + k \right\} \right) \\ &= \bigcup_{j=1}^{k} \left\{ \sum_{i=0}^{j-1} N'_{k-i} + j \right\} \cup \\ &\quad \left(\left(P + \sum_{i=1}^{k} N'_i + k \right) \cap \left\{ \sum_{i=1}^{k} N'_i + k, \ldots, F(P) + \sum_{i=1}^{k} N'_i + k \right\} \right) \\ &= \bigcup_{j=1}^{k} \left\{ \sum_{i=0}^{j-1} N'_{k-i} + j \right\} \cup \left((P \cap \{0, \ldots, F(P)\}) + \sum_{i=1}^{k} N'_i + k \right) \\ &= \bigcup_{j=1}^{k} \left\{ \sum_{i=0}^{j-1} N'_{k-i} + j \right\} \cup \left((P \cap \{1, \ldots, F(P)\}) + \sum_{i=1}^{k} N'_i + k \right) \\ &= \bigcup_{j=1}^{k} \left\{ \sum_{i=0}^{j-1} N'_{k-i} + j \right\} \cup \left(Core(P) + \sum_{i=1}^{k} N'_i + k \right) \end{aligned}$$

□

Corollary 9.1.16. *If S is an Arf numerical semigroup with $N(S) = (N'_k, \ldots, N'_1)$, then*

$$H(S) = \bigcup_{i=0}^{k} \left\{ \sum_{j=k+1-i}^{k} N'_j + i + 1, \sum_{j=k+1-i}^{k} N'_j + i + 2, \ldots, \sum_{j=k-i}^{k} N'_j + i \right\}$$

and $Core(S) = \bigcup_{j=1}^{k} \left\{ \sum_{i=0}^{j-1} N'_{k-i} + j \right\}$.

Proof. By 9.1.4, $S = \mathbb{N}_0^{(k)}$ for some $k \in \mathbb{N}_0$. Since, $H(\mathbb{N}_0) = \emptyset$ and $Core(\mathbb{N}_0) = \emptyset$, the result follows immediately from 9.1.14 and 9.1.15. □

9.2 2 generated Symmetric non-maximum embedding dimension with $m > 2$ M & N

We shift our attention to two generated sequences with multiplicity greater than 2. This way, we guarantee that they have non-maximal embedding dimension and they are roots in the semigroup forest.

For S a numerical semigroup with multiplicity m, suppose we are given M and N sequences $M(S) = (M_0, M_1, \ldots, M_k)$ and $N(S) = (N_0, N_1, \ldots, N_k)$. For a semigroup, we always know $M_0 = 1$ and $N_0 = m - 1$. For an indefinite amount of time, we can have that this pattern will continue. That is, for some maximal $j \in \mathbb{N}$,

$$\text{(property 1) } N_i = m - 1 \text{ and } M_i = 1 \text{ for } i < j$$
$$\text{(property 2) } N_i = m - 1 \text{ for } i < j \text{ and } M_i = 1 \text{ for } i \leq j$$

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This pattern must eventually terminate for every semigroup, whether it be because the M and N elements change or because the sequences terminate. In both of these cases, the pattern terminates because of the introduction of non-multiplicity generators. When only one new generator is allowed to be introduced, the M and N sequence is then completely determined. For all of the following, suppose property 1 holds for j, and j is the maximal such choice. We will often say that either M_j or N_j is the first to "break the pattern" to refer to property 1. We will also have to split the cases where M breaks the pattern or to when N breaks the pattern.

We will first consider the cases where M breaks first. That is, M_j is the first M not equal to 1, and $N_i = m - 1$ identically for $i < j$.

For the proof below, recall that $\{M_k\} = \{J, J+1, J+2, \ldots, J+M_k-1\}$ where $J = \sum_{i=0}^{k-1}(M_i + N_i)$

Lemma 9.2.1. *Suppose S is a numerical semigroup, and suppose we have for $j > 1$, M_j is the first M not equal to 1, and $N_i = m - 1$ identically for $i < j$. Then $\{M_j\}$ contains $M_j - 1$ generators.*

Proof. We first show that for $i \leq j$, $\{M_i\}$ contains im. This is because for $i \leq j$, then the M's are identically 1 before and the N's are identically m-1, so $\sum_{k=0}^{i-1} N_k + M_k = im$. From our hypothesis, since the pattern breaks at M we know $M_j \neq 1$, so there are M_j elements in $\{M_j\}$. We have just shown $jm \in \{M_j\}$, and $j > 1$ gives us its not a generator. The other elements of $\{M_j\}$, namely, $jm+1, jm+2, \ldots, jm+M_j-1$, are atoms since the only numbers preceding them are multiples of m, and therefore none of them can be combinations of smaller elements. Thus since all of $jm+1, jm+2, \ldots, jm+M_j-1$ are generators, there are $M_j - 1$ generators in $\{M_j\}$. □

In the case where j=1, it is not hard to see that there M_1 generators in $\{M_1\}$ since we now include the multiplicity.

Corollary 9.2.2. *If S is two generated, $M_j = 2$.*

Proof. If $M_j > 2$, then in light of 9.2.1, then there are at least 3 generators of S, but clearly this is contradiction. So $M_j = 2$. □

Lemma 9.2.3. *Let S be a numerical semigroup with $N(S) = (m-1, m-1, \ldots, m-1, N_j, N_{j+1}, \ldots, N_k)$ and $M(S) = (1, 1, \ldots, 1, M_j, M_{j+1}, \ldots, M_k)$ where $M_j \neq 1$. Then $N_j \neq m-1$.*

Proof. By 8.2.6, $M_j + N_j \leq m$, so since $M_j > 1$, $N_j \leq m - M_j < m - 1$. □

Lemma 9.2.4. *Let S be a two generated numerical semigroup with multiplicity $m \geq 3$ and $N(S) = (m-1, m-1, \ldots, m-1, N_j, N_{j+1}, \ldots, N_k)$ and $M(S) = (1, 1, \ldots, 1, M_j, M_{j+1}, \ldots, M_k)$ where $N_j \neq m-1$. Then the second generator of S is $g_2 = mj + 1$.*

Proof. In light of Theorem 9.2.1, then we know that the second generator is contained in $\{M_j\}$. Also, since S is two generated, then we know $M_j = 2$. Hence, since M_j breaks the pattern, $M_i = 1$ and $N_i = m - 1$ for all $i < j$, and $\sum_{k=0}^{j-1} N_k + M_k = (j-1)m + j = jm$ is the multiple of of m in $\{M_j\}$, so then $g_2 = jm + 1$. □

Corollary 9.2.5. *If S is semigroup with the M breaking the pattern, then for the second generator g_2, then $g_2 \equiv 1 \mod m(S)$.*

Lemma 9.2.6. *Let S be a two generated numerical semigroup as in the statement of Lemma 9.2.4. Then $F(S) = jm^2 - jm - 1$.*

Proof. By [5] if a numerical semigroup is two generated with generators s_1 and s_2, then $F(S) = s_1 s_2 - s_1 - s_2 - 1$. In our case, $s_1 = m$ and $s_2 = g_2 = mj + 1$, so

$$F(S) = (m)(mj + 1) - m - (mj + 1) - 1 = jm^2 + m - m - mj - 1 = jm^2 - jm - 1.$$

□

Lemma 9.2.7. *If S is a 2 generated numerical semigroup with multiplicity $m \geq 3$, and $N(S) = (m-1, m-1, \ldots, m-1, N_j, N_{j+1}, \ldots, N_k)$ and $M(S) = (1, 1, \ldots, 1, M_j, M_{j+1}, \ldots, M_k)$ where $M_j \neq 1$, then $Ap(S) = \{0, mj + 1, 2mj + 2, \ldots, (m-1)mj + (m-1)\}$.*

Proof. By 9.2.4, the second generator of S is $g_2 = mj + 1$. Suppose towards contradiction that there exists an $x \in Ap(S)$ such that $g_2 \not| x$. Then we must have $x = am + bg_2$ where $a \neq 0$, $a, b \in \mathbb{N}_0$. Notice that then $x - m = (a-1)m + bg_2 \in \langle m, g_2 \rangle = S$ since $a - 1 \geq 0$ so $a - 1, b \in \mathbb{N}_0$. This is a contradiction because since $x \in Ap(S)$ we must have $x - m \notin S$. Thus, everything in $Ap(S)$ is a multiple of g_2. So in particular, $Ap(S) = \{0, g_2, 2g_2, \ldots, (m-1)g_2\} = \{0, mj+1, 2mj+2, \ldots, (m-1)mj+(m-1)\}$. □

Theorem 9.2.8. *If $S = \langle m, q \rangle$ such that $q \equiv 1 \mod m$ and $m \geq 3$, then*

$$M(S) = (1, 1, \ldots, 1, 2, 2, \ldots, 2, \ldots, m-1, m-1, \ldots, m-1)$$

where the amount of repetition of each single element is the solution, j, for the equation $q = jm + 1$.

Proof. We have $q = jp + 1$ for some $j \in \mathbb{N}$. By 9.2.7, we know that the $Ap(S) = \{0, q, 2q, \ldots, (p-1)q\} = \{0, jp+1, 2jp+2, \ldots, (p-1)jp + (p-1)\}$. Until we reach q we only have multiples of p, so we have j steps with width one in the begining of $M(S) = (M_0, \ldots, M_k)$, i.e. $M(S) = (1, 1, \ldots, 1, M_j, M_{j+1}, \ldots, M_k)$. On the j^{th} step we have jm and $jm+1$ and we do not have $jm+2$ because we have nothing congruent to 2 modulo m until we reach the element of the Apery set congruent to 2, so until we reach $2jm + 2$. After $jm + 1$ we have everything congruent to 1 modulo m, so each of the j steps until the step with $2jm + 2$ has step width 2. So $M(S) = (1, 1, \ldots, 1, 2, 2, \ldots, 2, M_{2j}, M_{2j+1}, \ldots, M_k)$. Then after $2jm + 2$ we have everything congruent to 2 modulo m, but we do not have anything congruent to 3 until we reach $3jm + 3$, so the next j steps all have step width 3. This process continues until we have m elements in a row, so then we have everything after that point is in S. So, we have $M(S) = (1, 1, \ldots, 1, 2, 2, \ldots, 2, \ldots, m-1, m-1, \ldots, m-1)$ where there are p steps of each width. Since S is symmetric, we have that $N(S) = (m-1, m-1, \ldots, m-1, m-2, \ldots, m-2, \ldots, 1, \ldots, 1)$. ☐

Corollary 9.2.9. *Let S be a two-generated numerical semigroup with multiplicity $m \geq 3$ and $N(S) = (m-1, m-1, \ldots, m-1, N_j, N_{j+1}, \ldots, N_k)$ and $M(S) = (1, 1, \ldots, 1, M_j, M_{j+1}, \ldots, M_k)$ where $M_j \neq 1$, then $N_i + M_i = m$ for all $i \in \{0, \ldots, k\}$.*

Proof. By 9.2.8, $M(S) = (1, 1, \ldots, 1, 2, 2, \ldots, 2, \ldots, m-1, m-1, \ldots, m-1)$ and $N(S) = (m-1, m-1, \ldots, m-1, m-2, m-2, \ldots, m-2, \ldots, 1, 1, \ldots, 1)$. So $N_i + M_i = m$ for each i. ☐

Corollary 9.2.10. *Let S be a two generated numerical semigroup with multiplicity $m \geq 3$ and $N(S) = (m-1, m-1, \ldots, m-1, N_j, N_{j+1}, \ldots, N_k)$ and $M(S) = (1, 1, \ldots, 1, M_j, M_{j+1}, \ldots, M_k)$ with $M_j \neq 1$. Then $k = (m-1)j - 1$.*

Proof. By 9.2.6, $F(S) = jm^2 - jm - 1$ so the conductor of S is $jm^2 - jm = jm(m-1) = \sum_{i=0}^{k}(M_i + N_i)$. By 9.2.9, $M_i + N_i = m$ for all i so $\sum_{i=0}^{k}(M_i + N_i) = (k+1)m = jm(m-1)$. Then $k + 1 = j(m-1)$ and $k = j(m-1) - 1$. ☐

Now for when N_j is the first to break. That is, when we are working with M and N sequences with property 2.

Theorem 9.2.11. *If S is a numerical semigroup such that M and N obey* property 2 *up until N_j, then $g_2 = jm + N_j + 1$.*

Proof. We can locate g_2 since we know it is the first element of $\{M_{j+1}\}$(if this is unclear, argue how for $i \leq j$ that $\{M_i\}$ contains only a multiple of m). Hence, $g_2 = \sum_{i=0}^{j}(M_i + N_i) = (j+1) + ((m-1)j + N_j) = mj + N_j + 1$. ☐

Corollary 9.2.12. *If a semigroup obeys property 2, then $g_2 \equiv N_j + 1 \mod m(S)$.*

Corollary 9.2.13. *If S is a two generated semigroup with property 2, then $2 \leq N_j + 1 \leq m - 1$ and $N_j + 1 \nmid m$.*

Proof. We know $1 \leq N_j \leq m-1$ since S is a semigroup and due to 6.0.4. This gives us $2 \leq N_j + 1 \leq m$. Suppose $N_j + 1 = m$. Then the previous corollary gives $g_2 \equiv 0 \mod m$, but this means g_2 is a multiple of m and thus not a generator. Hence $2 \leq N_j + 1 \leq m - 1$.

Now suppose towards a contradiction that S is two generated and $N_j + 1 \mid m$. Then $g_2 = jm + N_j + 1 = jk(N_j + 1) + N_j + 1 = (N_j + 1)(jk + 1)$. In particular, $N_j + 1 \mid g_2$, but since $N_j + 1 \neq 1$, then this is contradiction to m and g_2 being relatively prime. Thus for the two generated case, $N_j + 1 \nmid m$. $\square$

Conjecture 9.2.14. *Suppose S is the semigroup generated by m and $jm + r$ where $j \in \mathbb{N}_0$ and $r \in \{1, \ldots, m-1\}$.*

- *If $r = 1$, $|N(S)| = |M(S)| = (m-1)j$.*
- *If $r = m-1$, $|N(S)| = |M(S)| = (m-1)j + (m-2)$.*
- *If $r = 2$, $|N(S)| = |M(S)| = \left(\frac{m-1}{2}\right)\left(\frac{m+1}{2}\right)j + \left(\frac{m-1}{2}\right)$.*
- *If $r = m-2$, $|N(S)| = |M(S)| = \left(\frac{m-1}{2}\right)\left(\frac{m+1}{2}\right)j + \left(\frac{m-1}{2}\right)\left(\frac{m+3}{2}\right)$.*
- *If $r = 3$,*
  - *If $m \equiv 1 \mod 3$, $|N(S)| = |M(S)| = \left(\frac{m-1}{3}\right)\left(\frac{2m+1}{3}\right)j + \left(\frac{2m-2}{3}\right)$.*
  - *If $m \equiv 2 \mod 3$, $|N(S)| = |M(S)| = \left(\frac{m+1}{3}\right)\left(\frac{2m-1}{3}\right)j + \left(\frac{2m-1}{3}\right)$.*
- *If $r = m-3$,*
  - *If $m \equiv 1 \mod 3$, $|N(S)| = |M(S)| = \left(\frac{m-1}{3}\right)\left(\frac{2m+1}{3}\right)j + \frac{\left(\frac{m-1}{3}\right)\left(\frac{m+8}{3}\right)}{2}$.*
  - *If $m \equiv 2 \mod 3$, $|N(S)| = |M(S)| = \left(\frac{m+1}{3}\right)\left(\frac{2m-1}{3}\right)j + \frac{\left(\frac{m-1}{3}\right)\left(\frac{m+10}{3}\right)}{2}$.*
- *If $r = 4$,*
  - *If $m \equiv 1 \mod 4$, $|N(S)| = |M(S)| = \left(\frac{m-1}{4}\right)\left(\frac{3m+1}{4}\right)j + \left(\frac{3m-3}{4}\right)$.*
  - *If $m \equiv 3 \mod 4$, $|N(S)| = |M(S)| = \left(\frac{m+1}{4}\right)\left(\frac{3m-1}{4}\right)j + \left(\frac{3m-1}{4}\right)$.*
- *If $r = m-4$,*
- *If $r = 5$,*
- *If $r = m-5$,*

Conjecture 9.2.15. *Suppose S is the semigroup generated by m and $jm + r$ where $j \in \mathbb{N}_0$ and $r \in \{1, \ldots, m-1\}$.*

- *If $m \equiv 1 \mod r$, then $|N(S)| = |M(S)| = \left(\frac{m-1}{r}\right)\left(\frac{(r-1)m+1}{r}\right)j + \left(\frac{(r-1)m-(r-1)}{r}\right)$*
- *If $m \equiv 1 \mod m-r$, then $|N(S)| = |M(S)| = \left(\frac{m-1}{m-r}\right)\left(\frac{(m-r-1)m+1}{m-r}\right)j + \frac{\left(\frac{m-1}{m-r}\right)\left(\frac{m+(m-r+1)(m-r-1)}{m-r}\right)}{m-r-1}$*

10 Conclusion

In this paper, we continued investigation into the complements of numerical sets and semigroups. Through this investigation we discovered exactly when S and $\tilde{S}$ are both numerical semigroups and characterized what the associated numerical semigroups are when only one is a numerical semigroup. This investigation gave rise to the M and N sequences which we were able to use to characterize a semigroup or set and find some of their connections to other invarients of numerical semigroups. M and N sequences also discovered a partial ordering on the set of all numerical semigroups while exploring the M and N sequences. This partial ordering created a forest that contains all numerical semigroups.

One open question we have deals with how the associated numerical semigroups of S and $\tilde{S}$ relate when they both start as numerical sets. This turned out to be a hard question to answer. In Example 7.2.8 we showed that when you fix the associated numerical semigroup of S, there is a wide variety of what the associated numerical semigroup of $\tilde{S}$ is. If you begin by fixing the associated numerical semigroup of $\tilde{S}$, there is even more variation because of the column and row extensions defined in section 9.

Our results have also inspired some question on the M and N sequences. For example, is there a way to tell if an M and N sequences correspond to a numerical set or semigroup definitively. We were able to classify a few necessary conditions on M and N for them to correspond to semigroups but they weren't sufficient conditions. In addition, how do our sequences relate to other invariants, such as the Apery Set, the embedding dimension, etc. Can that information of the numerical semigroup be extracted from just the M & N sequences. We also questioned whether you could classify the *Bras-Amoros Tree* with our sequences. There seems to be a pattern to how the sequences change while traversing the tree but it would need further investigation.

11 Miscellaneous

This is a section with somewhat random thoughts and possible ideas to pursue in the future. It probably does not follow any sort of logical progression.

Other questions:

- If $A(\tilde{S})$ is a column and $M_0 = 1$ and $N_i \leq m(S) - M_i$ for all i, is S necesadiky a semigroup? If not, what other restrictions do we need to put to make it a semigroup.
- If we left extend two things to the M and N sequences when do we get a semigroup?
- Look back at trees
- Can the family of sets with A(S) a column be grouped in a meaningful way (i.e. the complements of semigroups, other, and another).

11.1 Classifying Numerical Set Maps

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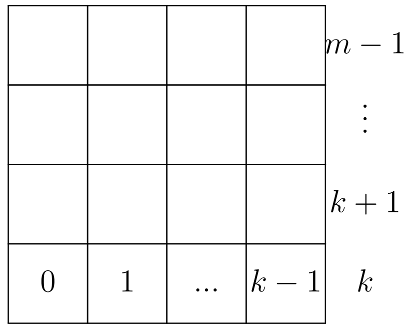
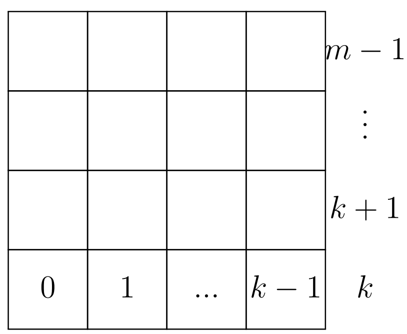
Theorem 11.1.1. *For all $m \in \mathbb{N}$ and for all k with $1 \leq k \leq m-1$ the numerical set $T = \{0, 1, \ldots, k-1, m, m+1, m+2, \ldots\}$ if and only if the complement partition of T is empty, so if and only if $\tilde{T} = \mathbb{N}_0$. Additionally, the numerical set T corresponds to the numerical semigroup $A(T) = \{0, m, m+1, m+2, \ldots\}$.*

Proof. Given $T = \{0, 1, .., k-1, m, \rightarrow\}$, $\lambda(T)$ corresponds to a young diagram that has the numbers 0 through $k-1$, doesn't include k through $(m-1)$, finally includes every number greater than m.



By definition $\tilde{\lambda} = \{r_1 - r_i : r_1 > r_i \;\; \& \;\; 1 < i \leq t\}$ where r_1 is the first and largest row in the partition and r_i is every sequential row less than r_1. Our λ is a rectangle where each row is equal to every other row. Therefore $\tilde{\lambda} = \emptyset$, because r_1 is never greater than r_i and our corresponding numerical set is $\mathbb{N}_0$.

If $\tilde{T} = \mathbb{N}_0$, then $\lambda(T)$ must look like



for some $k, m \in \mathbb{N}$ so $T = \{0, 1, \ldots, k-1, m \rightarrow\}$.

Now we show $A(T) = \{0, m, m+1, \ldots\}$ Let $i \in \{j \mid 1 \leq j < m\}$. There are then two cases.

Suppose $1 \leq i \leq k-1$. Then note that $k-(k-1) = 1 \leq k-i \leq k-1$, so $k-i \in$ T. Assume to the contrary that $i \in A(T)$. Then we must have $i+T \subseteq T$. However, since $k-i \in T$, then $k-i+i = k$ must be in $i+T$, but this is contradiction to the fact that $T = \{0, 1, .., k-1, m, \rightarrow\}$ (in particular, $k \notin T$). Hence any $i \in \{j \mid 1 \leq j \leq k-1\}$ must not be in $A(T)$.

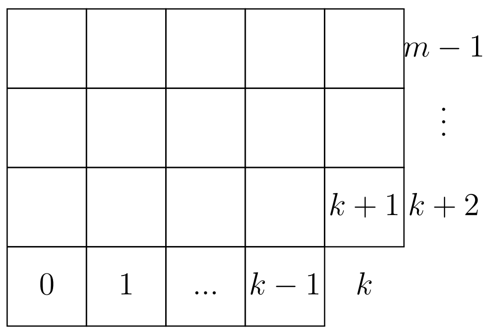
Now suppose $k \leq i \leq m-1$. Clearly $i \notin T$. Since $0 \in T$, then and with $i+0 = i \notin T$ then we conclude $i+T \nsubseteq T$, and no such i can be in $A(T)$.

It is straightforward to show $0 \in A(T)$. If $i \geq m$, then $i \in A(T)$ by definition of T. So we conclude $A(T) = \{0, m, m+1, \ldots\}$. $\square$

Theorem 11.1.2. *For all $m \in \mathbb{N}$, $m \geq 5$. For all k with $1 \leq k \leq m-3$. Consider the numerical set $T = \{0, 1, \ldots k-1, k+1, m \rightarrow\}$. The complement of this numerical set T is the numerical set $\{0, 2 \rightarrow\}$ and the numerical semigroup associated to this complement is also $\{0, 2 \rightarrow\}$. Additionally, the numerical set T corresponds to the numerical semigroup $A(T) = \{0, m \rightarrow\}$.*

Proof. For $1 \leq i \leq k-1$, $k-(k-1) = 1 \leq k-i \leq k-1$ so $k-i \in T$ and $i+k-i = k \notin T$. Thus, $i \notin A(T)$. Note $(k+1)+1 = k+2 \notin T$ so also $k+1 \notin A(T)$. For $k \leq j \leq m-1$, $j+0 = j \notin T$ so $j \notin A(T)$. However, for $n \geq m$, $n+t \geq m+0 = m \in T$ for every $t \in T$, so $n+T \subseteq T$ and $n \in A(T)$. Thus, $A(T) = \{0, m \rightarrow\}$.

Notice that the Young diagram of T would be k steps to the right to include $0, \ldots, k-1$, one step up to exclude k, one step right to include $k+2$, and $m-k-2$ steps up to exclude $k+2, \ldots, m-1$. So the Young diagram of T looks as below:



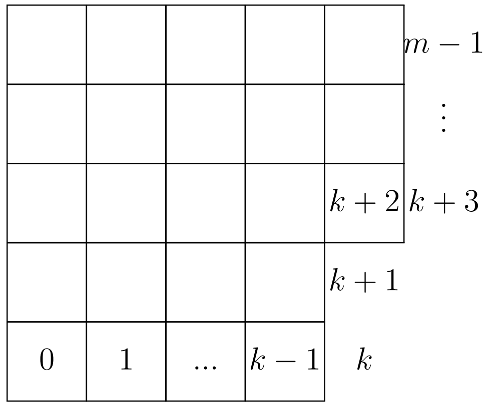
Then the first $m-k-2$ rows of the partition of T have row length $k+1$ and the $m-k-1^{\text{th}}$ row has length k, so the complement partition of T is $\tilde{\lambda} = \{k+1-k\} = \{1\}$ and the associated numerical set is $\{0, 2 \rightarrow\}$. The numerical set $\{0, 2 \rightarrow\}$ is also a numerical semigroup, so the semigroup associated to this set is also $\{0, 2 \rightarrow\}$. $\square$

Lemma 11.1.3. *Consider the numerical set $T = \{0, 1, \ldots, k-1, k+2, m, \rightarrow\}$. For $m < 6$, there exists no numerical set, T, that corresponds to $\{0, m, \rightarrow\}$.*

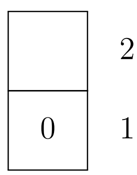
Proof. Consider the case where $m = 1$. Our $T = \{0, 1, \rightarrow\}$, so we have no space in between 0 and m to place our k's. Now consider $m = 2$, our $T = \{0, .., 2, \rightarrow\}$, this forces our $k = 1$, and by the structure of out T, 2 would have to be out of the set because it is $k+1$. Similarly for $m = 3$, our $T = \{0, .., 3, \rightarrow\}$. In this case our $k = 1$ or $k = 2$. If $k = 1$, then our $k+2 = 3 = m$ and our $k+3 = 4$ has to be excluded from the numerical set. But this is a contradiction because we have to include everything greater than 4. If $k = 2$, $k+1$ has to be excluded but $k+1 = 3 = m$. If you have $m = 4$, you can have $k = 1, 2, 3$. For all of these you end up excluding a number that is required in the set when you begin to construct T. Finally for $m = 5$, you can have $k = 1, 2, 3, 4$. And similarly, the structure of our T forces a number to be excluded that should not be when any of those k are selected. $\square$

Theorem 11.1.4. *For $m > 5$ and $1 \leq k \leq m-4$ consider the numerical set $T = \{0, 1, \ldots, k-1, k+2, m \rightarrow\}$. This numerical set T corresponds to the semigroup $A(T) = \{0, m \rightarrow\}$ and the complement of T corresponds to the semigroup $\{0, 3 \rightarrow\}$.*

Proof. To create the Young diagram of T we go right k spots to include $0, \ldots k-1$, up two to exclude k and $k+1$, right one to include $k+2$, and up $m-k-3$ to exclude $k+3, \ldots, m-1$. Below is the Young diagram of T:



Notice from the above depiction that the first $m-k-3$ rows of the Young diagram of T have length $k+1$ while the next and last two rows have length k, so the complement partition of T is $\tilde{\lambda} = \{k+1-k, k+1-k\} = \{1, 1\}$ and has diagram as depicted below:



So the numerical semigroup corresponding to the complement of T is $\{0, 3 \rightarrow\}$.

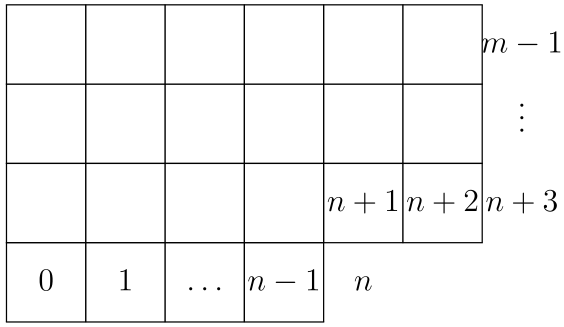
Here we show $A(T) = \{0, m \rightarrow\}$. Note that the proof for $i \in \{j \mid 0 \leq j \leq m-1\}$ given in Theorem 11.1.2 applies in this case as well, with the only difference being for $i = k+2$. We treat that case here.

If $k > 1$, then since $k+2+1 = k+3 \notin T$, then $(k+2) + T \nsubseteq T$. This along with the previous reasoning would give us $A(T) = \{0, m, \rightarrow\}$. All that remains is if $k = 1$. However, this gives us that $T = \{0, 3, m, \rightarrow\}$. Since $m > 6$, this guarantees that $3 + 3 = 6 \notin T$ and so $k+2 \notin A(T)$.

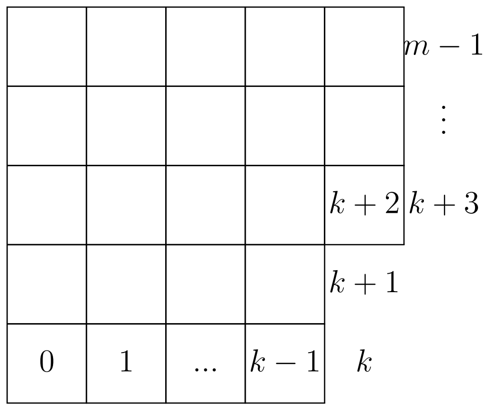
Hence, we conclude $A(T) = \{0, m, \rightarrow\}$. $\square$

Theorem 11.1.5. *Let $m > 5$, and let $1 \leq n \leq m-4$. Consider the numerical set $T = \{0, 1, \ldots, n-1, n+1, n+2, m \rightarrow\}$. This numerical set corresponds to the semigroup $A(T) = \{0, m \rightarrow\}$ and the complement of T corresponds to the semigroup $\{0, 3 \rightarrow\}$.*

Proof. Let $\lambda(T) = \lambda$ be the partition associated to the numerical semigroup T, λ is depicted below:



The conjugate partition of λ, $\lambda^* = \{(j,i) : (i,j) \in \lambda\}$ is depicted below. Let $k = m - n - 3$:



Note $1 = m-(m-4)-3 \leq k \leq m-1-3 = m-4$, so the numerical set corresponding to λ^* is of the form $T' = \{0, 1, \ldots, k-1, k+2, m \rightarrow\}$. By Theorem 11.1.4, $A(T') = \{0, m \rightarrow\}$ so by Propositon 8.0.11 $A(T) = \{0, m \rightarrow\}$. Also, note $\tilde{\lambda} = \{2\}$ and $(\widetilde{\lambda^*}) = \{1,1\} = (\tilde{\lambda})^*$, so again by Theorem 11.1.4 and Proposition 8.0.11 the complement numerical set of T corresponds to the numerical semigroup $\{0, 3 \rightarrow\}$. $\square$

Theorem 11.1.6. *Let $m > 6$ and $1 \leq k \leq m-5$. Consider $T = \{0, 1, \ldots, k-1, k+1, k+3, m \rightarrow\}$. Then $A(T) = \{0, m \rightarrow\}$ and the complement numerical set of T corresponds to the numerical semigroup $\{0, 2, 4 \rightarrow\}$.*

11.2 Looking at other families

This section will use Theorem 6.0.11 to analyze when semigroups that are members of other families are have that their complement is also a semigroup.

11.2.1 pseudo-arithmetic

Definition 11.2.1. *We call a numerical semigroup pseudo-arithmetic if $S = \langle a, ha + d, ha + 2d, \ldots, ha + kd \rangle$ for some $a \in \mathbb{N}_0$, $h \in \mathbb{N}$, $1 \leq k \leq a-1$, and $d \in \mathbb{Z} \backslash \{0\}$. [reference to Geogroup Union]*

Proposition 11.2.2. *Let S be a pseudo-arithmetic semigroup, i.e. $S\langle a, ha+d, ha+2d, \ldots, ha+kd\rangle$ for some $a \in \mathbb{N}$, $h \in \mathbb{N}$, $1 \leq k \leq a-1$, and $d \in \mathbb{Z}\backslash\{0\}$. Then $\tilde{S}$ is a semigroup if and only if $k = a-1$ and $d = 1$ or $a = 3$, $d = 2$, and $k = a-1$.*

Proof. Suppose S is a pseudo-arithmetic semigroup and $\tilde{S}$ is a semigroup. Then by 6.0.11, S is a truncated n-staircase with l steps for some $n, l \in \mathbb{N}$. So either $S = \langle n, ln+1, ln+2, \ldots, ln+a-1\rangle$ in which case $a = n$, $d = 1$, and $k = a-1$. Or

$$S = \langle n, ln-j, ln-j+1, \ldots, ln-1, ln+1, ln+2, \ldots, (l+1)n-j-1\rangle$$

for $j \in \mathbb{N}$, $j \neq 0$. We would still have $a = n$ and $k = n-1$, but now the difference between $ln-1$ and $ln+1$ is $d = 2$ but the difference between $ln-j$ and $ln-j+1$ is $d = 1$ so this only works when $n = a = 3$, $d = 2$, $k = n-1$. Thus, if S is an arithmetic semigroup with $\tilde{S}$ a semigroup if and only if $k = a-1$ and $d = 1$ or $a = 3$, $d = 2$, and $k = a-1$. □

11.2.2 Geometric Semigroups

Definition 11.2.3. *We call a numerical semigroup S geometric if $S = \left\langle a, ar, ar^2, \ldots, ar^k\right\rangle$ for some $a \in \mathbb{N}$, $r \in \mathbb{Q}^+\backslash\{0\}$, $1 \leq k \leq a-1$, and $ar, ar^2, \ldots, ar^k \in \mathbb{N}$.*

Proposition 11.2.4. *A geometric numerical semigroup S has $\tilde{S}$ is a semigroup if and only if S is a one or two staircase.*

Proof. $\Leftarrow$ If S is a one staircase (i.e. $S = \langle 1\rangle$) then clearly letting $a = 1$ and $r \in \mathbb{Q}^+\backslash\{0\}$ would show S is a geometric numerical semigroup. If S is a two staircase, then $S = \langle 2, j\rangle$ for some odd integer j, (in particular, k=B(S)+3). Let $a = 2$ and $r = j/2$, will produce S as a geometric semigroup.

$\Rightarrow$. It now suffices to show that no other instance of a geometric semigroup can be a truncated n-staircase. First note that since $ar^k \in \mathbb{Z} \forall 0 \leq k \leq a-1$, then their differences must also be integers. Or $ar^{k+1} - ar^k = ar^k(r-1) = c$ for $c \in \mathbb{Z}$. Since $ar^k > 0$, then (wait this is circular). First consider the case where there are at least three minimal generators. That is, $S = \langle a, ar, ar^2, \ldots\rangle$. If we consider the consecutive differences between the two generators $arandar^2$, then since they are both integers, their difference must be an integer. Hence □

11.2.3 pseudo-symmetric

Definition 11.2.5. *We call a semigroup S pseudo-symmetric if*

$$S = \left\langle \frac{p_1p_2\cdots p_k}{p_1}, \frac{p_1p_2\cdots p_k}{p_2}, \ldots, \frac{p_1p_2\cdots p_k}{p_k}\right\rangle$$

for some relatively prime $p_1, \ldots, p_k \in \mathbb{N}$.

Proposition 11.2.6. *The only pseudo-symmetric semigroup S with $\tilde{S}$ a semigroup is $S = \langle 1\rangle$.*

Proof. Without loss of generality, let $p_1 > p_2 > \cdots > p_k$ so $\frac{p_1p_2\cdots p_k}{p_1} < \frac{p_1p_2\cdots p_k}{p_2} < \ldots < \frac{p_1p_2\cdots p_k}{p_k}$. The semigroup S has $\tilde{S}$ is a semigroup if and only if S is a truncated staircase by 6.0.11. Then for all $i \in \{2,\ldots,k\}$, $\frac{p_1p_2\cdots p_k}{p_{i+1}} - \frac{p_1p_2\cdots p_k}{p_i} = q$ where $q = 1, 2$. So

$$\begin{aligned} p_i(p_1p_2\cdots p_k) - p_{i+1}(p_1p_2\cdots p_k) &= qp_ip_{i+1} \\ (p_1\cdots p_ip_{i+2}\cdots p_k) - (p_1\cdots p_{i-1}p_{i+1}\cdots p_k) &= q \\ (p_1\cdots p_{i-1}p_{i+2}\cdots p_k)(p_i - p_{i+1}) &= q \end{aligned}$$

If $q = 1$, then $(p_1\cdots p_{i-1}p_{i+2}\cdots p_k)|1$ so we must have $(p_1\cdots p_{i-1}p_{i+2}\cdots p_k) = 1$, then $k = 3$ and $p_1 = 1$ in which case $S = \langle 1\rangle$. If $q = 2$ suppose towards contradiction that $S \neq \langle 1\rangle$, then $p_1 \neq 1$ but $p_1|2$ so we must have $p_1 = 2$. Then there is not enough room for the other p's to be less than p_1 and relatively prime to p_1, so we have $k = 1$ and $S = \langle 2\rangle$ contradicting that S is a numerical semigroup. Thus the only pseud-symmetric semigroup S that has $\tilde{S}$ is a semigroup is $S = \langle 1\rangle$. □

11.2.4 Arf

Proposition 11.2.7. *Let S be a truncated n-staircase with $S = \{0, n, 2n, \ldots, (k-1)n, kn - j \rightarrow\}$ for some $j \in \{1, 2, \ldots, n - m\}$. Then S is Arf.*

Proof. Let $x, y, z \in S$ with $x \geq y \geq z$. If $x < kn - j$ then $x = m_1n$, $y = m_2n$, $z = m_3n$ for some $m_1, m_2, m_3 \in \{0, \ldots, k-1\}$. So $x + y - z = (m_1 + m_2 - m_3)n$ is a multiple of n so $x + y + z \in S$. If $x \geq kn - j$, then since $z \leq y$, $y - z \geq 0$, so $x + y - z \geq x \geq kn - j$ and $x + y - z \in S$. □

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