Invariants of Generalized Arithmetic Numerical Semigroups

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August 8, 2014

Abstract

A generalized arithemtic numerical monoid is of the form $S = \langle a, ah+d, ah+2d, \ldots, ah+xd\rangle$ where the $\gcd(a, d) = 1$ and $a > x$. Much is known for the arithmetic numerical monoid, when $h = 1$, due to known information for that specific monoid's length set. Therefore, this paper will explore various invariants of the generalized arithmetic numerical monoid.

1 Introduction and Preliminaries

Before we introduce a generalized arithmetic numerical monoid and present properties of this monoid, we must present some basic terms important to understanding numerical monoids in general.

Definition 1.1 ([8]). *A* ***numerical monoid*** *S is a subset of $\mathbb{N}$ such that*

- $0 \in S$
- *if $a, b \in S$, then $a + b \in S$*
- $gcd(S) = 1$

Definition 1.2 ([8]). *A* ***generator*** *or* ***atom*** *n_i of a numerical monoid S is an element of S that can not be written as the sum of elements of $S \setminus \{0\}$ and $\{n_0, n_1, \ldots, n_x\}$ is called the* ***set of atoms*** *of S.*

Definition 1.3 ([2]). *Define the* ***set of factorizations*** *of an element $s \in S$ to be the set of all solutions to $m = A_0n_0 + A_1n_1 + \cdots + A_xn_x$ for $A_i \in \mathbb{N}$. We will denote the set of factorizations of m as $(A_0, A_1, \ldots, A_x)$.*

Definition 1.4 ([2]). *Define the* ***length of factorization*** *of $s \in S$ to be $A_0 + \sum_{i=1}^{x} A_i$.*

Definition 1.5 ([2]). *Define the* ***length set of an element*** *in S, $\mathcal{L}(s)$, to be the set of all lengths of factorization of s. The* ***the length set of a numerical monoid****, $\mathcal{L}(S)$, is given by $\mathcal{L}(S) = \{\mathcal{L}(s) \mid s \in S\}$.*

We will now go on to define several important invariants of numerical monoids that will help us to later understand properties of generalized arithmetic numerical monoids.

Definition 1.6 ([1]). *The* ***elasticity of an element*** *$s \in S$, denoted $\rho(s)$, is given by $\rho(s) = \frac{L(s)}{\ell(s)}$ where $L(s) = max(\mathcal{L}(s))$ and $\ell(s) = min(\mathcal{L}(s))$. The* ***elasticity of a numerical monoid****, $\rho(S)$, is given by $\rho(S) = sup\{\rho(s) | s \in S\}$.*

We can define a more specific type of elasticity for numerical monoids, specialized elasticity, which we will later explore fully for generzlied arithmetic numerical semigroups.

Definition 1.7 ([4]). *The* ***specialized elasticity*** *of a numerical monoid S is given by $\rho_k(S) = sup\{L(s) \mid \ell(s) \leq k\}$*

Another invariant that we will explore later for generalized arithmetic sequences is delta sets.

Definition 1.8 ([2]). *Let $\mathcal{L}(s) = \{n_1, n_2, \ldots, n_t\}$ with $n_i < n_{i+1}$. Define the* ***delta set of an element*** *$s \in S$ as $\Delta(s) = \{n_{i+1} - n_i \mid 1 \leq i \leq t-1\}$ and define the* ***delta set of a numerical monoid*** *S as $\Delta(S) = \bigcup_{s \in S} \Delta(s)$*

The last invariant that we consider in this paper is omega-primality for arithmetic sequences. Before we define omega-primality, some other definitions are necessary.

Definition 1.9 ([7]). *For $s, t \in S$, we say that s* ***precedes*** *t ($s \preceq t$) if $t - s \in S$.*

Definition 1.10 ([7]). *A* ***bullet*** *for $s \in S$ is an expression $u_1 + \ldots + u_k$ such that*

- $s \preceq u_1 + \ldots + u_k$
- $s \not\preceq u_1 + \ldots + u_k - u_i$ *for all i.*

Now we can define omega-primality for a numerical monoid.

Definition 1.11 ([7]). *Define the* ***omega primality*** *of $s \in S$, $\omega(s)$, to be the smallest value of m such that if $s \preceq u_1 + \cdots + u_k$ where $k \geq m$, then there exists $T \subseteq \{1, \ldots, k\}$ with $|T| \leq m$ such that $s \preceq \sum_{i \in T} u_i$.*

2 Generalized Arithmetic Numerical Monoids

Much is known about arithmetic numerical monoids so much of our paper is focused on generalized arithmetic numerical monoids. However, we do also present some previously unknown results about arithmetic numerical monoids.

Definition 2.1 ([5]). *A* ***generalized arithmetic numerical monoid*** *is a monoid that is generated by a generalized arithmetic seqeunce, that is a numerical monoid of the form $S = \langle a, a+d, \ldots, ah+xd \rangle$, where $gcd(a,d) = 1$ and $a > x$. Also, $h = 1$ yields an* ***arithmetic numerical monoid****.*

It is essential that we have $gcd(a,d) = 1$ as otherwise $gcd(S) \neq 1$ and S will not be a numerical monoid. We also require $a > x$ as otherwise $\{a, ah+d, \ldots, ah+xd\}$ will not be a minimal generating set.

Throughout the remainder of this paper we may write $n_0 = a$ and $n_i = ah+id$. Furthermore, $s = (A_0, A_1, \ldots, A_x)$ will represent an element $s \in S$ that is the sum of A_0 copies of a and A_i copies of $ah + id$.

3 Specialized Elasticity of Generalized Arithmetic Numerical Monoids

We will now fully characterize the specialized elasticity of generalized arithmetic numerical monoids. Let $S = \langle a, a+d, \ldots, ah+xd \rangle$ with $gcd(a,d) = 1$ and $a > x$. Recall our definition of specialized elasticity,

$$\rho_k(S) = \sup\{L(s) \mid \ell(s) \leq k\}$$

Lemma 3.1 ([4]). $\rho_k(S) = sup\{L(s) \mid \ell(s) \leq k\} = sup\{L(s) \mid \ell(s) = k\}$

We will use the definition of specialized elasticity found in Lemma 3.1 for the remainder of this paper.

Consider an element t that can be written as the sum of k atoms. Assume we have N copies of a, and $(k-N)$ copies of the remaining atoms. We can then write

$$t = Na + (k-N)ah + \left(\sum_{i=1}^{k-N} \beta_i\right) d$$

Suppose the longest factorization of t is given by $(A_0, A_1, ..., A_x)$. Let $k_1 = A_0 + h\sum_{i=1}^{x} A_i$ and $k_2 = \sum_{i=1}^{x} iA_i$. We can now write

$$t = k_1 a + k_2 d = Na + (k-N)ah + \left(\sum_{i=1}^{k-N} \beta_i\right) d. \tag{1}$$

Looking at equation (1) modulo d yields $k_1 = N + (k-N)h + sd$ for some $s \in \mathbb{Z}$. Plugging this defenition of k_1 back into equation (2) also yields $k_2 = \sum_{i=1}^{k-N} \beta_i - sa$. Note that $k_1 = L(t) + (h-1)\sum_{i=1}^{x} A_i \implies$

$$L(t) = N + (k-N)h + sd - (h-1)\sum_{i=1}^{x} A_i = kh + sd + (1-h)\left(N + \sum_{i=1}^{x} A_i\right). \tag{2}$$

Lemma 3.2. *Suppose $a \leq kx$. Then $k \leq a\left\lfloor \frac{kx}{a} \right\rfloor$.*

Proof. First suppose $k \leq kx - a$. Then

$$k \leq kx - a = a\left(\frac{kx}{a} - 1\right) < a\left\lfloor \frac{kx}{a} \right\rfloor.$$

Now consider $k > kx - a$ and suppose $k > a\left\lfloor \frac{kx}{a} \right\rfloor$. Then

$$a \leq kx \implies a \leq a\left\lfloor \frac{kx}{a} \right\rfloor \implies a < k.$$

Furthermore,

$$k > kx - a \implies a > k(x-1) \geq k.$$

Thus $k < a < k$, a contradiction, so $k \leq a\left\lfloor \frac{kx}{a} \right\rfloor$. □

Lemma 3.3. *Suppose $a > kx$. Then*

$$kh - (h-1)\left(N + \sum_{i=1}^{x} A_i\right) \leq (h-1)\left\lfloor \frac{-k}{x} \right\rfloor + kh$$

Proof. Recall from above that $k_2 = \sum_{i=1}^{x} iA_i = \sum_{i=1}^{k-N} \beta_i - sa$. Notice

$$k - N \leq \sum_{i=1}^{k-N} \beta_i \implies k - N - sa \leq \sum_{i=1}^{k-N} \beta_i - sa.$$

Hence,

$$k - N - sa \leq \sum_{i=1}^{x} iA_i \implies k \leq \sum_{i=1}^{x} iA_i + sa + N.$$

But $a > kx$ and $\sum_{i=1}^{k-N} \beta_i - sa > 0 \implies s \leq 0$. Thus, we can write

$$k \leq \sum_{i=1}^{x} iA_i + N \leq x \left(\sum_{i=1}^{x} A_i + N \right)$$

But

$$k \leq x \left(N + \sum_{i=1}^{x} A_i \right) \implies -N - \sum_{i=1}^{x} A_i \leq \left\lfloor \frac{-k}{x} \right\rfloor$$

$$\implies kh - (h-1) \left(N + \sum_{i=1}^{x} A_i \right) \leq (h-1) \left\lfloor \frac{-k}{x} \right\rfloor + kh$$

□

We can now present our theorem characterizing specialized elasticity for all generalized arithmetic numerical monoids.

Theorem 3.1. *Let S be defined as above. Then*

$$\rho_k(S) = \begin{cases} kh + \left\lfloor \frac{kx}{a} \right\rfloor d & \text{if } a \leq kx \\ kh + (h-1) \left\lfloor \frac{-k}{x} \right\rfloor & \text{if } a > kx \end{cases}$$

Proof. First suppose $a \leq kx$. From equation (2), we can write

$$L(t) \leq kh + sd.$$

Note that

$$k_2 \geq 0 \implies \sum_{i=1}^{k-N} \beta_i \geq sa \implies sa \leq (k-N)x \leq kx \implies s \leq \left\lfloor \frac{kx}{a} \right\rfloor.$$

Hence,

$$L(t) \leq kh + \left\lfloor \frac{kx}{a} \right\rfloor d$$

Now let $n = a\lfloor \frac{kx}{a} \rfloor$. By Lemma 3.2, $k \leq n \leq kx$, so there exists $1 \leq i_j \leq x$ for all $1 \leq j \leq k$ such that $\sum_{i=1}^{k} i_j = n$. Then consider the element

$$(ah + i_1 d) + ... + (ah + i_k d) = kah + nd = a \left(kh + \frac{n}{a} d \right) = a \left(kh + \left\lfloor \frac{kx}{a} \right\rfloor d \right)$$

Thus, $L(t) \geq kh + \left\lfloor \frac{kx}{a} \right\rfloor d$ and $\rho_k(S) = kh + \left\lfloor \frac{kx}{a} \right\rfloor d$.

Now suppose $a > kx$. As before, we have $s \leq 0$. Then looking at equation (2) we have

$$L(t) \leq kh - (h-1) \left(N + \sum_{i=1}^{x} A_i \right) \leq (h-1) \left\lfloor \frac{-k}{x} \right\rfloor + kh$$

by Lemma 3.3.

Now consider the element $k(ah+d)$. First suppose $x \mid k$. Then

$$k(ah+d) = \left\lfloor \frac{k}{x} \right\rfloor (ah+xd) + \left(k - \left\lfloor \frac{k}{x} \right\rfloor\right) ha$$

But $x \mid k \implies \left\lfloor \frac{-k}{x} \right\rfloor = -\left\lfloor \frac{k}{x} \right\rfloor$, so

$$L(t) \geq \left\lfloor \frac{k}{x} \right\rfloor + \left(k - \left\lfloor \frac{k}{x} \right\rfloor\right) h = kh + (h-1)\left(-\left\lfloor \frac{k}{x} \right\rfloor\right) = (h-1)\left\lfloor \frac{-k}{x} \right\rfloor + kh.$$

Now suppose $x \nmid k$. Notice we can write $k = n_1 x + n_2$ where $0 < n_2 < x$. Then

$$k(ah+d) = n_1(ah+xd) + (ah+n_2 d) + (k-n_1-1)ha.$$

But $x \nmid k \implies \left\lfloor \frac{-k}{x} \right\rfloor = -\left\lfloor \frac{k}{x} \right\rfloor - 1 \implies \left\lfloor \frac{-k}{x} \right\rfloor = -\left\lfloor \frac{n_1 x + n_2}{x} \right\rfloor - 1 = -n_1 - 1$, so

$$L(t) \geq n_1 + 1 + (k-n_1-1)h = (h-1)\left\lfloor \frac{-k}{x} \right\rfloor + kh$$

Thus, $\rho(S) = kh + (h-1)\left\lfloor \frac{-k}{x} \right\rfloor$ □

Notice that by the definition of specialized elasticity, $\rho(S) = \lim_{k\to\infty} \frac{\rho_k(S)}{k}$. Also take note of the following theorem:

Proposition 3.1 ([3]). *Let $M = \langle a_1, \ldots, a_t \rangle$ be a numerical monoid where $a_1 < a_2 < \cdots < a_t$ is a minimal set of generators of M. Then $\rho(M) = \frac{a_t}{a_1}$.*

By Proposition 3.1, $\rho(S) = \frac{ah+xd}{a}$.

To see that our findings for specialized elasticity correspond to the known value of the generalized elasticity, consider $\lim_{k\to\infty} \frac{\rho_k(S)}{k}$. Notice that for large values of k we have $a \leq kx$ and $\rho_k(S) = kh + \left\lfloor \frac{kx}{a} \right\rfloor d$. Notice that

$$\lim_{k\to\infty} \frac{\rho_k(S)}{k} = \lim_{k\to\infty} \frac{kh + \left\lfloor \frac{kx}{a} \right\rfloor d}{k} = \lim_{k\to\infty} h + \frac{x}{a} d = \frac{ah+xd}{a},$$

so our results are consistent with previous findindings. This can be visualized in Figure 1.

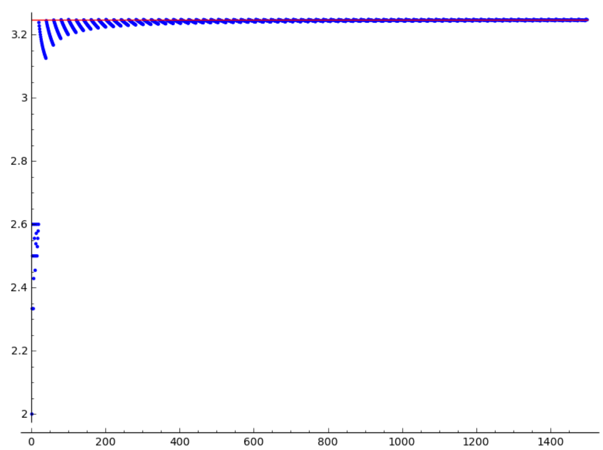

Figure 1: Plot of $\frac{\rho_k(S)}{k}$ versus k given in blue and $\rho(S)$ versus k given in red for for $S = \langle 101, 308, 313, 323, 328 \rangle$.

4 Delta Sets of Generalized Arithmetic Numerical Monoids

Lemma 4.1. *Let* $S = \langle a, ah+d, ah+2d, \dots, ah+xd\rangle$*. Let* $a > x$*,* $\gcd(a,d) = 1$*. Suppose* N *is an integer that can be factorized multiple ways. The difference between the lengths of any such factorizations is a multiple of* $\gcd(h-1, d)$*.*

Proof. Suppose that

$$\begin{aligned} &aA_0 + \sum_{i=1}^{x}(ah+id)A_i = aB_0 + \sum_{i=1}^{x}(ah+id)B_i \\ \iff\ & a(A_0 - B_0) + \sum_{i=1}^{x}(ah+id)(A_i - B_i) = 0 \\ \iff\ & a(A_0 - B_0) + \sum_{i=1}^{x}[a + a(h-1) + id](A_i - B_i) = 0 \\ \iff\ & a\sum_{i=0}^{x}(A_i - B_i) + \sum_{i=1}^{x}[a(h-1) + id](A_i - B_i) = 0 \end{aligned}$$

Let $g = \gcd(d, h-1)$. Taking both sides mod g, we get that $a\sum_{i=0}^{x}(A_i - B_i) \equiv 0 \mod g$. But since a, d are coprime, and d is divisible by g, a must also not share any prime factors with g. Thus dividing both sides by a gives us the desired $\sum_{i=0}^{x}(A_i - B_i) \equiv 0 \mod g$. □

Lemma 4.2. *Let* $S = \langle a+1, a+2, \dots, a+x\rangle$ *with* $x > 1$ *and* n *be a nonnegative integer.* $n \in S$ *if and only if there exists a nonnegative integer* j *such that* $j(a+1) \le n \le j(a+x)$*.*

Proof. Reverse direction is intuitive. The generators are all consecutive, so we can shimmy our way up from $j(a+1)$ to $j(a+x)$ using j atoms in each of our factorizations. For the forward direction, we let j be the number of atoms in one of our decompositions of n. □

Corollary 4.1. *As a corollary of lemma 2,* $n \notin S = \langle a+1, a+2, \dots, a+x\rangle$ *if and only if there is a nonnegative integer* j *such that* $j(a+x) < n < (j+1)(a+1)$*.*

Lemma 4.3. *If* x, y *are positive reals with* $y \le 1$*, then for any positive integer* $k' \le \lceil x\rceil$*, there is an integer* m *with* $0 \le m \le \lceil \frac{x}{y}\rceil - 1$ *such that* $\lceil x - my\rceil = k'$*.*

Proof. First note that for any positive reals x_1, x_2 with $x_2 \le 1$, we have $\lceil x_1 - x_2\rceil \ge \lceil x_1 - 1\rceil = \lceil x_1\rceil - 1 \implies \lceil x_1\rceil - \lceil x_1 - x_2\rceil \le 1$. Let $f(m) = \lceil x - my\rceil$. Note that between successive values of m (i.e. m to $m+1$), $f(m) - f(m+1) = \lceil x - my\rceil - \lceil x - (m+1)y\rceil = \lceil (x-my)\rceil - \lceil (x-my) - y\rceil \le 1$. Thus, between successive values of m, $f(m)$ cannot jump by more than 1. Note that if we take $m' = \lceil \frac{x}{y}\rceil - 1$, $f(m') = \lceil x - (\lceil \frac{x}{y}\rceil - 1)y\rceil \le \lceil x - (\frac{x}{y} - 1)y\rceil = \lceil x - \frac{x}{y}y + y\rceil = \lceil y\rceil \le 1$. Thus, increasing m from 0 to $\lceil \frac{x}{y}\rceil - 1$, $f(m)$ must traverse through all integers from 1 to $\lceil x\rceil$. Therefore, there is an integer m where $0 \le m \le \lceil \frac{x}{y}\rceil - 1$ such that $k' = \lceil x - my\rceil$. □

Lemma 4.4. *Let* $m \le \frac{a}{x-1}$ *be a nonnegative integer and* $S = \langle an+1, an+2, \dots, an+x\rangle$*, where* $n \ge 1, a > x$*. The smallest positive integer* k *such that* $ka + m(an+1) \in S$ *is* $n\lceil \frac{a - m(x-1)}{x}\rceil + 1$*.*

Proof. This is similar to the previous proof. Let $ka+m(an+1)=\sum_{i=1}^{x}(an+i)B_i$. First write $k=cn+r$, where $1\le r\le n$. Note that if k is divisible by n, $r=n$.

$$\begin{aligned}
&a(cn+r)+(an+1)m=\sum_{i=1}^{x}(an+i)B_i\ge(an+1)\sum_{i=1}^{x}B_i\\
&\implies \sum_{i=1}^{x}B_i\le\frac{a(cn+r)}{an+1}+m\le\frac{a(nc+n)}{an+1}+m\le\frac{an(c+1)}{an+1}+m<c+1+m\\
&\implies \sum_{i=1}^{x}B_i\le c+m \text{ (note that inequality is strict because } \frac{an}{an+1}<1)\\
&\implies a(cn+r)+(an+1)m=\sum_{i=1}^{x}(an+i)B_i\le(an+x)\sum_{i=1}^{x}B_i\le(an+x)(c+m)\\
&\iff acn+ar+anm+m\le acn+anm+xc+xm\\
&\iff \frac{ar-m(x-1)}{x}\le c\iff c\ge\lceil\frac{ar-m(x-1)}{x}\rceil\\
&\implies k=cn+r\ge n\lceil\frac{ar-m(x-1)}{x}\rceil+r\ge n\lceil\frac{a-m(x-1)}{x}\rceil+1
\end{aligned}$$

Note that the above inequality is true since $r\ge1$. It is necessary for the argument in the ceiling to be nonnegative to make this conclusion, which is the case when $m\le\frac{a}{x-1}$.

We will now use the intuitive lemma (Lemma 2) to implicitly show that if $k=n\lceil\frac{a-m(x-1)}{x}\rceil+1$ then $ka+m(an+1)\in S$. As in lemma 2, let $j=n\lceil\frac{a+m}{x}\rceil+1$. We wish to show that $j(an+1)\le ak+m(an+1)\le j(an+x)$. After some simplifying of the middle side of the inequality,

$$\begin{aligned}
&ak+m(an+1)=a(n\lceil\frac{a-m(x-1)}{x}\rceil+1)+m(an+1)\\
&=(n\lceil\frac{a+m}{x}-\frac{mx}{x}\rceil+1)a+(an+1)m\\
&=(n\lceil\frac{a+m}{x}\rceil-nm+1)a+(an+1)m=na\lceil\frac{a+m}{x}\rceil+a+m
\end{aligned}$$

We first show the higher bound.

$$na\lceil\frac{a+m}{x}\rceil+a+m\le\lceil\frac{a+m}{x}\rceil(an+x)\iff a+m\le\lceil\frac{a+m}{x}\rceil x$$

But note that $a+m=\frac{a+m}{x}x\le\lceil\frac{a+m}{x}\rceil x$, which proves the upper bound inequality.

Now we show the lower bound.

$$\lceil\frac{a+m}{x}\rceil(an+1)\le na\lceil\frac{a+m}{x}\rceil+a+m\iff\lceil\frac{a+m}{x}\rceil\le a+m$$

Note that $\lceil\frac{a+m}{x}\rceil\le\frac{a+m}{x}+1$, so it suffices to show that $\frac{a+m}{x}+1\le a+m\iff a+m+x\le x(a+m)\iff 1\le(x-1)(a+m-1)$. But this latter inequality is clearly true since $a>x\ge2$, thus proving the lower bound. Therefore, the intuitive lemma allows us to conclude that $ka+m(a+1)\in S$.

□

Lemma 4.5. *Let $S=\langle an+1,an+2,\dots,an+x\rangle$, where $x\ge2$. Let $1\le n'\le n$. The smallest $k>0$ such that $ak+(an+x)\lceil\frac{an+1-an'}{x-1}\rceil\in S$ is n'.*

Proof. We utilize lemma 2 and its corollary. Let $n_1 = \lceil \frac{an+1-an'}{x-1} \rceil$ (simplify the cloggy terms). We let $j = n_1 + 1$ as in lemma 2. Thus we want to show that

$$\begin{aligned}(an+1)(n_1+1) &\le an' + (an+x)n_1 \le (an+x)(n_1+1)\\ \iff an + n_1 + 1 &\le an' + xn_1 \le an + xn_1 + x\end{aligned}$$

The upper bound is immediately true by a comparison of terms since $n' < n$. For the lower bound, we have

$$\begin{aligned}an + n_1 + 1 \le an' + xn_1 &\iff an - an' + 1 \le (x-1)n_1\\ an - an' + 1 \le (x-1)\lceil \frac{an+1-an'}{x-1} \rceil &\text{ (this is true by the ceiling bound)}\end{aligned}$$

Now we have to show that for any positive n_0 with $n_0 < n'$, $an_0 + (an+x)\lceil \frac{an+1-an'}{x-1} \rceil \notin S$. We apply the corollary to lemma 2, with $j = n_1$. We wish to prove that $(an+x)n_1 < an_0 + (an+x)n_1 < (an+1)(n_1+1)$. The lower bound is definitely true by a comparison of terms since $an_0 > 0$. For the upper bound, we have

$$\begin{aligned}&an_0 + (an+x)n_1 < (an+1)(n_1+1)\\ &\iff an_0 + ann_1 + xn_1 < ann_1 + an + n_1 + 1\\ &\iff an_0 + xn_1 < an + n_1 + 1 \iff an_0 + (x-1)n_1 < an + 1\\ &\iff an_0 + (x-1)\lceil \frac{an+1-an'}{x-1} \rceil < an + 1\end{aligned}$$

Again, bounding the ceiling function by $\frac{an+1-an'}{x-1} + 1$, it suffices to show that

$$\begin{aligned}&an_0 + (x-1)(\frac{an+1-an'}{x-1} + 1) < an + 1\\ &\iff an_0 + an + 1 - an' + x - 1 < an + 1\\ &\iff an_0 - an' + x - 1 < 0 \iff x - 1 < a(n' - n_0)\end{aligned}$$

But note that since $n_0 < n'$, $n' - n_0 \ge 1$. Thus $a(n' - n_0) \ge a > x - 1$ since we know that $a > x$, thus proving the lemma.

□

Lemma 4.6. *$S = \langle an+1, an+2, \ldots, an+x \rangle$, and $x \ge 2$. If $m \ge \frac{a}{x-1}$, then the smallest k such that $ak + (an+1)m \in S$ is at most 1.*

Proof. It is enough to show that if $k = 1$, then $ak + (an+1)m \in S$. We again apply the intuitive lemma with $j = m$. Then we want $(an+1)m \le a(1) + (an+1)m \le (an+x)m$. The lower bound for sure holds. For the upper bound to hold, we need $a + (an+1)m \le (an+x)m \iff a \le (an+x-an-1)m = (x-1)m \iff \frac{a}{x-1} \le m$, which is true by assumption, as desired. □

Lemma 4.7. *Let $S = \langle an+1, an+2, \ldots, an+x \rangle$. Suppose that $B_j > 0$ for some $j > 1$. The smallest k such that $ak + \sum_{i=1}^{x}(an+i)B_i \in S$ is at most n.*

Proof. It is enough to show that $k = n$ will make $ak + \sum_{i=1}^{x}(an+i)B_i \in S$. Suppose we have $B_j > 0$. Then we want $an + (an+j)B_j + \sum_{i=1}^{j-1}(an+i)B_i + \sum_{i=j+1}^{x}(an+i)B_i = \sum_{i=1}^{x}(an+i)A_i$ for some nonnegative A_i's. But note that $an + (an+j)B_j = (an+1) + (an+j-1) + (an+j)(B_j-1)$. Then we let $A_1 = B_1 + 1$, $A_{j-1} = B_{j-1} + 1$, $A_j = B_j - 1$, and $A_i = B_i$ for all the other indices. Note that $A_j \ge 0$ since $B_j \ge 1$. □

Theorem 4.1. *Let $S = \langle a, ah+d, ah+2d, \ldots, ah+xd \rangle$. Let $a > x$, $\gcd a, d = 1$, and $h = nd+1$. Then $\Delta(S) = \{d, 2d, \ldots, nd\} \bigcup \{(n+1)d, (2n+1)d, \ldots, (n\lceil \frac{a}{x} \rceil + 1)d\}$.*

Proof. Suppose that we have an arbitrary N that can be factorized in multiple ways. We have

$$\begin{aligned} N &= aA_0 + [a(nd+1)+d]A_1 + [a(nd+1)+2d]A_2 + \ldots + [a(nd+1)+xd]A_x \\ &= aB_0 + [a(nd+1)+d]B_1 + [a(nd+1)+2d]B_2 + \ldots + [a(nd+1)+xd]B_x \end{aligned}$$

where A_i's and B_i's are the coefficients of their corresponding atoms.

and that $A_0 + A_1 + \ldots + A_x = L$, which corresponds to the length of the factorization. Suppose that the other factorization is longer. We have that $\gcd(h-1, nd) = \gcd(d, nd) = d$, so by lemma 1 above, $B_0 + B_1 + \ldots + B_x = L + kd$ for some positive integer k.

More compactly, these equations can be written as

$$N = aA_0 + \sum_{i=1}^{x} (a(nd+1)+id)A_i = aB_0 + \sum_{i=1}^{x} (a(nd+1)+id)B_i$$

$$\sum_{i=0}^{x} A_i = L, \qquad \sum_{i=0}^{x} B_i = L + kd$$

Now if this k is the smallest positive integer such that there exists no factorizations of N of length $L + k'd$ for $k' < k$, then this implies that $kd \in \Delta(S)$.

Let us keep manipulating these equations. Note that by subtracting two equations, we have $\sum_{i=0}^{x}(B_i - A_i) = (L+kd) - L = kd$. The two factorizations of N tells us that

$$a(B_0 - A_0) + \sum_{i=1}^{x} [a(nd+1)+id](B_i - A_i) = 0$$

$$a(B_0 - A_0) + \sum_{i=1}^{x} a(B_i - A_i) + \sum_{i=1}^{x} [(an+i)d](B_i - A_i) = 0 \text{ (extract out } a \text{ from the second summation)}$$

$$\sum_{i=0}^{x} a(B_i - A_i) + \sum_{i=1}^{x} [(an+i)d](B_i - A_i) = 0 \text{ (combine first term with first summation)}$$

$$akd + \sum_{i=1}^{x} [(an+i)d](B_i - A_i) = 0 \text{ (since } \sum_{i=0}^{x}(B_i - A_i) = kd)$$

$$ak + \sum_{i=1}^{x} (an+i)(B_i - A_i) = 0 \text{ (divide out } d)$$

$$ak + \sum_{i=1}^{x} (an+i)B_i = \sum_{i=1}^{x} (an+i)A_i \text{ (add our } A_i\text{'s to both sides)}$$

Recall that if k is the smallest positive integer such that $L + kd$ is the length of the next longest factorization after a length of L, then $kd \in \Delta(S)$. **Therefore, in order to show that $k'd \in \Delta(S)$, we will pick our B_i's $(B_1, B_2, \ldots, B_x)$ (each B_i nonnegative) such that the equation**

$$ak' + \sum_{i=1}^{x} (an+i)B_i = \sum_{i=1}^{x} (an+i)A_i \tag{3}$$

admits a nonnegative x-tuple integer solution $(A_1, A_2, \ldots, A_x)$, and that this k' is the smallest positive integer that does so.

We have now shown a way to reinterpret the problem. Despite the heavy algebra, the advantage of this interpretation is that we do not have to carry the d term around. In fact, our next arguments are independent of d. Equation 1 strongly resembles the form of the lemmas above, which will be exploited. In fact, our lemmas will show precisely how to choose our B_i's.

First, we will show how to attain $\{n+1, 2n+1, \ldots, n\lceil \frac{a}{x} \rceil + 1\}$. Note that these values are of the form $nu + 1$ for some positive integer $u \leq \lceil \frac{a}{x} \rceil$. First, for such a u, choose our $m \leq \lceil \frac{a}{x-1} \rceil - 1 < \frac{a}{x-1}$ such that $\lceil \frac{a-m(x-1)}{x} \rceil = u$. This is possible by lemma 3, where our x is lemma 3 is interchanged with $\frac{a}{x}$ and our y is interchanged with $\frac{x-1}{x}$. If we choose $B_1 = m, B_2 = B_3 = \ldots = B_x = 0$, then by lemma 4, the smallest positive k such that $ka + \sum_{i=1}^{x}(an+i)B_i = ka + m(an+1) = \sum_{i=1}^{x}(an+1)A_i$ is solvable in A_i's is $n\lceil \frac{a-m(x-1)}{x} \rceil + 1 = nu + 1$.

Now we show how to attain $\{1, 2, \ldots, n\}$. Let's suppose we want to attain some arbitrary $n' \leq n$. Now we set $B_1 = B_2 = \ldots = B_{x-1} = 0$ and $B_x = \lceil \frac{an+1-an'}{x-1} \rceil$. Now we consider the minimal k such that $ka + \sum_{i=1}^{x}(an+i)B_i = ka + (an+x)\lceil \frac{an+1-an'}{x-1} \rceil = \sum_{i=1}^{x}(an+i)A_i$ is solvable in A_i's. But by lemma 5, the smallest $k > 0$ that admits a solution is n', as desired.

We're almost done! Now as an essential step, we need to see what additional k can possibly arise from some arbitrary x-tuple $(B_1, B_2, \ldots, B_x)$. Suppose that we have $B_j > 0$ for some $j \geq 2$. Now consider the equation $ak + \sum_{i=1}^{x}(an+i)B_i = \sum_{i=1}^{x}(an+i)A_i$. But by lemma 7, $k = n$ already admits a solution! Thus the smallest possible k that can yield a solution in this case is at most n, which does not add anything new to our Δ set. We can thus restrict our analysis to B_1 being arbitrary and $B_2 = B_3 = \ldots = B_x = 0$. We have $ak + (an+1)B_i = \sum_{i=1}^{x}(an+i)A_i$. But for $B_1 \leq \frac{a}{x-1}$, lemma 4 immediately gives us all $k \in \{n+1, 2n+1, \ldots, n\lceil \frac{a}{x} \rceil + 1\}$. If $B_1 \geq \frac{a}{x-1}$, lemma 6 already tells us that $k \leq 1$, and we've shown $k = 1$ admits a solution of A_i's in lemma 5.

Lastly, since we found a new way to interpret the problem, we need to go back to our old ways. Recall that we had d, A_0, B_0 involved.

$$N = aA_0 + \sum_{i=1}^{x}(a(nd+1) + id)A_i = aB_0 + \sum_{i=1}^{x}(a(nd+1) + id)B_i$$

$$\sum_{i=0}^{x} A_i = L, \qquad \sum_{i=0}^{x} B_i = L + kd$$

The structure of the proof involved choosing our $B_1, B_2, \ldots, B_x$ to obtain a specific k. This predetermines the values $A_1, A_2, \ldots, A_x$ (determined by $ka + \sum_{i=1}^{x} B_i(an+id) = \sum_{i=1}^{x} A_i(an+id)$, with $(A_1, A_2, \ldots, A_x)$ the solution corresponding to the value of k).

Now using our equations as laid out above, we can manipulate them to obtain $B_0 = L + kd - \sum_{i=1}^{x} B_i = A_0 + kd + \sum_{i=1}^{x}(A_i - B_i)$. Now we choose a nonnegative A_0 large enough such that $B_0 = A_0 + kd + \sum_{i=1}^{x}(A_i - B_i) \geq 0$. This well-defines N in terms of the generators and $B_0, B_1, \ldots, B_x$, as given in the equation above.

Thus, after exhausting all possibilities for $(B_1, B_2, \ldots, B_x)$,

$$\Delta(S) = \{d, 2d, \ldots, nd\} \bigcup \{(n+1)d, (2n+1)d, \ldots, (n\lceil \frac{a}{x} \rceil + 1)d\}$$

$\square$

5 Omega Primality of ja for Arithmetic Numerical Monoids of Embedding Dimension 3

We will now consider omega primality for all multiples of a for an arithmetic numerical monoid of embedding dimension three. Suppose $j \geq 1 \in \mathbb{N}$ and $S = \langle a, a+d, a+2d \rangle$.

The following two propositions will be very useful in the following sections regarding omega-primality.

Proposition 5.1 ([6, Theorem 3.1]). *Let $n = qa + id$ where $q, i \in \mathbb{N}$ and $0 \leq i \leq a-1$. Then $n \in S$ if and only if $\lceil \frac{i}{x} \rceil \leq q$.*

Proposition 5.2 ([7]). *$\omega(s) = \max\{k | n_1 + \cdots + n_k \text{ a bullet for } s\}$*

Lemma 5.1. *Let $m = \lceil \frac{a}{2} \lceil \frac{2j}{a+2d} \rceil \rceil$. Then $(0, 0, m)$ is a bullet for ja.*

Proof. We need to find the smallest positive integer m such that $(a+2d)m - ja \in S$. Thus we have $aA_0 + (a+d)B_0 + (a+2d)C_0 = (a+2d)m - ja$. First note that if in such a representation, $C_0 \geq 1$, then $(a+2d)m$ is not a bullet to begin with, as we can delete out C_0 copies of $a+2d$ on both sides. Thus, we can assume $C_0 = 0$ so we have

$$aA_0 + (a+d)B_0 = (a+2d)m - ja = (m-j) + 2md$$

Now we take both sides of the above equation mod d, giving us $a(A_0 + B_0) \equiv a(m-j) \mod d \iff A_0 + B_0 \equiv m - j \mod d \iff A_0 + B_0 = m - j + kd$, where k is an integer. Plugging this back in we get

$$\begin{aligned} &a(m-j+kd) + B_0 d = a(m-j) + 2md \\ &B_0 = 2m - ak \\ &A_0 = k(a+d) - m - j \end{aligned}$$

Since we know $B_0 \geq 0$, we have $m \geq \frac{ak}{2}$. Since $A_0 \geq 0$, we have $m \leq k(a+d) - j$. Thus we have $\frac{a}{2}k \leq m \leq k(a+d) - j \implies \frac{a}{2}k \leq k(a+d) - j \iff 0 \leq \frac{ak}{2} + kd - j \iff j \leq k(\frac{a}{2} + d) \iff k \geq \frac{2j}{a+2d} \iff k \geq \lceil \frac{2j}{a+2d} \rceil$. (Note that in our chain of inequalities, we've shown $k \geq 0$ and hence all the upcoming inequalities do not receive a change in direction)

But then we know that $m \geq \frac{ak}{2} \iff m \geq \lceil \frac{a}{2}k \rceil \implies m \geq \lceil \frac{a}{2} \lceil \frac{2j}{a+2d} \rceil \rceil$.

Now that we've showed a tight bound, we need to show the equality can be achieved (i.e. that we can actually make $A_0, B_0 \geq 0$). We will let $k = \lceil \frac{2j}{a+2d} \rceil$ and $m = \lceil \frac{a}{2}k \rceil$. Clearly, $B_0 = 2m - ak = 2\lceil \frac{a}{2}k \rceil - ak$ is nonnegative. Now we have $A_0 = k(a+d) - m - j = k(a+d) - \lceil \frac{ak}{2} \rceil - j$. But note that $\lceil \frac{ak}{2} \rceil = \frac{ak}{2} + r$, where $r \in \{0, 1/2\}$. Then $A_0 = k(a + d - \frac{a}{2}) - j = k\frac{a+2d}{2} - j - r = \lceil \frac{2j}{a+2d} \rceil \frac{a+2d}{2} - j - r \geq \frac{2j(a+2d)}{2(a+2d)} - j - r = -r \geq -\frac{1}{2}$. Thus we have $A_0 \geq -\frac{1}{2}$, but since our expression for A_0 is known to be an integer, it must be at least 0. Thus, we've shown our lemma.

□

Lemma 5.2. *Let $S = \langle a, a+d, a+2d \rangle$. Suppose that j is a positive integer that can be written as $(a+d)r_0 + r_1$ where $1 \leq r_1 \leq a+d$. Then $(0, m, 0)$ is a bullet for ja, where*

$$m = \begin{cases} a\lceil \frac{j}{a+d} \rceil & \text{if } r_1 \geq \frac{a}{2} \\ 2j - (a+2d)\lfloor \frac{j}{a+d} \rfloor & \text{if } r_1 \leq \frac{a}{2} \end{cases}$$

Proof. We need to see when $aA_0 + (a+2d)C_0 = (a+d)m - ja = a(m-j) + md$ admits a nonnegative solution A_0, C_0 (we assumed $B_0 = 0$ for same reasons as last proof) As usual, we have $A_0 + C_0 = m - j + kd$, so a bit of computation shows that

$$C_0 = \frac{m - ak}{2} \qquad A_0 = k(\frac{a}{2} + d) + \frac{m}{2} - j$$

C_0 tells us that $m = ak + 2k_1$ for some nonnegative integer k_1. Thus $A_0 = \frac{ka+m}{2} + kd - j = \frac{ak+ak+2k_1}{2} + kd - j = ak + kd + k_1 - j \geq 0$. Thus we have $k(a+d) \geq j - k_1 \iff k \geq \frac{j-k_1}{a+d} \iff k \geq \lceil \frac{j-k_1}{a+d} \rceil$.

Now we have $m = ak + 2k_1 \geq a\lceil \frac{j-k_1}{a+d} \rceil + 2k_1$. Now let us try to minimize the expression on the RHS. Write $j = r_0(a+d) + r_1$ where $1 \leq r_1 \leq a+d$. Then we have $m \geq a\lceil \frac{r_0(a+d)+r_1-k_1}{a+d} \rceil + 2k_1 = a(r_0 + \lceil \frac{r_1-k_1}{a+d} \rceil) + 2k_1$.

We need to minimize the expression $a\lceil \frac{r_1-k_1}{a+d} \rceil + 2k_1$ in order to minimize m by choosing an optimal $k_1 \geq 0$. We analyze the components of this sum. The left-hand term is a from $k_1 = 0$ to $r_1 - 1$ while the right-hand term increases with k_1, so clearly $k_1 \in \{1, 2, \ldots, r_1 - 1\}$ cannot possibly minimize the sum. Now for $k_1 = r_1$ to $a + d$, the LH term is zero (note that $1 \leq r_1 \leq a + d$) while the RH term increases, so therefore $k_1 \in \{r_1 + 1, r_1 + 2, \ldots, a + d\}$ also does not minimize the sum.

Now suppose that more generally, $k_1 = k_2(a+d) + k_3$, where $0 \leq k_3 \leq a + d - 1$. Our sum is thus equal to $a\lceil \frac{r_1-(k_2(a+d)+k_3)}{a+d} \rceil + 2(k_2(a+d)+k_3) = a(-k_2 + \lceil \frac{r_1-k_3}{a+d} \rceil) + 2k_2a + 2k_2d + 2k_3 = k_2(a+2d) + a\lceil \frac{r_1-k_3}{a+d} \rceil + 2k_3$. This sum is minimized when $k_2 = 0$, but we've shown above that $a\lceil \frac{r_1-k_3}{a+d} \rceil + 2k_3$ cannot be minimal for $k_3 \in \{1, 2, \ldots, r_1 - 1\} \cup \{r_1 + 1, r_1 + 2, \ldots, a + d\}$.

Thus our problem boils down to a comparison of $k_3 = 0$ with $k_3 = r_1$. When plugging these values in, their respective expressions are $a\lceil \frac{r_1}{a+d} \rceil = a$ and $2r_1$. The latter expression (when $k_3 = r_1$) is smaller when $r_1 \leq \frac{a}{2}$, and the former expression is larger when $r_1 \geq \frac{a}{2}$. Thus, for our expression $m = ak + 2k_1$ (recall that $k \geq \lceil \frac{j-k_1}{a+d} \rceil$), so we have $m \geq \lceil \frac{j-k_1}{a+d} \rceil + 2k_1$). When $k_1 = 0$ (whenever $r_1 \geq \frac{a}{2}$), we have $m \geq \lceil \frac{j}{a+d} \rceil$. When $k_1 = r_1$ (whenever $1 \leq r_1 \leq \frac{a}{2}$), we have $m \geq \lceil \frac{j-r_1}{a+d} \rceil + 2r_1$. But note that since $1 \leq r_1 \leq \frac{a}{2}$, r_1 is the remainder when dividing j by $a + d$, so $r_1 = j - (a+d)\lceil \frac{j}{a+d} \rceil$. Thus we have $m = ak + 2r_1 \geq a\lceil \frac{j-r_1}{a+d} \rceil + 2r_1 = a\lceil \frac{j-(j-(a+d)\lceil \frac{j}{a+d} \rceil)}{a+d} \rceil + 2(j - (a+d)\lceil \frac{j}{a+d} \rceil) = a\lfloor \frac{j}{a+d} \rfloor + 2(j - (a+d)\lceil \frac{j}{a+d} \rceil) = 2j - (a+2d)\lfloor \frac{j}{a+d} \rfloor$, after some simplification.

Now we need to show that our bound for m (which is a piecewise function of j) is tight. We will do so by explicitly constructing A_0, C_0. As above, we let $C_0 = k_1$, which is nonnegative (recall it's either 0 or r_1). Now let $A_0 = k(a+d) + k_1 - j$, where m is defined piecewise as above, and k_1 is given above. Then let $k = \lceil \frac{j-k_1}{a+d} \rceil$. Also take note that $j - k_1 = r_0(a+d) + r_1 - k_1 \geq r_1 - k_1 \geq 0$, so again, no change of directions invoked in our inequalities. This implies that $A_0 = \lceil \frac{j-k_1}{a+d} \rceil (a+d) + k_1 - j \geq 0$, as desired, thus proving our lemma.

□

Lemma 5.3. *Let $S = \langle a, a+d, a+2d \rangle$. Then $(j, 0, 0)$ is a bullet for ja.*

Proof. Don't think too hard on this. It's right in front of your eyes. □

Definition 5.1. *A pure bullet is a bullet (A_0, A_1, A_2) such that only one component is nonzero.*

Lemma 5.4. *Suppose a even and $j \neq \frac{a}{2}$. Let s be the sum of any $j + \frac{a}{2}$ atoms. Then $ja \preceq s$.*

Proof. Note we can write $s = (j + \frac{a}{2})a + Nd$. Then $s - ja = (\frac{a}{2})a + Nd$. Then we can write $N = qa + r$ for $0 \le r < a$ and $q \ge 0$ and

$$s - ja = \left(\frac{a}{2} + qd\right) a + rd$$

But $\lceil \frac{r}{2} \rceil \le \frac{a}{2} + qd$ so by Proposition 5.1, $s - ja \in S$ and $ja \preceq s$. $\square$

Lemma 5.5. *Suppose a even and $j \neq \frac{a}{2}$. Let s be the sum of any $j + \frac{a-2}{2}$ atoms. Then $ja \preceq s$ unless s is of the form $\left(j + \frac{a-2}{2}\right) a + (a-1)d$.*

Proof. Say we have $s = aA + (a+d)B + (a+2d)C = a(A+B+C) + d(B+2C)$ where $A + B + C = j + \frac{a}{2} - 1$. Now $s - ja = a(j + \frac{a}{2} - 1) - ja + (B + 2C)d = a(\frac{a}{2} - 1) + (B + 2C)d$. Now let $B + 2C = q_1 a + q_2$ where $q_1 \ge 0, 0 \le q_2 \le a - 1$. We use this substitution and get that $s - ja = a(\frac{a}{2} - 1 + q_1 d) + q_2 d$. Now in order for $s - ja \in S$, by the membership criterion, this is equivalent to stating that $2(\frac{a}{2} - 1 + q_1 d) \ge q_2 \iff a - 2 + q_1 d \ge q_2$. We know that the LHS is at least $a - 2$, and the RHS is at most $a - 1$. The only way the inequality fails is if $q_1 = 0, q_2 = a - 1$, giving us $s = a(A + B + C) + d(a - 1) = a(j + \frac{a}{2} - 1) + (a - 1)d$. $\square$

Proposition 5.3. *Suppose a even and $j \neq \frac{a}{2}$. $\omega(ja) \le j + \frac{a-2}{2}$.*

Proof. By Lemma 5.4, it is clear that $\omega(ja) \le j + \frac{a}{2}$. For sake of contradiction, suppose s is the sum of $j + \frac{a}{2}$ atoms.

Suppose our bullet is (A, B, C) so that we have $A + B + C = j + \frac{a}{2}$. To prove this supposition false, we need to show that we can somehow remove an element from the bullet and still remain in S.

Description of Process: For a factorization (A_0, B_0, C_0), we will need to check whether $m = aA_0 + (a + d)B_0 + (a + 2d)C_0 - ja = a(A_0 + B_0 + C_0 - j) + d(B_0 + 2C_0) \in S$. First write $B_0 + 2C_0 = q_1 a + q_2$ where $q_1 \ge 0, 0 \le q_2 \le a - 1$. Then $m = a(A_0 + B_0 + C_0 - j + q_1 d) + dq_2$. Then by membership criterion, this is equivalent to checking whether $q_2 \le 2(A_0 + B_0 + C_0 - j + q_1 d)$.

Now take our bullet (A, B, C) and remove an element from it (so you're left with either $(A - 1, B, C), (A, B - 1, C), (A, B, C - 1)$). By membership criterion, we need to see when $q_2 \le 2(A + B + C - 1 - j + q_1 d) = 2(\frac{a}{2} - 1 + q_1 d)$. We know that $q_2 \le a - 1$. If $q_2 < a - 1$, then the RHS strictly dominates the LHS. Also, if $q_1 > 0$, then we'd have $q_2 \le a - 1 \le 2(\frac{a}{2} - 1 + d) = a - 2 + d$, which clearly, RHS bounds the LHS from above. Thus $q_1 > 0$ or $q_2 < a - 1$ will show that we can remove an element from the bullet and still remain in S. Thus, we can further assume that $q_1 = 0$ and $q_2 = a - 1$.

Suppose that two of the bullet components are nonzero.

1. Case 1: $A, B > 0$. We either need to show that $(A - 1, B, C)$ or $(A, B - 1, C)$ is in S. For the first factorization, we'd have to see whether $(A - 1 + B + C - j)a + (B + 2C)d \in S$, so we want to show that $q_2 \le 2(A - 1 + B + C + q_1 d - j)$ (refer to description of process above). We want to show that $a - 1 \le 2(j + \frac{a}{2} - 1 - j) = a - 2$. We have $A - 1 + B + C - j = \frac{a}{2} - 1$ and $B + 2C = a - 1$ (recall last sentence of previous paragraph). Now consider $(A, B - 1, C)$. We have $A + B + C - 1 - j = \frac{a}{2} - 1$ and $B - 1 + 2C = a - 1$. This implies both $B + 2C$ and $B + 2C - 1$ are $a - 1$, which clearly is false.

2. Case 2: $A, C > 0$. Here, the only way $a(A - 1 + B + C - ja) + d(B + 2C)$ cannot be a member of S is if $B + 2C = a - 1$, and the only way $a(A + B + C - 1 - ja) + d(B + 2(C - 1)) \notin S$ is if $B + 2(C - 1) = B + 2C - 2 = a - 1$. But since $a \ge 3$, both of these constraints cannot be met.

3. Case 3: $B, C > 0$. Let us interchange $B \to B - 1$. Again, the only way $a(A + B - 1 + C - ja) + d(B - 1 + 2C) \notin S$ is if $B - 1 + 2C = a - 1$. Now we interchange $C \to C - 1$. $a(A + B + C - 1 - ja) + d(B + 2(C - 1)) \notin S$ if $B + 2C - 2 = a - 1$. These two conditions cannot be simultaneously satisfied.

Now suppose that we have a pure bullet of length $\frac{a}{2} + j$. Clearly, it can't be $(A, B, C) = (\frac{a}{2} + j, 0, 0)$ since $(j, 0, 0)$ is already a bullet for ja. Suppose it is $(A, B, C) = (0, 0, \frac{a}{2} + j)$. Well, we again need to have $A + B + C - 1 = \frac{a}{2} + j - 1$ and $B + 2C - 2 = a - 1$. Plugging in $A = B = 0$, these equations simplify to $C - 1 = \frac{a}{2} + j - 1$ and $2(C - 1) = a - 1$. The first equation requires us to have $2(C - 1) = a + 2j - 2$. The second equation tells us that $a + 2j - 2 = a - 1 \iff 2j = 1$, which can't be satisfied, hence a contradiction.

Now for our final case, we have $(A, B, C) = (0, \frac{a}{2} + j, 0)$. We need to have $A + B + C = \frac{a}{2} + j$ and $B - 1 + 2C = a - 1 \iff B + 2C = a$. Since $A = C = 0$, we have $B = \frac{a}{2} + j = a$. If $j \neq \frac{a}{2}$, then there cannot be a pure bullet in the middle component. But if $j = \frac{a}{2}$, then by lemma above, it is a bullet of length $\frac{a}{2}$. □

Theorem 5.1. *For a odd, $\omega(ja) = j + \frac{a-1}{2}$.*

Proof. Let a be odd.

First suppose $j < \frac{a+1}{2}$. Consider the element

$$t = (2j - 1)(a + d) + \left(\frac{a+1}{2} - j\right)(a + 2d) = a\left(j + \frac{a-1}{2} + d\right).$$

Then $t - ja = \left(\frac{a-1}{2} + d\right) a \in S$ and $ja \preceq t$.
Now consider

$$t' = (2j - 2)(a + d) + \left(\frac{a+1}{2} - j\right)(a + 2d) = \left(j + \frac{a-3}{2}\right) a + (a - 1)d.$$

Then $t' - ja = (\frac{a-3}{2})a + (a - 1)d$. But $\lceil \frac{a-1}{2} \rceil > \frac{a-3}{2}$ so by Proposition 5.1 $t' - ja \notin S$ and $ja \not\preceq t'$.
Also consider

$$t'' = (2j - 1)(a + d) + \left(\frac{a+1}{2} - j - 1\right)(a + 2d) = \left(j + \frac{a-3}{2}\right) a + (a - 2)d.$$

Then $t'' - ja = (\frac{a-3}{2})a + (a - 2)d$. But $\lceil \frac{a-2}{2} \rceil > \frac{a-3}{2}$ so by Proposition 5.1 $t'' - ja \notin S$ and $ja \not\preceq t''$.

Thus t is a bullet for ja

Now suppose $j \geq \frac{a+1}{2}$. Consider the element

$$s = \left(j - \frac{a-1}{2}\right) a + (a - 1)(a + d) = \left(\frac{a-1}{2} + j\right) a + (a - 1)d.$$

Then $s - ja = \left(\frac{a-1}{2}\right) a + (a - 1)d$. But $\lceil \frac{a-1}{2} \rceil = \frac{a-1}{2}$ so by Proposition 5.1, $s - ja \in S$ and $s \preceq ja$.
Now consider

$$s' = \left(j - \frac{a-1}{2} - 1\right) a + (a - 1)(a + d) = \left(\frac{a-3}{2} + j\right) a + (a - 1)d.$$

Then $s' - ja = \left(\frac{a-3}{2}\right) a + (a-1)d$. But $\lceil \frac{a-1}{2} \rceil > \frac{a-3}{2}$ so by Proposition 5.1 $s' - ja \notin S$ and $ja \not\preceq s'$.
Also consider

$$s'' = \left(j - \frac{a-1}{2}\right) a + (a-2)(a+d) = \left(\frac{a-3}{2} + j\right) a + (a-2)d.$$

Then $s'' - ja = \left(\frac{a-3}{2}\right) a + (a-2)d$. But $\lceil \frac{a-2}{2} \rceil > \frac{a-3}{2}$ so by Proposition 5.1 $s'' - ja \notin S$ and $ja \not\preceq s''$.

Thus s is a bullet for ja.

□

Theorem 5.2. *For a even and $j \neq \frac{a}{2}$, $\omega(ja) = j + \frac{a-2}{2}$.*

Proof. Let a be even.

First suppose $j < \frac{a}{2}$. Consider the element

$$t = (2j-2)(a+d) + \left(\frac{a}{2} - j + 1\right)(a+2d) = \left(\frac{a-2}{2} + j + d\right) a.$$

Then $t - ja = \left(\frac{a-2}{2} + d\right) a \in S$ and $ja \preceq t$.
Now consider

$$t' = (2j-3)(a+d) + \left(\frac{a}{2} - j + 1\right)(a+2d) = \left(\frac{a-4}{2} + j\right) a + (a-1)d.$$

Then $t' - ja = \left(\frac{a-4}{2}\right) a + (a-1)d$. But $\lceil \frac{a-1}{2} \rceil > \frac{a-4}{2}$ so by Proposition 5.1 $t' - ja \notin S$ and $ja \not\preceq t'$.
Also consider

$$t'' = (2j-2)(a+d) + \left(\frac{a}{2} - j\right)(a+2d) = \left(\frac{a-4}{2} + j\right) a + (a-2)d.$$

Then $t'' - ja = \left(\frac{a-4}{2}\right) a + (a-2)d$. But $\lceil \frac{a-2}{2} \rceil > \frac{a-4}{2}$ so by Proposition 5.1 $t'' - ja \notin S$ and $ja \not\preceq t''$.

Thus t is a bullet for ja.

Now suppose $j > \frac{a}{2}$. Consider the element

$$s = \left(j - \frac{a}{2} + 1\right) a + (a-2)(a+d) = \left(\frac{a-2}{2} + j\right) a + (a-2)d$$

Then $s - ja = \left(\frac{a-2}{2}\right) a + (a-2)d$. But $\lceil \frac{a-2}{2} \rceil = \frac{a-2}{2}$ so by Proposition 5.1 $s - ja \in S$ and $ja \preceq s$.
Now consider

$$s' = \left(j - \frac{a}{2}\right) a + (a-2)(a+d) = \left(\frac{a-4}{2} + j\right) a + (a-2)d$$

Then $s' - ja = \left(\frac{a-4}{2}\right) a + (a-2)d$. But $\lceil \frac{a-2}{2} \rceil > \frac{a-4}{2}$ so by Proposition 5.1 $s' - ja \notin S$ and $ja \not\preceq s'$.
Also consider

$$s'' = \left(j - \frac{a}{2} + 1\right) a + (a-3)(a+d) = \left(\frac{a-4}{2} + j\right) a + (a-3)d$$

Then $s'' - ja = \left(\frac{a-4}{2}\right) a + (a-3)d$. But $\left\lceil \frac{a-3}{2} \right\rceil > \frac{a-4}{2}$ so by Proposition 5.1 $s'' - ja \notin S$ and $ja \not\preceq s''$.

Thus s is a bullet for ja.

$\square$

Definition 5.2. *Define $\beta(n)$ to be the set of all bullets of n.*

Lemma 5.6. *Let a be even then $\{(0, 2i, \frac{a}{2} - i)\}_{i=0}^{\frac{a}{2}} \bigcup \{(i, a-2i, 0)\}_{i=1}^{\frac{a}{2}} \subseteq \beta(\frac{a^2}{2})$.*

Proof. Let $n \in \mathbb{Z} \ni 0 \leq n \leq \frac{a}{2}$. Then we claim that $(0, 2n, \frac{a}{2} - n) \in \beta(\frac{a^2}{2})$, so we want to show that $\frac{a^2}{2} \preceq 2n(a+d) + (\frac{a}{2} - n)(a+2d)$, $\frac{a^2}{2} \not\preceq (2n-1)(a+d) + (\frac{a}{2} - n)(a+2d)$, and $\frac{a^2}{2} \not\preceq 2n(a+d) + (\frac{a}{2} - n - 1)(a+2d)$.
Now

$$\begin{aligned} 2n(a+d) + (\frac{a}{2} - n)(a+2d) - \frac{a^2}{2} &= 2na + 2nd + \frac{a^2}{2} + ad - an - 2nd - \frac{a^2}{2} \\ &= a(n+d) \\ &\in S \end{aligned}$$

Suppose $\frac{a^2}{2} \preceq (2n-1)(a+d) + (\frac{a}{2} - n)(a+2d)$. Then $(2n-1)(a+d) + (\frac{a}{2} - n)(a+2d) - \frac{a^2}{2} \in S$. This implies

$$\begin{aligned} a(A_0 + A_1 + A_2) + d(A_1 + 2A_2) &= (2n-1)(a+d) + (\frac{a}{2} - n)(a+2d) - \frac{a^2}{2} \\ &= an + ad - a - d. \end{aligned}$$

It follows that

$$\begin{aligned} a(A_0 + A_1 + A_2) &\equiv an - a \pmod{d} \\ \implies A_0 + A_1 + A_2 &\equiv n - 1 \pmod{d} \\ \implies A_0 + A_1 + A_2 &= n - 1 + kd \text{ for some } k \in \mathbb{Z} \end{aligned}$$

Substituting $A_0 + A_1 + A_2 = n - 1 + kd$, it is clear that $A_1 + 2A_2 = a - 1 - ak$. Since $A_1 + 2A_2 \geq 0$, we see that $k \leq 0$.

Now,

$$\begin{aligned} \frac{a-1}{2} &\leq \frac{a-1-ak}{2} \\ &= \frac{A_1}{2} + A_2 \\ &\leq A_0 + A_1 + A_2 \\ &= n - 1 + kd \\ &\leq n - 1 \\ &\leq \frac{a}{2} - 1 \end{aligned}$$

Thus we have $\frac{a-1}{2} \leq \frac{a-2}{2}$, a contradiction. Thus $\frac{a^2}{2} \not\preceq (2n-1)(a+d) + (\frac{a}{2}-n)(a+2d)$.

Suppose $\frac{a^2}{2} \preceq 2n(a+d) + (\frac{a}{2}-n-1)(a+2d)$. Note that in order for this condition to be true, n is forced to be at most $\frac{a}{2}-1$. Now $2n(a+d)+(\frac{a}{2}-n-1)(a+2d)-\frac{a^2}{2} \in S$.
Hence

$$\begin{aligned} a(A_0+A_1+A_2)+d(A_1+A_2) &= 2n(a+d)+(\frac{a}{2}-n-1)(a+2d)-\frac{a^2}{2} \\ &= an-a-2d+ad. \end{aligned}$$

Taking both sides modulo d yields

$$\begin{aligned} a(A_0+A_1+A_2) &\equiv an-a(\text{mod } d) \\ \implies A_0+A_1+A_2 &\equiv n-1(\text{mod } d) \ \ \text{since}\,\gcd(a,d)=1 \\ \implies A_0+A_1+A_2 &= n-1+kd \ \ \text{for some } k\in\mathbb{Z}. \end{aligned}$$

This implies

$$\begin{aligned} akd+d(A_1+2A_2) &= ad-2d \\ \implies A_1+2A_2 &= a-ak-2 \end{aligned}$$

Since $A_1+2A_2 \geq 0$ and $a \geq 2$, we have $k \leq 0$.

$$\begin{aligned} \frac{a-2}{2} &\leq \frac{a-ak-2}{2} \\ &= \frac{A_1}{2}+A_2 \\ &\leq A_0+A_2+A_3 \\ &= n-1+kd \\ &\leq n-1 \\ &\leq \frac{a}{2}-2 \end{aligned}$$

So $\frac{a}{2}-1 \leq \frac{a}{2}-2$, a contradiction! Thus $\frac{a^2}{2} \not\preceq 2n(a+d)+(\frac{a}{2}-n-1)(a+2d)$, and so we have $\{(0,2i,\frac{a}{2}-i)\}_{i=0}^{\frac{a}{2}} \subseteq \beta(\frac{a^2}{2})$.

All that is left to show is that $\{(i,a-2i,0)\}_{i=1}^{\frac{a}{2}} \subseteq \beta(\frac{a^2}{2})$.
Let $n \in \mathbb{Z} \ni 1 \leq n \leq \frac{a}{2}$. Consider $(n, a-2n, 0)$. We want to show that $\frac{a^2}{2} \preceq na+(a-2n)(a+d)$, $\frac{a^2}{2} \not\preceq (n-1)a+(a-2n)(a+d)$, and $\frac{a^2}{2} \not\preceq na-(a-2n-1)(a+d)$.
Now,

$$\begin{aligned} na+(a-2n)(a+d)-\frac{a^2}{2} &= \frac{a^2}{2}-na+ad-2nd \\ &= a(\frac{a}{2}-n)+2d(\frac{a}{2}-n) \\ &= (a+2d)(\frac{a}{2}-n) \\ &\in S. \end{aligned}$$

Suppose $\frac{a^2}{2} \preceq (n-1)a + (a-2n)(a+d)$. Then $(n-1)a + (a-2n)(a+d) - \frac{a^2}{2} \in S$.

Using a similar argument as above we see that $A_0 + A_1 + A_2 = \frac{a}{2} - n - 1 + kd, A_1 + 2A_2 = a - 2n - ak$ where $k \in \mathbb{Z}$ and $k \leq 0$.

$$\begin{aligned} \frac{a}{2} - n &\leq \frac{a-2n-ak}{2} \\ &= \frac{A_1}{2} + A_2 \\ &\leq A_0 + A_1 + A_2 \\ &= \frac{a}{2} - n - 1 + kd \end{aligned}$$

Thus $\frac{a}{2} - n \leq \frac{a}{2} - n - 1 + kd \implies 0 \leq kd - 1$ a contradiction as $k \leq 0$. Therefore, $\frac{a^2}{2} \npreceq (n-1)a + (a-2n)(a+d)$.

Lastly, suppose $\frac{a^2}{2} \preceq na-(a-2n-1)(a+d)$. Then $na-(a-2n-1)(a+d)-\frac{a^2}{2} \in S$. Then going through the same argument we find that $A_0+A_1+A_2 = \frac{a}{2}-n-1+kd, A_1+2A_2 = a-2n-1-ak$, where $k \in \mathbb{Z}$ and $k \leq 0$.

It follows that $\frac{a}{2}-n-\frac{1}{2} \leq \frac{a}{2}-n-1$, a contradiction. Therefore, $\frac{a^2}{2} \npreceq na-(a-2n-1)(a+d)$ and so we have $\{(i, a-2i, 0)\}_{i=1}^{\frac{a}{2}} \subseteq \beta(\frac{a^2}{2})$. □

Definition 5.3. *Let $A = (A_1, A_2, \ldots, A_x)$ and $B = (B_1, B_2, \ldots, B_x)$, and $A \neq B$ Then we say A majorizes B if $\forall_{1 \leq i \leq x} A_i \geq B_i$.*

Lemma 5.7. *Let A and B be as above. Suppose $B \in \beta(n)$. If A majorizes B, then $A \notin \beta(n)$.*

Proof. Trivial, and very easy to see. □

Theorem 5.3. *Let $a \in S$ be even. Then $\omega(\frac{a^2}{2}) = a$.*

Proof. Recall $\{(0, 2i, \frac{a}{2} - i)\}_{i=0}^{\frac{a}{2}} \bigcup \{(i, a-2i, 0)\}_{i=1}^{\frac{a}{2}} \subseteq \beta(\frac{a^2}{2})$. Now $(0, a, 0) \in \beta(\frac{a^2}{2})$ so $\omega(\frac{a^2}{2}) \geq a$. Let $(A, B, C) \in \beta(\frac{a^2}{2})$. Then note if $B = a$, then $A = C = 0$ for if not then (A, B, C) majorizes $(0, a, 0)$. So suppose $B < a$, is even, then $B \leq a-2$. Then $B = 2j$ or $B = a - 2j$ for some $1 \leq j \leq \frac{a}{2} - 1$. Now if $B = 2j$ then $C \leq \frac{a}{2} - j$. This gives $B + 2C \leq a$. If $B = a - 2j$, then $A \leq j$ which implies $2A + B \leq a$. It follows that $A + B + C \leq a$, and so we have $\omega(\frac{a^2}{2}) \leq a$. So $\omega(\frac{a^2}{2}) = a$ when B is even. Now suppose B is odd, then $B \leq a - 1$. Now realize since B is odd, it can be written as $B = 2k + 1$ where $1 \leq k \leq \frac{a}{2}$. Now notice $C < \frac{a}{2} - k$ for if not then (A, B, C) majorizes $(0, 2k, \frac{a}{2} - k)$. Thus $C \leq \frac{a}{2} - k - 1$, implying $2C + B \leq a$. Likewise $2A + B \leq a$. After adding our two equations we get $A + B + C \leq a$. Therefore $\omega(\frac{a^2}{2}) = a$. □

Lemma 5.8. *Let S be as above, $a \geq 3$ be odd and $k \geq 2$. Whenever $A_0+A_1+A_2 \geq \lceil \frac{a}{2} \rceil + k - 1$, then $ak \preceq a(A_0 + A_1 + A_2) + d(A_1 + 2A_2)$.*

Proof. Realize by the division algorithm, $A_1 + 2A_2 = q_1 a + q_2$ where $q_1 > 0$ and $0 \leq q_2 \leq a - 1$. Let $t = a(A_0 + A_1 + A_2 - k) + d(A_1 + 2A_2)$.
Now

$$t = a(A_0 + A_1 + A_2 + q_1 d - k) + q_2 d.$$

Then

$$\begin{aligned} q_2 &\leq a-1 \\ &= 2(\frac{a}{2}+\frac{1}{2}-1) \\ &= 2(\left\lceil\frac{a}{2}\right\rceil-1) \\ &\leq 2(\left\lceil\frac{a}{2}\right\rceil+k-1+q_1d-k) \\ &\leq 2(A_0+A_1+A_2+q_1d-k). \end{aligned}$$

Therefore $t \in S$ by the membership criterion. Hence $ak \preceq a(A_0+A_1+A_2)+d(A_1+2A_2)$. $\square$

6 Omega Primality of Generators of Arithmetic Numerical Monoids

Let $S = \langle a, a+d, \ldots a+xd \rangle$ be an arithmetic numerical monoid. We will find the omega primality for all generators of S, $\omega(a+id)$ for $0 \leq i \leq x$.

Before we begin, it is interesting to note that omega-primality is a measurement of how far an element is from being prime. To see this, first consider the natural numbers. In $\mathbb{N}, a|b$ if there exists some $c \in \mathbb{Z}$ such that $ac = b$ and $x \in \mathbb{N}$ is prime if whenever $x|ab$, then $x|a$ or $x|b$. Now consider our numerical monoid S. In S, $a \preceq b$ if there exists some $c \in S$ such that $a+c=b$ and if $x \preceq n_1+n_2$, x is prime if $x \preceq n_1$ or $x \preceq n_2$. Elements with higher ω values are further away from prime.
It is also worth noting that by this definition no element of a numerical monoid will be prime, so $\min(\omega) = 2$.

6.1 $\omega(a)$

let $k = \left\lceil \frac{a-2}{x} \right\rceil + 1$

Lemma 6.1. *Let $s \in S$ be the sum of any $k+1$ atoms. Then $a \preceq s$.*

Proof. Note that we can write

$$s = (k+1)a + Md \text{ for } 0 \leq M \leq k+1.$$

Hence

$$s - a = ka + Md.$$

We can write $M = qa + r$ with $0 \leq r < a$. Thus

$$s - a = (k+qd)a + rd.$$

Clearly $\left\lceil \frac{r}{x} \right\rceil \leq \left\lceil \frac{a-1}{x} \right\rceil \leq k+qd$, thus by Theorem 0.1 we have that $s-a \in S$ and we are done. $\square$

Lemma 6.2. *Let s be the sum of any k atoms of S. Then $a \preceq s$ unless the following happens: $a \equiv 2 \mod x$ and s is of the form $ka+(a-1)d$.*

Proof. Note that we can write

$$s = ka + Md \text{ for } 0 \leq M \leq k.$$

Hence

$$s - a = (k-1)a + Md.$$

As above, write $M = qa + r$ with $0 \le r < a$. Thus

$$s - a = (k - 1 + qd)a + rd$$

Clearly when $r \le a-2$, $\lceil \frac{r}{x} \rceil \le k - 1 + qd$ so $s - a \in S$ by Theorem 0.1. Hence assume $r = a - 1$. Write $a - 2 = ux + v$ with $0 \le v < x$. Then $k - 1 = u$ when $a \equiv 2 \pmod{x}$ and $k-1 = u+1$ otherwise. Note that $\lceil \frac{r}{x} \rceil = \lceil \frac{a-1}{x} \rceil = u+1$, so if $a \not\equiv 2 \pmod{x}$, then $s - a \in S$ by Theorem 0.1. Hence $s - a \notin S$ only when $a \equiv 2 \pmod{x}$ and $q = 0$ (if q is positive then clearly $s - a \in S$). Hence, $s - a \notin S$ is only possible when $a \equiv 2 \pmod{x}$ and $M = a - 1$, meaning $s = ka + (a-1)d$. □

Definition 6.1. *Call the numerical semigroup $S = \langle a, a+d, \ldots, a+xd \rangle$ sporadic if both of the following hold: $a \equiv 2 \pmod{x}$ and $a \equiv 1 \pmod{k}$.*

Lemma 6.3. *Suppose S is not sporadic. Then $\omega(a) \le k$.*

Proof. Let S be not sporadic and let $s = A_1 + ... + A_{k+1}$. If $a \not\equiv 2 \pmod{x}$ then $a \preceq s - A_i$ for any A_i by Lemma 1.2 and we obtain $\omega(a) \le k$. Hence, assume that $a \equiv 2 \pmod{x}$ (meaning $a \not\equiv 1 \pmod{k}$ as S is not sporadic). Without loss of generality, remove A_{k+1}, yielding

$$t = A_1 + ... + A_k.$$

By Lemma 1.2, we know that $a \preceq t$ unless $t = ka + (a-1)d$, so suppose t is of this form. First suppose that there exists some $A_i \ne A_{k+1}$ and consider the element $t' = t + A_{k+1} - A_i$. This can be written as

$$t' = ka + (a - 1 + j_{k+1} - j_i)d = ka + Nd$$

Note that as $A_i \ne A_{k+1}$ we have that $j_{k+1} - j_i \ne 0$ so $0 \le N \le 2a - 2$ and $N \ne a - 1$. But then by Lemma 1.2, we have that $a \preceq t'$.
Now suppose that there does not exist any $A_i \ne A_{k+1}$. Then we can write

$$t = ka + kjd \text{ for } 0 \le j \le x.$$

But $a \not\equiv 1 \pmod{k} \implies kj \ne a - 1$, thus by Lemma 1.2, we have that $a \preceq t$. □

Theorem 6.1. *Let $S = \langle a, a+d, ..., a+xd \rangle$, then we have:*

$$\begin{cases} \omega(a) = k+1 & \text{if } S \text{ is sporadic} \\ \omega(a) = k & \text{otherwise} \end{cases}$$

Proof. First assume that S is sporadic. By Lemma 1.1 it is clear that $\omega(a) \le k+1$. We have that $\frac{a-2}{x} \in \mathbb{Z}$ and $j = \frac{a-1}{k} \in \mathbb{Z}$, yielding

$$j = \frac{a-1}{k} = \frac{a-1}{\frac{a-2}{x} + 1} = x\left(\frac{a-1}{a-2+x}\right),$$

so clearly $j < x$. Hence, $a + jd$ is an atom. Note that by Lemma 1.1 we have $a \preceq (k+1)(a+jd)$. But we also know that $a \not\preceq k(a+jd) = ka + (a-1)d$ by Lemma 1.2, so $(k+1)(a+jd)$ is a bullet for a.

Now assume that S is not sporadic. First suppose $a \not\equiv 2 \pmod{x}$. By Lemma 1.2 it is clear that $\omega(a) \le k$. Let $j = \left\lceil \frac{a-2}{k-1} \right\rceil$ and notice that $j = \left\lceil \frac{a-2}{\lceil \frac{a-2}{x} \rceil} \right\rceil \le \lceil x \rceil$, so clearly $1 \le j \le x$. Consider

$$s = [(a-1) - (k-1)(j-1)](a + jd) + [(k-1)j - (a-2)](a + (j-1)d).$$

$(a-1)-(k-1)(j-1) \ge (a-1)-(k-1)(\frac{a-2}{k-1}-1) > 0$ and $(k-1)j-(a-2) \ge (k-1)\frac{a-2}{k-1}-(a-2) = 0$ so s is the sum of k atoms. By Lemma 1.2 we have that $a \preceq s$. Now consider $s' = s - (a + jd)$. We can write

$$s' = (k-1)a + (a-2)d \implies s' - a = (k-2)a + (a-2)d$$

But $k-2 = \lceil \frac{a-2}{x} \rceil - 1 < \lceil \frac{a-2}{x} \rceil$, so by Theorem 0.1 $s' - a \notin S$ and $a \not\preceq s'$. Also consider $s' = s - (a + (j-1)d)$. We can write

$$s'' = (k-1)a + (a-1)d \implies s'' - a = (k-2)a + (a-1)d$$

But $k-2 = \lceil \frac{a-2}{x} \rceil - 1 < \lceil \frac{a-1}{x} \rceil$, so by Theorem 0.1 $s'' - a \notin S$ and $a \not\preceq s''$. Thus s is a bullet of a.

Now suppose $a \equiv 2 \pmod{x}$. Notice this means that $a \not\equiv 1 \pmod{k}$ as S is not sporadic. Consider $t = k(a + xd) = ka + kxd$. But $kx = a - 2 + x > a - 1$, so by Lemma 1.2 we have that $a \preceq t$. Now consider $t' = t - (a + xd) = (k-1)a + (k-1)xd$. But then $t' - a = (k-2)a + (k-1)xd$. But $\left\lceil \frac{(k-1)x}{x} \right\rceil = k - 1 < k - 2$, so by Theorem 0.1 $t' - a \notin S$ and $a \not\preceq t'$. Thus t is a bullet of a. □

6.2 $\omega(a + id)$

Let $t = \lceil \frac{a-2}{x} \rceil + d + 1$.

Lemma 6.4. *Let s be the sum of any $t+1$ atoms of S. Then $(a + id) \preceq s$.*

Proof. Note we can write $s = (t+1)a + Md$. Then $s - (a + id) = ta + (M - i)d$.
Suppose $M - i \ge 0$. Then we can write $(M - i) = Aa + r$ for $0 \le r \le a - 1$ and

$$s - (a + id) = \left(d + \left\lceil \frac{a-2}{x} \right\rceil + 1 + Ad\right) a + rd$$

Now, $\lceil \frac{r}{x} \rceil \le d + \lceil \frac{a-2}{x} \rceil + 1 + Ad$ and $s - (a + id) \in S$ by the membership criteria.
Next suppose $M - i < 0$. Then $s - (a + id) = \left(\lceil \frac{a-2}{x} \rceil + 1\right) a + (a + M - i)d$. We can write $(a + M - i) = r$ for $0 < r \le a - 1$ and

$$s - (a + id) = (\left\lceil \frac{a-2}{x} \right\rceil + 1)a + rd$$

Now, $\lceil \frac{r}{x} \rceil \le \lceil \frac{a-2}{x} \rceil + 1$ and $s - (a + id) \in S$ by the membership criteria. □

Lemma 6.5. *Let s be the sum of any t atoms of S. Then $(a + id) \preceq s$ unless the following happens: $a \equiv 2 \mod x$ and s is of the form $ta + Nd$ where $N - i = -1$*

Proof. Let s be the sum of t atoms, i.e, $s = ta + Nd$.Then

$$ta + Nd - (a + id) = a(t-1) + d(N - i)$$

First suppose $N \ge i$. Notice we can write $N - i = qa + r$ where $0 \le r < a$ and $q \ge 0$. Then

$$a(t-1) + d(N-1) = a(t - 1 + qd) + dr.$$

which is clearly in S by membership criterion. Now suppose $N < i$. Notice we can write $a(t-1)+d(N-i) = a(t-1-d)+d(N-i+a)$. Then we can write $N-i+a = r$ where $0 < r \leq a-1$. Then

$$a(t-1-d)+d(N-i+a) = a(t-1-d)+dr = \left\lceil \frac{a-2}{x} \right\rceil a + rd$$

first of all if $a \not\equiv 2 \pmod{x}$, then $\lceil \frac{r}{x} \rceil \leq \lceil \frac{a-2}{x} \rceil$. The only way this element fails to be in S is if $\frac{a-2}{x} \in \mathbb{Z}$ and $r = a-1$, which precisely says $N - i = -1$. □

Theorem 6.2. *Let $S = \langle a, a+d, ..., a+xd \rangle$, then we have:*

$$\begin{cases} \omega(a+id) = t+1 & \textit{if } a \equiv 2 \pmod{x} \textit{ and } i \equiv 1 \pmod{t} \\ \omega(a+id) = t & \textit{otherwise} \end{cases}$$

Proof. First suppose $a \equiv 2 \pmod{x}$ and $i \equiv 1 \pmod{t}$. By Lemma 2.1 it is clear that $\omega(a+id) \leq t+1$. Let $m = \frac{i-1}{t} \in \mathbb{Z}$. Notice $1 \leq i \leq x \implies 0 \leq m \leq x$, so $a+md$ is an atom. Consider $s = (t+1)(a+md)$. Notice $a+id \preceq s$ by Lemma 2.1. Now consider $s' = s-(a+md) = ta+(i-1)d$. But $(i-1)-i = -1$ so by Lemma 3.2, $a+id \not\preceq s'$. Thus s is a bullet for $a+id$ and $\omega(a+id) = t+1$.

Now assume that S does not satisfy the first case. First we show the upper bound. If $a \not\equiv 2 \pmod{x}$ then we have that $\omega(a+id) \leq t$ by the above lemma. If $a \equiv 2 \pmod{x}$ (which implies that $i \not\equiv 1 \pmod{t}$) then we have that if s is the sum of $t+1$ atoms:

$$s = A_1 + ... + A_{t+1}$$

We want to show that we can always remove one, A_i such that $a+id$ precedes $s - A_i$. Let $p = s - A_{t+1}$. By above work, we know that $(a+id) \preceq p$ most of the time, and fails when $N - i = -1$, hence assume this happens. Choose A_i for $1 \leq i \leq t$ such that $A_i \neq A_{t+1}$ (this can be done unless all the atoms are the same). Consider

$$\tilde{p} = p - A_i + A_{t+1} = ta + \tilde{N}d - (a+id).$$

Now, $\tilde{N} - i \neq -1$ and therefore, $(a+id) \preceq \tilde{p}$. If all the atoms are the same, then $p = t(a+jd)$, so we have:

$$p - (a+id) = (t-1)a + (tj-i)d$$

this fails to be in S if and only if $tj - i = -1$, but this implies $i \equiv 1 \pmod{t}$ which we said does not happen. Hence, $\omega(a+id) \leq t$.

To show lower bounds we will exhibit bullets that attain the desired values. If $a \equiv 2 \pmod{x}$ and $i \not\equiv 1 \pmod{t}$, then we have that $(a+id) \preceq ta$ by the previous lemma. Also, note that $(a+id) \not\preceq (t-1)a$ since

$$(t-2-d)a + (a-i)d \notin S$$

by an easy application of the membership criterion.

For the bullet when $a \equiv 2 \pmod{x}$ consider the following construction: First recall the definition of k from earlier sections (so we have $t = k+d$). Then write $a-1 = k_1x + k_2$ with $0 \leq k_2 < x$. Note that we have that $k_2 \neq 1$. First we do the case when $k_2 \geq 2$. Let $j = \lfloor \frac{i-1}{d+k-1} \rfloor$. Let m, n satisfy the following system of equations:

$$m + n = d + k \qquad mj + n(j+1) = i - 1 + j$$

Solving the system yields

$$m = (d+k-1)(j+1) - (i-2) \qquad n = i - 1 - (d+k-1)j$$

We claim that $N = m(a+jd) + n(a+(j+1)d)$ is a bullet for $a+id$. Like before, we want to show this decomposition is well-defined (i.e. $0 \le j \le x-1, m \ge 0, n \ge 0$).

Clearly $j \ge 0$. Now we also have that $j = \lfloor \frac{i-1}{d+k-1} \rfloor \le i - 1 \le x - 1$.

Now $n = i - 1 - (d+k-1)j = i - 1 - (d+k-1)\lfloor \frac{i-1}{d+k-1} \rfloor \ge i - 1 - (d+k-1)\frac{i-1}{d+k-1} = 0$.

Now $m = (d+k-1)(j+1) - (i-2) = (d+k-1)(\lfloor \frac{i-1}{d+k-1} \rfloor + 1) - (i-2) \ge (d+k-1)\frac{i-1}{d+k-1} - (i-2) = 1$, so $m \ge 0$.

We need to show that these actually are bullets. We proved earlier that an upper bound for $\omega(a+id)$ is as given in the lemma (so we already know $N - (a+id) \in S$), and now we have to show that $N_1 = N - (a+jd) - (a+id)$ and $N_2 = N - (a+(j+1)d) - (a+id)$ are not in S. First, recall our handy dandy system of equations for m and n, which tells us that $m + n - 2 = d + k - 2$ and $jm + (j+1)n - j - i = -1$.

For the first case, $N_1 = N - (a+jd) - (a+id) = (m+n-2)a + d(jm + (j+1)n - j - i) = (d+k-2)a + d(-1) = (k-2)a + d(a-1)$, which is in canonical form. Now note that $a - 1 = x\frac{a-1}{x} > x(\lceil \frac{a-1}{x} \rceil - 1) = (k-2)x$, so by membership criterion, $N_1 \notin S$.

Before we take a step further, first note that $a - 1 = k_1 x + k_2$, with $2 \le k_2 \le x - 1$. We note that $\lceil \frac{a-1}{x} \rceil = \lceil k_1 + \frac{k_2}{x} \rceil = k_1 + \lceil \frac{k_2}{x} \rceil = k_1 + 1$.

But we also have that $a - 2 = k_1 x + k_2 - 1$, where $1 \le k_2 - 1 \le x - 2$. Then we have $\lceil \frac{a-2}{x} \rceil = k_1 + \lceil \frac{k_2 - 1}{x} \rceil = k_1 + 1$, so $\lceil \frac{a-1}{x} \rceil = \lceil \frac{a-2}{x} \rceil$.

For the second case, $N_2 = N - (a+(j+1)d) - (a+id) = (m+n-2)a + d(-2) = (d+k-2)a + d(-2) = (k-2)a + d(a-2)$. We have $k - 2 = \lceil \frac{a-1}{x} \rceil - 1 = \lceil \frac{a-2}{x} \rceil - 1$. So then we have $a - 2 = x\frac{a-2}{x} > x(\lceil \frac{a-2}{x} \rceil - 1) = x(\lceil \frac{a-1}{x} \rceil - 1) = (k-2)x$, so by membership criterion, $N_2 \notin S$.

Thus, we've constructed an explicit bullet for $a + id$.

The only remaining case is when $a - 1 = k_1 x$ (meaning $k_2 = 0$). In this case the bullet is much easier and it is given by $(t, 0, ..., 0)$. First of all it is clear by our previous lemma that $a + id \preceq ta$, and if we have:

$$(t-1)a - (a+id) = (t-2-d)a + (a-i)d$$

but $t - 2 - d = \lceil \frac{a-2}{x} \rceil - 1 < \lceil \frac{a-2}{x} \rceil = \left\lceil \frac{a-(x-1)}{x} \right\rceil \le \lceil \frac{a-i}{x} \rceil$ and by membership criteria we have that this elements is not in S, i.e., $(t, 0, ..., 0)$ is a bullet for $a + id$. $\square$

7 Omega Primality of Generators of Generalized Arithmetic Numerical Monoids

In the section below let $S = \langle a, ah+d, \ldots, ah+xd \rangle$ where $\gcd(a,d) = 1, h \ge 2$, and $x < a$.

7.1 $\omega(a)$

let $k = \lceil \frac{a-2}{x} \rceil + 1$

Definition 7.1. *If $S = \langle a, ah+d, ah+2d, \ldots, ah+xd \rangle$, define a large atom to be an atom in the set $S - \{a\}$. In other words, it's of the form $ah + bd$.*

Lemma 7.1. *Let $s \in S$ be the sum of any $k+1$ atoms. Then $a \preceq s$.*

Proof. First note that if s has any atoms equal to a, then clearly, $a \preceq s$. Thus assume the $k+1$ atoms are large atoms. Note that we can write

$$s = (k+1)ah + Md \text{ for } 0 \leq M \leq (k+1)x.$$

Hence

$$s - a = ((k+1)h - 1)a + Md.$$

We can write $M = qa + r$ with $0 \leq r < a$. Thus

$$s - a = ((k+1)h - 1 + qd)a + rd.$$

Clearly $\lceil \frac{r}{x} \rceil \leq \lceil \frac{a-1}{x} \rceil \leq k \leq k + 1 - \frac{1}{h} \leq k + 1 - \frac{1}{h} + \frac{qd}{h}$, thus by Theorem 0.1 we have that $s - a \in S$ and we are done. □

Lemma 7.2. *Let s be the sum of any k large atoms of S. Then $a \preceq s$ unless the following happens: $a \equiv 2 \mod x$ and s is of the form $kah + (a-1)d$.*

Proof. Note that we can write

$$s = kah + Md \text{ for } 0 \leq M \leq kx.$$

Hence

$$s - a = (kh - 1)a + Md.$$

As above, write $M = qa + r$ with $0 \leq r < a$. Thus

$$s - a = (kh - 1 + qd)a + rd$$

Clearly when $r \leq a - 2$, $\lceil \frac{r}{x} \rceil \leq k - 1 \leq \frac{kh-1}{h} \leq \frac{kh-1+qd}{h}$ so $s - a \in S$ by Theorem 0.1. Hence assume $r = a - 1$. Write $a - 2 = ux + v$ with $0 \leq v < x$. Then $k - 1 = u$ when $a \equiv 2 \pmod{x}$ and $k - 1 = u + 1$ otherwise. Note that $\lceil \frac{r}{x} \rceil = \lceil \frac{a-1}{x} \rceil = u + 1$, so if $a \not\equiv 2 \pmod{x}$, then $s - a \in S$ by Theorem 0.1. Hence $s - a \notin S$ only when $a \equiv 2 \pmod{x}$ and $q = 0$ (if q is positive then clearly $s - a \in S$). Hence, $s - a \notin S$ is only possible when $a \equiv 2 \pmod{x}$ and $M = a - 1$, meaning $s = kah + (a-1)d$. □

Definition 7.2. *Call the numerical semigroup $S = \langle a, ah+d, \ldots, ah+xd \rangle$* ***sporadic*** *if both of the following hold: $a \equiv 2 \pmod{x}$ and $a \equiv 1 \pmod{k}$.*

Lemma 7.3. *Suppose S is not sporadic. Then $\omega(a) \leq k$.*

Proof. Let S be not sporadic and let $s = A_1 + ... + A_{k+1}$. Clearly, we can assume that the A_i's are all large atoms. If $a \not\equiv 2 \pmod{x}$ then $a \preceq s - A_i$ for any A_i by Lemma 1.2 and we obtain $\omega(a) \leq k$. Hence, assume that $a \equiv 2 \pmod{x}$ (meaning $a \not\equiv 1 \pmod{k}$ as S is not sporadic). Without loss of generality, remove A_{k+1}, yielding

$$t = A_1 + ... + A_k.$$

By Lemma 1.2, we know that $a \preceq t$ unless $t = ka + (a-1)d$, so suppose t is of this form. First suppose that there exists some $A_i \neq A_{k+1}$ and consider the element $t' = t + A_{k+1} - A_i$. This can be written as

$$t' = kah + (a - 1 + j_{k+1} - j_i)d = kah + Nd$$

Note that as $A_i \neq A_{k+1}$ we have that $j_{k+1} - j_i \neq 0$ so $0 \leq N \leq 2a-2$ and thus $N \neq a-1$. But then by Lemma 1.2, we have that $a \preceq t'$.
Now suppose that there does not exist any $A_i \neq A_{k+1}$. Then we can write

$$t = kah + kjd \text{ for } 0 \leq j \leq x.$$

But $a \not\equiv 1 \pmod{k} \implies kj \neq a-1$, thus by Lemma 1.2, we have that $a \preceq t$. □

Theorem 7.1. *Let $S = \langle a, ah+d, ..., ah+xd \rangle$, then we have:*

$$\begin{cases} \omega(a) = k+1 & \textit{if } S \textit{ is sporadic} \\ \omega(a) = k & \textit{otherwise} \end{cases}$$

Proof. First assume that S is sporadic. By Lemma 1.1 it is clear that $\omega(a) \leq k+1$. We have that $\frac{a-2}{x} \in \mathbb{Z}$ and $j = \frac{a-1}{k} \in \mathbb{Z}$, yielding

$$j = \frac{a-1}{k} = \frac{a-1}{\frac{a-2}{x}+1} = x\left(\frac{a-1}{a-2+x}\right),$$

so clearly $j < x$. Hence, $ah+jd$ is an atom. Note that by Lemma 1.1 we have $a \preceq (k+1)(ah+jd)$. But we also know that $a \not\preceq k(ah+jd) = kah + (a-1)d$ by Lemma 1.2, so $(k+1)(a+jd)$ is a bullet for a.

Now assume that S is not sporadic. First suppose $a \not\equiv 2 \pmod{x}$. By Lemma 1.2 it follows that $\omega(a) \leq k$. Let $j = \left\lceil \frac{a-2}{k-1} \right\rceil$ and notice that $j = \left\lceil \frac{a-2}{\lceil \frac{a-2}{x} \rceil} \right\rceil \leq \lceil x \rceil$, so clearly $1 \leq j \leq x$.

Clearly, $j \geq 1$, being the ceiling of a positive number. Now we will show that when $a > 3$, then $j \geq 2$. Since j is an integer, it suffices to show that $j = \left\lceil \frac{a-2}{\lceil \frac{a-2}{x} \rceil} \right\rceil > 1$. We can prove this by showing that $\frac{a-2}{\lceil \frac{a-2}{x} \rceil} > 1 \iff a-2 > \lceil \frac{a-2}{x} \rceil$. It then suffices to show that $a-2 > \frac{a-2+x-1}{x} = \frac{a-3}{x} + 1 \iff a-3 > \frac{a-3}{x} \iff x > 1$.

Now let's see what happens when $a = 3$. Then $k-1 = \lceil \frac{a-2}{x} \rceil = \lceil \frac{1}{x} \rceil = 1$ and $j = \left\lceil \frac{1}{k-1} \right\rceil = 1$. Then $(k-1)j - (a-2) = 1 - 1 = 0$.

Consider

$$s = [(a-1) - (k-1)(j-1)](ah+jd) + [(k-1)j - (a-2)](ah + (j-1)d).$$

$(a-1) - (k-1)(j-1) \geq (a-1) - (k-1)(\frac{a-2}{k-1} + 1 - 1) = 1 > 0$ and $(k-1)j - (a-2) \geq (k-1)\frac{a-2}{k-1} - (a-2) = 0$ so s is the sum of k atoms. Note that in the derivation above, $ah + (j-1)d$ is an atom when $j > 1$, which is the case when $a > 3$. But when $a = 3$, we showed that $(k-1)j - (a-2) = 0$, so the coefficient of $ah + (j-1)d$ in the expression for s is 0, so no harm done there.

By Lemma 1.2 we have that $a \preceq s$. Now consider $s' = s - (ah+jd)$. We can write

$$s' = (k-1)ha + (a-2)d \implies s' - a = (kh - h - 1)a + (a-2)d$$

But $kh - h - 1 = h\lceil \frac{a-2}{x} \rceil - 1 < h \lceil \frac{a-2}{x} \rceil$, so by Theorem 0.1 $s' - a \notin S$ and $a \not\preceq s'$.

Also consider $s'' = s - (ah + (j-1)d)$. We can write

$$s'' = (k-1)ha + (a-1)d \implies s'' - a = (kh - h - 1)a + (a-1)d$$

But $kh - h - 1 = h\left\lceil \frac{a-2}{x} \right\rceil - 1 < h\left\lceil \frac{a-1}{x} \right\rceil$, so by Theorem 0.1 $s'' - a \notin S$ and $a \not\preceq s''$. Thus s is a bullet of a.

Now suppose $a \equiv 2 \pmod{x}$. Notice this means that $a \not\equiv 1 \pmod{k}$ as S is not sporadic. Consider $t = k(ah + xd) = kah + kxd$. But $kx = a - 2 + x > a - 1$, so by Lemma 1.2 we have that $a \preceq t$. Now consider $t' = t - (ah + xd) = (k-1)ha + (k-1)xd$. But then $t' - a = (kh - h - 1)a + (k-1)xd$. But $\left\lceil \frac{(k-1)x}{x} \right\rceil = k - 1 > k - 1 - \frac{1}{h} = \frac{kh-h-1}{h}$, so by Theorem 0.1 $t' - a \notin S$ and $a \not\preceq t'$. Thus t is a bullet of a. □

7.2 $\omega(ah + id)$

Let $S = \langle a, ah+d, \ldots ah+xd \rangle$ be a generalized arithmetic progression numerical semigroup. We will find the omega primality for all generators of the form $ah + id$ where $h \geq 2$ and $1 \leq i \leq x$. Let $k_i = \left\lceil \frac{a-i+x}{x} \right\rceil$.

Lemma 7.4. *For $(ah + id)$ where $h \geq 2$ and $1 \leq i \leq x$, the following is a bullet:*

$$(hk_i + d, 0, 0, \ldots, 0)$$

which means $\omega(ah + id) \geq hk_i + d$.

Proof. Consider the element $(hk_i + d, 0, 0, \ldots, 0)$. Then

$$a(hk_i + d) - (ah + id) = ah\left\lceil \frac{a-i}{x} \right\rceil + (a-i)d$$

Notice, $\left\lceil \frac{a-i}{x} \right\rceil \leq \left\lceil \frac{a-i}{x} \right\rceil$ by Omidali's characterization. Thus $ah\left\lceil \frac{a-i}{x} \right\rceil + (a-i)d \in S$ and $ah + id \preceq (hk_i + d, 0, 0, \ldots, 0)$.
To show that this is a bullet, consider the element $(hk_i + d - 1, 0, 0, \ldots, 0)$. Then

$$a(hk_i + d - 1) - (ah + id) = a(h\left\lceil \frac{a-i}{x} \right\rceil - 1) + (a-i)d$$

Notice, $h\left\lceil \frac{a-i}{x} \right\rceil > h\left\lceil \frac{a-i}{x} \right\rceil - 1$ by Omidali's characterization. Thus $a(h\left\lceil \frac{a-i}{x} \right\rceil - 1) + (a-i)d \notin S$ and $ah + id \not\preceq (hk_i + d - 1, 0, 0, \ldots, 0)$.

□

Lemma 7.5. *Let s be the sum of $hk_i + d$ elements. Then $ah + id \preceq s$ unless s is of the form $(hk_i + d - 1)a + (ah + \beta d)$ with $\left\lceil \frac{a+\beta-i}{x} \right\rceil = \left\lceil \frac{a-i}{x} \right\rceil + 1$*

Proof. Let p be the number of copies of a, and let t be such that $p + t = hk_i + d$. We can then write $s = ap + aht + (\sum_{j=1}^{t} \beta_j)d$. Consider the element

$$s - (ah + id) = ap + aht + (\sum_{j=1}^{t} \beta_j)d - (ah + id) = a(p + h(t-1)) + (\sum_{j=1}^{t} \beta_j - i)d \qquad (4)$$

First consider the case when $\sum \beta_j - i \geq 0$: In this case write $\sum \beta_j - i = Aa + r$ with $0 \leq r \leq a - 1$. Hence we can rewrite above as:

$$a(p + h(t-1) + Ad) + rd$$

Note $h\left\lceil\frac{r}{x}\right\rceil \leq h\left\lceil\frac{a-1}{x}\right\rceil$, we also have that

$$p + h(t-1) + Ad \geq h\left\lceil\frac{a-i+x}{x}\right\rceil + Ad$$

because the left hand side is minimized when t is as low as possible. Since $\sum_{j=1}^{t}\beta_k - i \geq 0$ we have that $t \geq 1$, so this quantity is minimized when $t = 1$ and the inequality holds. Hence we have that

$$h\left\lceil\frac{r}{x}\right\rceil \leq h\left\lceil\frac{a-1}{x}\right\rceil \leq h\left\lceil\frac{a-x+1}{x}\right\rceil \leq h\left\lceil\frac{a-x+1}{x}\right\rceil + Ad \leq p + h(t-1) + Ad$$

and by Omidali's characterization we have that this is in S.

For the case $\sum\beta_j - i < 0$, note that we can rewrite the equation (1) as

$$a(p + h(t-1) - d) + (a + \sum_{j=1}^{t}\beta_j - i)d \tag{5}$$

Let $r = a + \sum_{j=1}^{t}\beta_j - i$. First of all note that the case $t = 0$ is done by the previous lemma. If $t \geq 2$ then we have that

$$p + h(t-1) - d \geq (hk_i + d - 2) + h - d = h\left\lceil\frac{a-i+x}{x}\right\rceil - 2 + h$$

since the left hand side is minimized when t is as low as possible. Note that $\left\lceil\frac{r}{x}\right\rceil \leq \left\lceil\frac{a-i+x}{x}\right\rceil$ and since $h - 2 \geq 0$ we have that

$$p + h(t-1) - d \geq h\left\lceil\frac{a-i+x}{x}\right\rceil - 2 + h \geq \left\lceil\frac{r}{x}\right\rceil$$

and we get that $ah + id$ does indeed preceed s.

The last case is when $t = 1$, in this case equation (2) reads:

$$a(p-d) + (a + \beta - i)d = a(hk_i - 1) + (a + \beta - i)d$$

this is in S if and only if $h\left\lceil\frac{a+\beta-i}{x}\right\rceil \leq h\left\lceil\frac{a-i+x}{x}\right\rceil - 1 = h\left\lceil\frac{a-i}{x}\right\rceil + (h-1)$. Note that $\left\lceil\frac{a+\beta-i}{x}\right\rceil$ is equal to $\left\lceil\frac{a-i}{x}\right\rceil$ or $\left\lceil\frac{a-i}{x}\right\rceil + 1$. In the first case the inequality holds trivialy and in the second case the inequality does not hold, which is precisely what the theorem states. □

Lemma 7.6. *Let s be the sum of any $hk_i + d + 1$ atoms. Then $ah + id \preceq s$.*

Proof. Let $s = A_1 + ... + A_{hk_i+d+1}$ and let $s' = A_1 + ... + A_{hk_i+d}$. By above we know that $ah + id \preceq s'$ unless s is of the form $(hk_i + d - 1)a + (ah + \beta d)$ with $\left\lceil\frac{a+\beta-i}{x}\right\rceil = \left\lceil\frac{a-i}{x}\right\rceil + 1$. If this is the case then define $s" = s' + A_{hk_i+d+1} - A_q$ (with $A_q = a$). Then by the above lemma we have $ah + id \preceq s" \preceq s$, and we are done.

□

Lemma 7.7. *Let s be the sum of m atom, with $m > hk_i + d$ and $ah + id \preceq s$, then we can always find $hk_i + d$ of them such that $ah + id$ precedes their sum. Hence, $\omega(ah + id) \leq hk_i + d$*

Proof. By the above lemma it suffices to assume that $m = hk_i + d + 1$. Then we can do as in the proof above and obtain that either $ah+id \preceq s'$ or $ah+id \preceq s$" and this finishes the proof. □

Theorem 7.2. *Let* $S = \langle a, ah + d, .., ah + xd\rangle$*, then* $\omega(ah + id) = hk_i + d$*.*

Proof. Last lemma proved the upper bound for $\omega(ah + id)$ and our first construction proved the lower bound. Hence we have that $\omega(ah + id) = hk_i + d$.

□

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