Group Final Report

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1 Introduction

In the following paper, we will explore various properties of numerical semigroups, such as specialized elasticity, delta sets, omega primality, and catenary degree. In particular, we will focus on extreme cases of these invariants, such as when specialized elasticity or the size of the delta set is very small, and show what kinds of numerical semigroups have these properties. We will begin with some definitions, which, unless otherwise stated, are adapted from [11]:

Let $S = \langle n_0, n_1, \ldots, n_x \rangle$ be a numerical semigroup, and let $m \in S$. Any element in the set $\{n_0, n_1, \ldots, n_x\}$ is a *generator* or *atom* of S. Since there are $x+1$ generators of S, we say S has *embedding dimension* $x+1$. S is *minimally generated* if no proper subset of $\{n_0, n_1, \ldots, n_x\}$ generates S. We say that S is*primitive*, if $\gcd(n_0, n_1, \ldots, n_x) = 1$. We can write m as a non-negative linear combination of $\{n_0, n_1, \ldots, n_x\}$. That is, $m = \sum_{i=0}^{x} c_i n_i$ is a *factorization* of m in S. If we let z be this factorization of m, then the *length* of z, or $|z|$, is $\sum_{i=0}^{x} c_i$. We denote the set of all factorizations of m as $\mathbb{F}(m)$.

As explained in [2], the *length set* of m, written as $\mathcal{L}(m)$, is $\{|z| : z \in \mathbb{F}(m)\}$. We write $\ell(m)$ for $\min\{l : l \in \mathcal{L}(m)\}$ and $L(m)$ for $\max\{l : l \in \mathcal{L}(m)\}$. If $\mathcal{L}(m) = \{l_1, l_2, \ldots, l_k\}$ where $l_i < l_{i+1}$, then the *delta set* of m is $\Delta(m) = \{l_{i+1} - l_i\}$ for $1 \leq i \leq k-1$. The *delta set* of S is the union of the delta sets of all the elements in S. That is, $\Delta(S) = \bigcup_{m \in S} \Delta(m)$.

Using the notation of [4], for every element $m \in S$, let G_m be the graph with vertices

$$V_m = \{m_i : m - m_i \in S\}$$

and edges

$$E_m = \{m_i m_j : m - (m_i + m_j) \in S\}.$$

Any m for which G_m is a disconnected graph is a *Betti element* of S.

Now, suppose $z = \sum_{i=0}^{x} z_i n_i$ and $z' = \sum_{i=0}^{x} z'_i n_i$ are two factorizations of m in S. If

$$\gcd(z, z') = (\min\{z_0, z'_0\}, \min\{z_1, z'_1\}, \ldots, \min\{z_x, z'_x\}),$$

then define the *distance* between z and z' to be:

$$d(z, z') = \max\{|z - \gcd(z, z')|, |z' - \gcd(z, z')|\}.$$

There is an N-*chain* of factorizations connecting z and z' if there is a sequence of factorizations $z_0, z_1, \ldots, z_k$ such that $z_0 = z$ and $z_k = z'$ and $d(z_i, z_{i+1}) \leq N$ for all i. The *catenary degree* of m, $c(m)$, is the minimal $N \in \mathbb{N} \cup \infty$ such that for any two factorizations of m, z and z', there exists an N-chain from z to z'. The *catenary degree* of S, $c(S)$, is defined to be

$$c(S) = \sup\{c(m) : m \in S\}.$$

According to [8], the *elasticity* of m, denoted $\rho(m)$, is equal to the longest factorization of m divided by the shortest factorization of m. The elasticity of the entire semigroup, $\rho(S)$, is $\max\{\rho(m) : m \in S\}$. For some $k \in \mathbb{N}$, the *specialized elasticity* of a semigroup, which we denote as $\rho_k(S)$, is $\max\{L(m) : \ell(m) \leq k\}$. It can be shown that this is equivalent to finding $\max\{L(m) : \ell(m) = k\}$. (Suppose $\ell(m) < k$ and $L(m) = l$. Then by adding some generator n_i to m we have $\ell(m + n_i) \leq k$ and $L(n + n_i) > l$.) So throughout the paper, when determining $\rho_k(S)$ we will only consider those elements m in S such that $\ell(m) = k$.

Given some generator n_i of S, the *Apéry set of S with respect to n_i* is

$$Ap(S, n_i) = \{s \in S : s - n_i \notin S\}.$$

An equivalent description of the Apéry set is

$$Ap(S, n_i) = \{w_0, \ldots, w_{n_i - 1}\},$$

where each $w_j = \min\{s \in S : s \equiv j \pmod{n_i}\}$. The *Frobenius number* of S, denoted $F(S)$, is the largest natural number that is not in S. It follows from the definition of the Apéry set that $F(S) = \max\{Ap(S, n_i)\} - n_i$.

For any element $n \in S$, we say m *precedes* n, or $m \preceq n$, if $n - m \in S$. As defined in [9], the *omega primality* of m, denoted $\omega_S(m)$ (or simply $\omega(m)$ if it is clear from context which semigroup we are referring to), is the smallest positive integer k such that whenever $m \preceq \sum_{i=0}^{r} a_i$ for $a_i \in \{n_0, n_1, \ldots, n_x\}$ and $r > k$, there exists a subset $T \subset \{1, 2, \ldots, r\}$ with $|T| \le k$ such that $m \preceq \sum_{i \in T} a_i$. A *bullet* of m is an expression $a_1 + a_2 + \cdots + a_l$ such that $m \preceq a_1 + a_2 + \cdots + a_l$ and $m \not\preceq a_1 + a_2 + \cdots + a_l - a_i$ for every $i \in [1, l]$. We denote the set of bullets of m with $\mathrm{bul}(m)$.

Let $S = \langle n_0, n_1, \ldots, n_x \rangle$. If there exist natural numbers d_1 and d_2 and primitive numerical semigroups $A = \langle a_0, a_1, \ldots, a_i \rangle$ and $B = \langle b_{i+1}, b_{i+2}, \ldots, b_x \rangle$ with the following properties, then S is a *gluing* of A and B by $d_1 d_2$, which is the *glue*, and we write $S = d_1 A + d_2 B$:

1. $d_1 \in \langle b_{i+1}, \ldots, b_x \rangle$ but $d_1 \notin \{b_{i+1}, \ldots, b_x\}$,
2. $d_2 \in \langle a_0, \ldots, a_i \rangle$ but $d_2 \notin \{a_0, \ldots, a_i\}$,
3. $\gcd(d_1, d_2) = 1$,
4. $n_j = d_1 a_j$ for each $j \in [0, i]$,
5. $n_j = d_2 a_j$ for each $j \in [i+1, x]$.

An important property of the glue, $d_1 d_2$, is that any factorization of $d_1 d_2$ in S is either a factorization in $d_1 A$ or a factorization in $d_2 B$.

In embedding dimension 2, the numerical semigroup S is *free* if it is the gluing of $\mathbb{N}$ with $\mathbb{N}$, that is, if $S = d_1\langle 1 \rangle + d_2 \langle 1 \rangle$. In any higher embedding dimension n, S is *free* if it is the gluing of a free semigroup in embedding dimension $n-1$ with $\langle 1 \rangle$.

2 Catenary Degree

The following proposition was proven in [10] and restated in [6] in the following way:

Proposition 2.1. *Let S be a minimally generated numerical semigroup. Then*

$$c(S) = \max\{c(b) : b \in \mathrm{Betti}(S)\}.$$

The following theorem was given in [3]:

Theorem 2.2. *If S is the gluing of S_1 and S_2 by d, then* $\mathrm{Betti}(S) = \mathrm{Betti}(d_1 S_1) \cup \mathrm{Betti}(d_2 S_2) \cup \{d\}$.

The following corollary was given in [3] as a consequence of Theorem 2.2. However, this corollary turns out to be false, as shown by the counterexample that follows:

Corollary 2.3. *Assume that S is the gluing of S_1 and S_2 by d, then*

$$c(S) = \max\{c(S_1), c(S_2), c(d)\}.$$

Counterexample 2.4. Consider the numerical semigroup $S = \langle 6, 9, 10, 14 \rangle$, where $S = 2 \cdot \langle 3, 5, 7 \rangle + 9\langle 1 \rangle$. Then, $c(S) = 3$ and $c(\langle 3, 5, 7 \rangle) = 4$. According to Corollary 2.3, we should have $c(S) \ge c(\langle 3, 5, 7 \rangle)$, but this is clearly not the case.

Here is a corrected version of Corollary 2.3:

Corollary 2.5. *Assume that S is the gluing of S_1 and S_2 by d, then*

$$c(S) \le \max\{c(S_1), c(S_2), c(d)\}.$$

Proof. We know from Proposition 2.1 that the catenary degree of a semigroup is the maximum of the catenary degrees of the Betti elements. By Theorem 2.2, the Betti elements of S are the Betti elements of $d_1 S_1$, the Betti elements of $d_2 S_2$, and d. Therefore, $c(S)$ is the maximum of the catenary degrees of $\mathrm{Betti}(d_1 S_1)$, $\mathrm{Betti}(d_2 S_2)$, and d. Let $b \in \mathrm{Betti}(d_1 S_1)$. Then, the catenary degree of b in $d_1 S_1$ is equal to the catenary degree of b in S_1. The catenary degree of b in S is no greater than the catenary degree of b in d_1S_1 because any N-chain that exists between two factorizations of b in d_1S_1 also exists in S. However, the catenary degree of b in S may be smaller that the catenary degree of b in d_1S_1, as shown by Counterexample 2.4, because there may be new factorizations of b in S. The same is true for any Betti element of d_2S_2. Therefore, $c(S) \leq \max\{c(d_1S_1), c(d_2S_2), c(d)\} = \max\{c(S_1), c(S_2), c(d)\}$. □

Lemma 2.6. *If $S = \langle a, b\rangle$ is a primitive numerical semigroup in embedding dimension 2 and $a < b$, then $c(S) = b$.*

Proof. According to Proposition 2.1, $c(S) = \max\{c(b) : b \in \mathrm{Betti}(S)\}$. The only Betti element of S in this case is ab. There are exactly 2 factorizations of ab: ab and ba, and the distance between these factorizations is b. Therefore, the smallest N for which there is an N-chain connecting ab and ba is b, so $c(ab) = c(S) = b$. □

Lemma 2.7. *If S is a primitive numerical semigroup with embedding dimension ≥ 2, then $c(S) \geq 3$.*

Proof. Suppose S is a numerical semigroup with $c(S) \leq 2$. We will show that this implies that S has embedding dimension 1. If $c(S) = 0$, then every element in S has exactly one factorization, so $S = \langle 1\rangle$. If $c(S) = 1$, then for some element $m \in S$, there are two factorizations z and z' such that $d(z, z') = 1$. However, this never happens for any element. Finally, if $c(S) = 2$, then S is half-factorial. However, the only half-factorial semigroup is $\langle 1\rangle$, which has catenary degree 0, as shown in [8]. Therefore, if S is such that $c(S) \leq 2$, then S has embedding dimension 1. The contrapositive is also true: If S has embedding dimension ≥ 2, then $c(S) \geq 3$. □

Theorem 2.8. *Let S_n be a primitive free numerical semigroup in embedding dimension n of the form*

$$S_n = \begin{cases} \langle 2, 3\rangle & n = 2 \\ \langle bs_1, \ldots, bs_{n-1}, i\rangle, b \in \{2, 3\}, i \in \left\{ \sum\limits_{j=1}^{n-1} a_j s_j \not\equiv 0 (\mathrm{mod}\ b) : \sum\limits_{j=1}^{n-1} a_j \in \{2, 3\} \right\} & n > 2 \end{cases}$$

where $S_{n-1} = \langle s_1, \ldots, s_{n-1}\rangle$ is a free semigroup of dimension $n - 1$ with $c(S_{n-1}) = 3$. Then, $c(S_n) = 3$.

Proof. Let S have the form above. First suppose that $n = 2$. By Lemma 2.6, $c(S) = 3$. In fact, $\langle 2, 3\rangle$ is the only numerical semigroup of embedding dimension two with catenary degree three.
Now suppose $n > 2$. Then S_n is the gluing of S_{n-1} and $\langle 1\rangle$ by bi. By Corollary 2.5,

$$c(S_n) \leq \max\{c(S_{n-1}), c(\langle 1\rangle), c(bi)\}.$$

We know that $c(S_{n-1}) = 3$ and that $c(\langle 1\rangle) = 0$.
Since S_n is a gluing with glue bi, any factorization of bi in S_n is either a factorization in bS_{n-1} or in $\langle i\rangle$. Let z be a factorization of bi in bS_{n-1}, so $bi = \sum_{j=1}^{n-1} a_j bs_j$ where $\sum_{j=1}^{n-1} a_j \leq 3$. Let z' be a factorization of bi in $\langle i\rangle$, so $bi = bi$. The greatest common divisor of z and z' is trivial, so $d(z, z') = \max\{|z|, |z'|\} = \max\{\sum_{j=1}^{n-1} a_j, b\} \leq 3$. Hence, $c(bi) \leq 3$.
Therefore, we have that $c(S) \leq \max\{3, 0, c(bi)\} = 3$. In fact, since $n > 2$, by Lemma 2.7, $c(S) = 3$. □

Unfortunately, the converse of Theorem 2.8 is not true, since there are other free numerical semigroups with catenary degree 3 that do not have the form above. (See Counterexample 2.4 for one such semigroup.) However, an important consequence of this theorem is that there are numerical semigroups in every embedding dimension greater than or equal to 2 that have catenary degree 3.

Theorem 2.9. *Let $S = \langle n_1, \ldots, n_p\rangle$ be a numerical semigroup of catenary degree three, and let $\sigma = \{(\alpha_1, \beta_1) \ldots, (\alpha_t, \beta_t)\}$ be a minimal presentation. Pick a subset $\{(\alpha_1, \beta_1), \ldots, (\alpha_{p-1}, \beta_{p-1})\}$ which is linearly independent. Then*

$$\det(X, \alpha_1 - \beta_1, \ldots, \alpha_{p-1} - \beta_{p-1}) = \lambda(n_1x_1 + \cdots + n_px_p)$$

for some $\lambda \in \mathbb{Z}$.

Proof. Let $M = \{(x_1, \ldots, x_p) \in \mathbb{Z}^p : n_1x_1 + \cdots + n_px_p = 0\}$. Now, consider the vector spaces $L_{\mathbb{Q}}(M)$ and $L_{\mathbb{Q}}(\{\alpha_1 - \beta_1, \ldots, \alpha_{p-1} - \beta_{p-1}\})$ (the set of linear combinations with rational coefficients). Note that the second of these is included in the first, and since both have dimension $p-1$ they must be equal. The space $L_{\mathbb{Q}}(\{\alpha_1 - \beta_1, \ldots, \alpha_{p-1} - \beta_{p-1}\})$ is defined by the equation $n_1x_1 + \cdots + n_px_p = 0$. Furthermore, we have $X \in L_{\mathbb{Q}}(M)$ if and only if $\det(X, \alpha_1 - \beta_1, \ldots, \alpha_{p-1} - \beta_{p-1}) = 0$, so $L_{\mathbb{Q}}(M)$ is defined by $m_1x_1 + \cdots + m_px_p = 0$ for integers $m_1, \ldots, m_p$. Therefore, there exists nonzero $\lambda \in \mathbb{Q}$ such that we have $m_j = \lambda n_j$ for $j = 1, \ldots, p$. But since we have $m_j, n_j \in \mathbb{Z}$ and $\gcd(n_1, \ldots, n_p) = 1$, we know $\lambda \in \mathbb{Z}$. □

We can use Theorem 2.9 to find all numerical semigroups in embedding dimension n that have catenary degree 3 using the following steps:

1. Compile a list of all possible equations for a minimal presentation that could produce a semigroup of embedding dimension 3. (The sum of the coefficients on each side of the equations must be less than or equal to 3.)
2. Form an $n \times n$ matrix where the first row contains n variables and where every other row contains the coefficients of an equation from step (1) (the coefficients on one side of the equation will be positive and the coefficients on the other side will be negative).
3. Compute the determinant. The coefficients in front of the n variables are generators for a semigroup. Check to see if that semigroup has catenary degree 3.
4. Repeat steps (2) and (3), taking all possible combinations of rows. Although not every combination will result in a semigroup of catenary degree 3, all semigroups of catenary degree 3 can be found in this way.

Using this method, we were able to find all semigroups in embedding dimension 3 that have catenary degree 3. They are:

$$\langle 3,4,5\rangle, \langle 4,5,6\rangle, \langle 4,5,7\rangle, \langle 4,6,7\rangle, \langle 4,6,9\rangle, \langle 5,6,9\rangle, \langle 6,7,9\rangle, \langle 6,8,9\rangle$$

Note: $\langle 3,4,5\rangle$ and $\langle 4,5,7\rangle$ are the only semigroups that were not found using gluing.

Here is the code that we used:

```
permlist=[]
perm1=permutations([-2,0,3])
perm2=permutations([-2,1,2])
perm3=permutations([-3,1,2])
perm4=permutations([-2,1,1])
perm5=permutations([-3,1,1])

for i in range(6):
    permlist.append(perm1[i])
    permlist.append(perm2[i])
    permlist.append(perm3[i])
for i in range(3):
    permlist.append(perm4[i])
    permlist.append(perm5[i])
print(permlist)

def determinant(l1,l2):
    c1=l1[0]*l2[1]-l1[1]*l2[0]
    c2=-l1[0]*l2[2]+l1[2]*l2[0]
    c3=l1[1]*l2[2]-l1[2]*l2[1]
    return [c1,c2,c3]
```

```
def numlist(List):
    numlist=[]
    for i in range(24):
        for j in range(24):
            numlist.append(determinant(List[i],List[j]))
    return numlist

def narrow(List):
    newSet=[]
    for i in List:
        keep = true
        Gcd=max(1,gcd(i))
        for k in range(3):
            i[k]=i[k]/Gcd
        for j in range(3):
            if i[j]<2:
                keep=false
        if i[0]>=i[1] or i[1]>=i[2] or i[0]>=i[2]:
            keep=false
        if keep:
            newSet.append(i)
    return newSet

Narrow=narrow(numlist(permlist))
Narrow

for l in Narrow:
    S=NumericalSemigroup(l)
    print l,S.CatenaryDegree()

sorted(Narrow)
```

Using this same method, we discovered 157 semigroups in embedding dimension 4 that have catenary degree 3. These semigroups are listed in Appendix 7.1.

3 Special Elasticity

The elasticity of numerical semigroups has been a popular invariant to look at over the years. This invariant describes the largest ratio between the maximum and minimum of the longest factorization lengths. What has not been looked at as much is the special elasticity of a numerical semigroup. The special elasticity of a monoid S or $\rho_k(S)$ is defined [8] to be the following

$$\rho_k(S) = \sup\{\mathcal{L}(n) : n \in S \text{ and } \ell(n) = k\}.$$

A problem posed by Alfred Geroldinger was to investigate extreme cases for special elasticity in semigroups. Specifically, characterize $\rho_2(S) \leq 3$ for a numerical semigroup S. This problem is one that deals with minimal cases for special elasticity. This section discusses semigroups of embedding dimension three and the process taken to characterize a portion of $\rho_2(S) = 2$ and all of $\rho_2(S) = 3$. Also, we will show that removing a single generator from a semigroup generated by a general arithmetic sequence rarely changes the special elasticity.

3.1 Preliminaries

Throughout this section we will use S as a numerical semigroup. Suppose $S = \langle n_1, n_2, n_3 \rangle$ is a numerical semigroup where $\gcd(n_1, n_2, n_3) = 1$. Also suppose that (r, s, t) represents the semigroup element $r(n_1) +$

$s(n_2) + t(n_3)$. For $\rho_2(S)$ we are considering the six elements $(2,0,0)$, $(0,2,0)$, $(0,0,2)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$ and looking for their longest factorizations. Let $\vec{v}$ be one of these elements, and $\vec{u}$ be its longest factorization. Then I claim that if $u \neq v$, then $\vec{v} \cdot \vec{u} = 0$, where $\cdot$ denotes the dot product. The reason is that if both $\vec{u}$ and $\vec{v}$ are strictly positive in some coordinate, then we may subtract one from that coordinate to get again two factorizations that agree – except now one of them is a single irreducible, which is a contradiction.

The following proposition limits the number of factorizations we need to look at when $\rho_2(S) \leq 3$.

Proposition 3.1. *Suppose $S = \langle a, b, c\rangle$ is a numerical semigroup with minimal generators. The only possibilities for sums of pairs of factorizations such that $\rho_2(S) \leq 3$ are unique factorizations and*

- $(1,0,1) = (0,2,0)$ *or* $(0,3,0)$
- $(0,2,0) = (3,0,0)$ *or* $(2,0,1)$
- $(0,1,1) = (3,0,0)$
- $(0,0,2) = (3,0,0)$ *or* $(2,1,0)$ *or* $(1,2,0)$ *or* $(0,3,0)$

Proof. Suppose $S = \langle a, b, c\rangle$ is a numerical semigroup with minimal generators where $a < b < c$. As described above, (r, s, t) denotes the element $r(a) + s(b) + t(c)$ in S. I will first show that $(2,0,0)$ and $(1,1,0)$ are always unique factorizations. After that I will show the rest of possible equalities

Case 1:
Consider $(2,0,0)$ and suppose it has a factorization $\vec{u} = (0, s, t)$ where $s, t \in \mathbb{N}$. Since b and c are atoms and all three generators are distinct, $\vec{u}$ cannot be $(0,1,0)$, $(0,0,1)$, $(0,2,0)$, nor $(0,0,2)$. This gives us $2a = sb+tc$. But since a is smaller than both b and c, $2a < 3b \leq sb$, $2a < 3c \leq tc$, and $2a < (b + c) \leq (xb + yc)$ for $x, y \in \mathbb{Z}^+$. Hence $2a < (sb + tc)$, this is a contradiction. Therefore $(2,0,0)$ has no factorizations other than itself.

Case 2:
Consider $(1,1,0)$ and suppose it has a factorization $\vec{v} = (0,0,t)$. Since c is an atom $\vec{v} \neq (0,0,1)$. This gives us that $a + b = tc$ for $t \geq 2$. But since c is greater than both a and b, $a + b < 2c \leq tc$. Thus $(1,1,0)$ has no factorizations other than itself.

Case 3:
Consider $(1,0,1)$ and it's possible factorizations $\vec{v} = (0,s,0)$. If $s = 1$, then $a+c = b$ which is a contradiction because b is an atom. If $s \geq 4$, then the length of the factorization would be $\rho_2(S) = s > 3$ which is a contradiction. Thus $s \in \{2,3\}$. Therefore the possible factorizations of $(1,0,1)$ are $(0,2,0)$ and $(0,3,0)$.

Case 4:
Consider $(0,2,0)$ and it's possible factorizations $\vec{u} = (r,0,t)$. If $t = 0$, then $2b = ra$. Since $a < b < 2b$, $r \notin \{1,2\}$. From similar reasoning in case 3, $r \leq 3$. Thus $r = 3$. Now if $t = 1$, then $2b = ra + c$. The previous paragraph covers when $r = 1$. Again by similar reasoning in case 3, $r \leq 2$. Hence $r = 2$. Since $2b < 2c$, $t < 2$. Therefore the possible factorizations for $(0,2,0)$ are $(3,0,0)$, $(2,0,1)$, and $(1,0,1)$.

Case 5:
Consider $(0,1,1)$ and it's possible factorizations $\vec{v} = (r,0,0)$. Since $a < 2a < b + c$, $r \notin \{1,2\}$. From similar reasoning in case 3, $r \leq 3$. Thus $r = 3$. Therefore the only possible factorization for $(0,1,1)$ is $(3,0,0)$.

Case 6:
Consider $(0,0,2)$ and it's possible factorizations $\vec{u} = (r,s,0)$. From similar reasoning in case 3, $r, s \leq 3$. If $s = 0$, then $ra = 2c$. Since $a < 2a < 2c$, $r \notin \{1,2\}$. Thus $r = 3$. If $s = 1$, then $ra + s = 2c$. Since $a + b < 2c$, $r \neq 1$. Again by similar reasoning in case 3, $r \leq 2$. Hence $r = 2$. If $s = 2$, then $ra + 2b = 2c$. From similar reasoning in case 3, $r \leq 1$. So $r = 1$. If $s = 3$, then by similar reasoning in case 3, $r = 0$. Therefore the possible factorizations for $(0,0,2)$ are $(3,0,0)$, $(2,1,0)$, $(1,2,0)$, and $(0,3,0)$. ☐

This proposition is key in proving many of our results. It cuts down on the amount of cases that we must look at. Note that this proposition can only be used when the generators are specifically ordered. Now we will draw from other results on special elasticity. We use the following theorem [1] which characterizes ρ_k for semigroups generated by a general arithmetic sequences.

Theorem 3.2. *Let S be a numerical semigroup generated by a general arithmetic sequence $a, ah+d, ah+2d, ..., ah+xd$ with $\gcd(a,d)=1$. Then*

$$\rho_k(S) = \begin{cases} kh + \left\lfloor \frac{kx}{a} \right\rfloor d & \text{if } a \le kx \\ kh + (h-1)\left\lfloor \frac{-k}{x} \right\rfloor & \text{if } a > kx \end{cases}$$

From this theorem we can extrapolate the instances when $\rho_2(S) \le 3$ form this theorem in the following corollaries.

Corollary 3.3. *If $S = \langle a, a+1, \ldots, a+x\rangle$ is a numerical semigroup and $a \le 2x$, then $\rho_2(S) = 3$*

Corollary 3.4. *If $S = \langle a, a+d, a+2d, \ldots, a+xd\rangle$ is a numerical semigroup and $a > 2x$, then $\rho_2(S) = 2$*

Corollary 3.5. *If $S = \langle a, 2a+d, 2a+2d, \ldots, 2a+xd\rangle$ is a numerical semigroup and $a > 2x$, then $\rho_2(S) = 3$*

A useful tool to quickly check if a semigroup has $\rho_k \le m$ for some $m > k$ is to show that m times the smallest generator is greater than k times the largest generator.

Lemma 3.6. *If $S = \langle a, a+x, a+y\rangle$ is a numerical semigroup where $x \in \{1, 2, \ldots, a-2\}$ and $y \in \{2, 3, \ldots, a-1\}$ and $x < y$, then $\rho_2(S) \le 3$.*

Proof. Let $S = \langle a, a+x, a+y\rangle$ be a numerical semigroup where $x \in \{1, 2, \ldots, a-2\}$ and $y \in \{2, 3, \ldots, a-1\}$.

BWOC assume that two generators can be written as 4 generators. Then $r_1+r_2 = r_3+r_4+r_5+r_6$ where r_i are generators. This equation is bounded by $2(a+y)$ and $4a$. Hence $2(a+y) \ge 4a$ simplifies to $y \ge a$. This is a contradiction from our hypothesis and can be generalized for all $n \ge 4$. Therefore $\rho_2(S) \le 3$. □

Example 3.7. Consider $\langle 7, 12, 13\rangle$. Using Proposition 3.1, we need to look at the factorizations $(1,0,1) = 20$, $(0,1,1) = 25$, $(0,2,0) = 24$, and $(0,0,2) = 26$. The first factorization is not a multiple of 12 so we can that $(1,0,1)$ is unique. The second factorization is not a multiple of 7 so we can say that $(0,1,1)$ is unique. The third factorization cannot be written as a linear combination of 7 and 13 so we can say that $(0,2,0)$ is unique. Finally we see that $26 = 12 + 2 \cdot 7$ so we can say that $(0,0,2) = (2,1,0)$. Thus $\rho_2(\langle 7,12,13\rangle) = 3$.

3.2 $\rho_2(S) = 2$

Now we will be delving into the minimal value for $\rho_2(S)$. Although this is the smallest value, this is in no way the most trivial case. From Proposition 3.1 we can partition $\rho_2(S) = 2$ into two categories.

Definition 3.8. If $\rho_2(S) = 2$ and all pairwise factorizations are unique, then we call S *2-unique*

Definition 3.9. If $\rho_2(S) = 2$, the factorizations $(1,0,1)$ and $(0,2,0)$ are equal, and all other pairwise factorizations are unique, then we call S *2-non-unique*

The following theorem characterizes all 2-non-unique semigroups.

Theorem 3.10. *Suppose $S = \langle n_1, n_2, n_3\rangle$ is a minimally generated numerical semigroup where the generators are in ascending order. S is 2-non-unique if and only if $S = \langle n_1, n_1+d, n_1+2d\rangle$ where $\gcd(d, n_1) = 1$ and $n_1 > 4$.*

Proof. Let $S = \langle n_1, n_2, n_3\rangle$ is a minimally generated numerical semigroup where the generators are in ascending order.

($\Rightarrow$) Suppose $S = \langle n_1, n_1+d, n_1+2d\rangle$. By Corollary 3.4 $\rho_2(S) = 2$. We can see that $(0,2,0) = 2(n_1+1) = n_1 + n_1 + 2 = (1,0,1)$. Thus by Proposition 3.1 we only need to show that the factorizations $(0,1,1)$ and $(0,0,2)$ are unique. We are going to let $[(r,s,t)]$ represent the residue of $r(n_1) + s(n_1+d) + t(n_1+2d) \pmod{n_1}$ for $r, s, t > 0$.

Consider $(0,1,1)$, and suppose its longest factorization is $(r,0,0)$. $[(0,1,1)] = [3d]$ and $[(r,0,0)] = [0]$. Hence $3d \equiv 0 \pmod{n_1}$. Since $\gcd(d, n_1) = 1$ we can divide by d. We have $3 \equiv 0 \pmod{n_1}$, but $n_1 \ge 5$ which means we have a contradiction.

Consider $(0,0,2)$, and suppose its longest factorization is $(r,s,0)$. $[(0,0,2)] = [4d]$ and $[(r,s,0)] = [sd]$. Hence $4d \equiv sd \pmod{n_1}$. Similarly we divide by d to get that $4 \equiv s \pmod{n_1}$, and in particular $4 \geq s$. But $(r,s,0) \leq (r,4,0) = rn_1 + 4n_1 + 4d = 2n_1 + 4d = (0,0,2)$ implies $r = -2$ which is a contradiction.

Therefore by Definition 3.9, S is 2-non-unique.

($\Leftarrow$) Suppose S is 2-non-unique. Then we know that $(1,0,1) = n_1 + n_3 = 2n_2 = (0,2,0)$. Hence n_2 is the midpoint of n_1 and n_3 so the only form possible is $S = \langle n_1, n_1 + d, n_1 + 2d\rangle$. Now it is necessary to show that $\gcd(d, n_1) = 1$. Since $d = n_2 - n_1 = n_3 - n_2$, we have $\gcd(d, n_1) = \gcd(n_3, n_2, n_1)$, so the semigroup is reduced exactly when $\gcd(d, n_1) = 1$. This completes the second direction of the if and only if. $\square$

Furthermore, arithmetic sequences of this form are the only semigroups that have the factorization $(1,0,1) = (0,2,0)$.

Corollary 3.11. *Suppose $S = \langle n_1, n_2, n_3\rangle$ is a minimally generated numerical semigroup where the generators are in ascending order. $S = \langle n_1, n_1 + d, n_1 + 2d\rangle$ where $\gcd(d, n_1) = 1$ if and only if the factorization $(1,0,1) = (0,2,0)$ holds.*

Interestingly enough there are only 2 semigroups of the above form that have $\rho_2 = 3$. They are $\langle 3,4,5\rangle$ and $\langle 4,5,6\rangle$. This results directly from Corollary 3.3.

Unfortunately, *2-unique* semigroups are much harder to characterize. We have only begun to scratch the surface of this type. Although we have found some notable forms. The first form is where at least two atoms share a common factor, the largest common factor between any two atoms is 3, and that the multiplicity of the semigroup is at least 8. The second form characterizes all semigroups which have multiplicity 7.

Proposition 3.12. *If $S = \langle ba, ca, n\rangle$ is a minimally generated numerical semigroup where $a \geq 3$ is the largest common factor between any two atoms, $2 < b < c$, and $2c < n$, then S is 2-unique.*

Proof. Let $S = \langle ba, ca, n\rangle$ where $a \geq 3$. We can use the division algorithm to get $n = qa + r$. Let (j,k,l) be the semigroup element $j(ba) + k(ca) + l(qa + r)$, and let $[(j,k,l)] = [lr]$ be the residue modulo a. Note we have not designated any order between generators.

Consider $(1,1,0)$ and suppose its longest factorization is $(0,0,l)$. $[(1,1,0)] = [0] = [rl] = [(0,0,l)]$. But since $\gcd(r,a) = 1$ we must have $l \geq a$. Hence $ba + ca = l(n) \geq an$, so $n \leq b + c$, contradiction. Thus $(1,1,0)$ is a unique factorization.

Consider $(1,0,1)$ and suppose its longest factorization is $(0,k,0)$. $[(1,0,1)] = [r] \neq [0] = [(0,k,0)]$ and thus $(1,0,1)$ is a unique factorization.

Consider $(0,1,1)$ and suppose its longest factorization is $(j,0,0)$. $[(0,1,1)] = [r] \neq [0] = [(j,0,0)]$ and thus $(0,1,1)$ is a unique factorization.

Consider $(2,0,0)$ and suppose its longest factorization is $(0,k,l)$. $[(2,0,0)] = [0] = [rl] = [(0,k,l)]$. Either $l = 0$ (impossible since $ba < ca$) or $l \geq a$. Hence $2(ba) = k(ca) + l(n) \geq k(ca) + an$, so $n \leq 2b$, contradiction. Thus $(2,0,0)$ is a unique factorization.

Consider $(0,2,0)$ and suppose its longest factorization is $(j,0,l)$. $[(0,2,0)] = [0] = [rl] = [(j,0,l)]$. Either $l = 0$ (impossible since $b \geq 3$) or $l \geq a$. Hence $2(ca) = j(ba) + l(n) \geq j(ba) + an$, so $n \leq 2c$, contradiction. Thus $(0,2,0)$ is a unique factorization.

Consider $(0,0,2)$ and suppose its longest factorization is $(j,k,0)$. $[(0,0,2)] = [2r] \neq [0] = [(j,k,0)]$ and thus $(0,0,2)$ is a unique factorization.

Therefore S is 2-unique. $\square$

In the following proposition we fix the multiplicity to be 7 and form the other two generators by congruence classes modulo 7. From analyzing data we noticed that when we look at the residues modulo 7, certain relationships appeared between the two larger generators.

Proposition 3.13. *Let $S = \langle 7, c, d\rangle$ be a minimally generated numerical semigroup where 7 is the multiplicity. Then S is 2-unique if and only if $S = \langle 7, c, 3c - 7\gamma\rangle$ where $\gcd(7, c) = 1$ and $\gamma \in \{2, 3, \ldots, \lceil \frac{c}{3} \rceil - 1\}$.*

Proof. Suppose that $S = \langle 7, c, d\rangle$ be a minimally generated numerical semigroup and S is not of the form from Theorem 3.10. Notice that we do not restrict $c < d$ or alternatively $d < c$. They may be in either order.

($\Rightarrow$) Assume S is 2-unique. Since S is minimally generated, we know that the $\gcd(7, c, d) = 1$. In order to show that S is of the form $\langle 7, c, 3c - 7\gamma\rangle$ we must show that $d \equiv 3c \pmod 7$. To do this we will show that the residue of d modulo 7 cannot be $0, c, 2c, 4c, 6c$. Also when $d \equiv 5c \pmod 7$ we get that

$$3d \equiv 15c \equiv c \pmod 7. \tag{1}$$

Therefore we can rename the second and third generators so that we have our desired form. Now we will begin our 5 cases. For any of the following cases we will use that $d \equiv lc \pmod 7$ implies that $d = lc + 7j$ for $l \in \{0, 1, 2, 4, 6\}$ and $j \in \mathbb{Z}$. In every case j must be negative or else we can simply write d in terms of multiples of the first and second generators. Hence we will refer to $d = lc - 7k$ for $l \in \{0, 1, 2, 4, 6\}$ and $k \in \mathbb{Z}^+$.

Case 1: $l = 0$
Suppose that $d = -7k$. This means that $d < 0$, thus a contradiction.

Case 2: $l = 1$
Suppose that $d = c - 7k$. Then we can write $(0, 1, 0) = c = 7k + c - 7k = (k, 0, 1)$. We have a contradiction since c is an atom.

Case 3: $l = 2$
Suppose that $d = 2c - 7k$. Then we can write $(0, 2, 0) = 2c = 7r + 2c - 7k = (r, 0, 1)$ for some $r \in \mathbb{Z}^+$. This leads us to $r = k$. We assumed that $\rho_2(S) = 2$, so $r = k = 1$. This means that S is overlapping, which is a contradiction since we assumed S to be definite.

Case 4: $l = 4$
Suppose that $d = 4c - 7k$. Then we can write $(0, 0, 2) = 2(4c - 7k) = c + 7(c - 2k) = (c - 2k, 1, 0)$. If $c - 2k > 0$, then $c - 2k = 1$ because $\rho_2(S) = 2$. Again this implies that S is overlapping, so contradiction. If $c - 2k = 0$, then $c = 2(4c - 7k) = (0, 0, 2)$. But c is an atom so contradiction. If $c - 2k < 0$, then $c = 2(4c - 7k) + (2k - c)7 = (2k - c, 0, 2)$. This isn't possible since c is an atom.

Case 5: $l = 6$
Suppose that $d = 6c - 7k$. Consider $(0, 1, 1) = c + 6c - 7k = 7(c - k)$. If $c \leq k + 1$, then $7(c - k) <\leq 7$ which is a contradiction since 7 is the multiplicity. Hence we consider when last possible choice that $c = k + 2$. This gives us $S = \langle 7, k + 2, 12 - k\rangle$. From $k + 2$ we get that $k > 5$, but from $12 - k$ we get that $k < 5$. This is a contradiction. Therefore $S = \langle 7, c, 3c - 7k\rangle$. If $\gcd(c, 7)z > 1$, then $7|z$. This implies that $\gcd(7, c, 3c - 7k) \neq 1$. Hence $\gcd(c, 7) = 1$.

Now we must show that $k \in \{2, 3, \ldots, \lceil \frac{c}{3} \rceil - 1\}$. It will be shown that $2 \leq k$ and $k \leq \lceil \frac{c}{3} \rceil - 1$.

Case i: $k = 1$
Then $(1, 0, 1) = 7 + 3c - 7 = 3c = (0, 3, 0)$. Whence, $\rho_2(S) = 3$ which is a contradiction.

Case ii: $k \geq \lceil \frac{c}{3} \rceil$
Then we can write $k = \lceil \frac{c}{3} \rceil + y$ where $y \geq 0$. Choose x to be the smallest non-negative integer such that $3|(c + x)$. This bounds $x \in \{0, 1, 2\}$. Then we can say that $\lceil \frac{c}{3} \rceil = \frac{c+x}{3}$. Hence

$$\begin{aligned}
(0, 2, 0) = 2c &= 2c + 7c - 7c + 7x - 7x + 7(3y) - 7(3y) \\
&= 9c - 7(c + x + 3y) + 7(x + 3y) \\
&= 3[3c - 7(\frac{c + x}{3} + y)] + 7(x + 3y) \\
&= (x + 3y, 0, 3).
\end{aligned}$$

Therefore $\rho_2(S) \geq 3$ which is a contradiction. This completes our bounds on k and the forward direction.

($\Rightarrow$) Now let $S = \langle 7, c, 3c - 7\gamma\rangle$ where $\gcd(7, c) = 1$ and $\gamma \in \{2, 3, \ldots, \lceil \frac{c}{3} \rceil - 1\}$. From Proposition 3.1 we need only concern ourselves with $(0, 1, 1)$, $(1, 0, 1)$, $(0, 0, 2)$, and $(0, 2, 0)$.

Consider $[(0, 1, 1)]$ and suppose its longest factorization is $[(r, 0, 0)]$. Then $[(0, 1, 1)] = [4c] = [0] = [(r, 0, 0)]$, but since $\gcd(7, c) = 1$, $[4c] \neq [0]$.

Consider $[(1, 0, 1)]$ and suppose its longest factorization is $[(0, s, 0)]$. Then $[(1, 0, 1)] = [3c] = [sc] = [(0, s, 0)]$, in particular $s \geq 3$. We have $(0, s, 0) = sc \geq 3c > 3c + 7(1 - k) = (1, 0, 1)$ since $k > 1$, which is a contradiction.

Consider $[(0, 0, 2])$ and suppose its longest factorization is $[(r, s, 0)]$. Then $[(0, 0, 2)] = [6c] = [sc] = [(r, s, 0)]$, in particular $s \geq 6$. We have $(r, s, 0) \geq (0, 6, 0) = 6c > 6c - 7(2k) = (0, 0, 2)$, which is a contradiction.

Consider $[(0,2,0)]$ and suppose its longest factorization is $[(r,0,)t]$. Then $[(0,2,0)] = [2c] = [3ct] = [(r,0,t)]$, in particular $3t \geq 2$. The smallest t that satisfies this is 3, so $t \geq 3$. Again assume that x is the smallest non-negative integer such that $3|(c+x)$ and $\lceil \frac{c}{3} \rceil = \frac{c+x}{3}$. From our bounds on k, we can write $k = \lceil \frac{c}{3} \rceil - z = \frac{c+x}{3} - z$ for $1 \leq z \leq \lceil \frac{c}{3} \rceil - 2$. Then

$$\begin{aligned} (r,0,t) \geq (0,0,3) = 9c - 7(3k) &= 2c = (0,2,0) \\ 7c - 7(3k) &= 0 \\ c &= 3k \\ c &= c + x - 3z \\ 3z &= x \end{aligned} \tag{2}$$

Equation (2) is a contradiction since $x < 3$ and $3z \geq 3$. Therefore S is 2-unique and we have completed our proof. □

Example 3.14.

These are two different methods to look at *2-unique* semigroups. Looking at them in the symmetric/non-symmetric sense and organizing them by multiplicity. We had difficulties in both directions. As you will see later it is very difficult to work with non-symmetric semigroups and it seemed that there were gaps or missing elements in the patterns we were looking at. While characterizing by multiplicity we were hoping to find a more general form by using the congruence class of one of the other two generators. As the multiplicities grew, the patterns seemed to use more and more congruence classes.

3.3 $\rho_2(S) = 3$

We are going to split this up into two cases. The first will be semigroups where at least two generators share a common factor greater than 1. The Second will be when all generators are pairwise coprime.

Definition 3.15. We call $S = \langle x, y, z \rangle$ *symmetric*, if at least one pair of atoms share a common factor greater than 1.

Definition 3.16. We call $S = \langle x, y, z \rangle$ *non-symmetric*, if x, y, and z are pairwise coprime.

These definitions are not to be confused with symmetric from [11]. Our definitions deal strictly with common factors between generators of semigroups.

3.3.1 Symmetric Semigroups

Suppose $S = \langle x, y, z \rangle$ is a minimally generated numerical semigroup. The following lemmas and propositions will characterize many numerical semigroups of embedding dimension 3 when $\rho_2(S) = 3$. The goal is to characterize all semigroups where the atoms are pairwise not coprime, two pairs of atoms have a common factor greater than 1 but the other pair doesn't, and where exactly one pair of atoms has a common factor greater than 1. To do this we will change our semigroup form to $S = \langle ab, ac, n \rangle$ where a is the largest common factor between any two atoms and the atoms are not necessarily in order. This leaves n to fall into any of the three partitions above. Also, this forces b and c to be coprime. For simplicity, we make $b < c$. We will split up these 3 types for $\rho_2(S) = 3$ into five different cases.

Lemma 3.17 will characterize $\rho_2(S) = 3$ for our smallest possible values for a, b, and c. Proposition 3.18 will characterize $\rho_2(S) = 3$ for $a = 2$ and non-minimal values of b and c. Lemma 3.20 will characterize $\rho_2(S) = 3$ for $2 < a$ and any values of b and c, where $n \leq 2c$. Lemma 3.22 will characterize $\rho_2(S) = 3$ for $2 < a$ and the minimal values of b and c, where $2c < n$. Proposition 3.12 showed that for $2 < a$ and non-minimal values of b and c, where $2c < n$, there are only 2-unique semigroups.

Consider the general form of $\langle ab, ac, n \rangle$ and let $b = 2$. Now we can write the factorization $(0,2,0) = 2(ac) = c(2a) = (c,0,0)$. Thus we can say that $\rho_2(\langle 2a, ac, n \rangle) \geq c$. Thus when $b = 2$ we must also have $c = 3$. The following Lemma is the smallest case, where $a = 2$, $b = 2$, and $c = 3$.

Lemma 3.17. *Suppose $S = \langle 4, 6, n \rangle$ is a minimally generated numerical semigroup. $\rho_2(S) = 3$ if and only if $n \in \{5, 7\}$.*

Proof. Let $S = \langle 4, 6, n\rangle$ be a minimally generated numerical semigroup. We can use the division algorithm to get $n = 2q + 1$. Let (j, k, l) be the semigroup element $j(4) + k(6) + l(2q + 1)$, and let $[(j, k, l)] = [l]$ be the residue modulo 2. Note we have not designated any order between generators.

($\Rightarrow$) Suppose that $\rho_2(S) = 3$.

Consider $(1, 1, 0)$ and suppose its longest factorization is $(0, 0, l)$. $[(1, 1, 0)] = [0] = [l] = [(0, 0, l)]$. But since $\gcd(r, 2) = 1$ we must have $3 \geq l \geq 2$. This leaves $l = 2$. Hence $4 + 6 = 2(n)$, so $n = 5$.

Consider $(1, 0, 1)$ and suppose its longest factorization is $(0, k, 0)$. $[(1, 0, 1)] = [1] \neq [0] = [(0, k, 0)]$ and thus $(1, 0, 1)$ is a unique factorization.

Consider $(0, 1, 1)$ and suppose its longest factorization is $(j, 0, 0)$. $[(0, 1, 1)] = [1] \neq [0] = [(j, 0, 0)]$ and thus $(0, 1, 1)$ is a unique factorization.

Consider $(2, 0, 0)$ and suppose its longest factorization is $(0, k, l)$. $[(2, 0, 0)] = [0] = [l] = [(0, k, l)]$. Either $l = 0$ (impossible since $2b < 2c$) or $3 \geq l \geq 2$. This leaves $l = 2$. Hence $2(4) = k(6) + 2(n)$, so either $j = 0$ (impossible since n is odd) or $j = 1$. Thus $n = 4 - 3 = 1$ which is a contradiction since S is minimally generated.

Consider $(0, 2, 0)$ and suppose its longest factorization is $(j, 0, l)$. $[(0, 2, 0)] = [0] = [l] = [(j, 0, l)]$. Either $l = 0$ (which gives $j = 3 = \rho_2(S)$) or $3 \geq l \geq 2$. This leaves $l = 2$. Hence $2(6) = j(4) + 2(n)$, so either $j = 0$ (impossible since n is odd) or $j = 1$. Thus $n = 6 - 2 = 4$, a contradiction.

Consider $(0, 0, 2)$ and suppose its longest factorization is $(j, k, 0)$. $[(0, 0, 2)] = [0] = [0] = [(j, k, 0)]$. So either $j = k = 1$ or $j = 2$ and $k = 1$ or $j = 1$ and $k = 2$. The case where $j = k = 1$ is covered in the $(1, 0, 1)$ factorization. Thus either $n = 6 + 2 = 8$ (contradiction) or $n = 4 + 3 = 7$.

Therefore $n \in \{5, 7\}$.

($\Leftarrow$) Consider $S = \langle 4, 5, 6\rangle$, by Corollary 3.3, $\rho_2(S) = 3$. Consider $S = \langle 4, 6, 7\rangle$, by Lemma 3.6, $\rho_2(S) \leq 3$. But since $(0, 0, 2) = (2, 1, 0)$, $\rho_2(S) = 3$. $\square$

The next proposition is the case where $a = 2$ and $2 < b < c$.

Proposition 3.18. *Suppose $S = \langle 2b, 2c, n\rangle$ is a minimally generated numerical semigroup where 2 is the largest common factor between any two atoms and $3 \leq b < c$. Then $\rho_2(S) = 3$ if and only if $n \in \{2c \pm b, 2b \pm c\}$*

Proof. Let $S = \langle 2b, 2c, n\rangle$ be a minimally generated numerical semigroup where 2 is the largest common factor between any two atoms and $b < c$. This implies that $\gcd(b, c) = 1$. We can use the division algorithm to get $n = 2q + 1$. Let (j, k, l) be the semigroup element $j(2b) + k(2c) + l(2q + 1)$, and let $[(j, k, l)] = [l]$ be the residue modulo 2. Note we have not designated any order between generators.
($\Rightarrow$) Suppose that $\rho_2(S) \leq 3$.

Consider $(1, 1, 0)$ and suppose its longest factorization is $(0, 0, l)$. $[(1, 1, 0)] = [0] = [l] = [(0, 0, l)]$. Since $\gcd(r, 2) = 1$ we must have $3 \geq l \geq 2$. This leaves $l = 2$. Hence $2b + 2c = 2(n)$, so $n = b + c$. But in this case we could write $S = \langle x, x + d, x + 2d\rangle$ where $x = 2b$ and $d = c - b$. By Corollary 3.4, $\rho_2(S) = 2$, which is a contradiction.

Consider $(1, 0, 1)$ and suppose its longest factorization is $(0, k, 0)$. $[(1, 0, 1)] = [1] \neq [0] = [(0, k, 0)]$ and thus $(1, 0, 1)$ is a unique factorization.

Consider $(0, 1, 1)$ and suppose its longest factorization is $(j, 0, 0)$. $[(0, 1, 1)] = [1] \neq [0] = [(j, 0, 0)]$ and thus $(0, 1, 1)$ is a unique factorization.

Consider $(2, 0, 0)$ and suppose its longest factorization is $(0, k, l)$. $[(2, 0, 0)] = [0] = [l] = [(0, k, l)]$. Either $l = 0$ (impossible since $2b < 2c$) or $3 \geq l \geq 2$. This leaves $l = 2$. Hence $2(2b) = k(2c) + 2(n)$, so either $j = 0$ (impossible since n is odd) or $j = 1$. Thus $n = 2b - c$.

Consider $(0, 2, 0)$ and suppose its longest factorization is $(j, 0, l)$. $[(0, 2, 0)] = [0] = [l] = [(j, 0, l)]$. Either $l = 0$ (impossible since $b \geq 3$) or $3 \geq l \geq 2$. This leaves $l = 2$. Hence $2(2c) = j(2b) + 2(n)$, so either $j = 0$ (impossible since n is odd) or $j = 1$. Thus $n = 2c - b$.

Consider $(0, 0, 2)$ and suppose its longest factorization is $(j, k, 0)$. $[(0, 0, 2)] = [0] = [0] = [(j, k, 0)]$. So either $j = k = 1$ or $j = 2$ and $k = 1$ or $j = 1$ and $k = 2$. The case where $j = k = 1$ is covered in the $(1, 0, 1)$ factorization. Thus either $n = 2b + c$ or $n = 2c + b$.
($\Leftarrow$) It is only necessary to show that for the five values of n listed above, $\rho_2(S) \leq 3$.

Case 1: $S = \langle 2b, 2c, 2c + b\rangle$
This implies that b must be odd. By Proposition 3.1, we only need to check four factorizations.

Consider $(1,0,1)$ and suppose its longest factorization is $(0,k,0)$. $[(1,0,1)] = [1] \neq [0] = [(0,k,0)]$ and thus $(1,0,1)$ is a unique factorization.

Consider $(0,1,1)$ and suppose its longest factorization is $(j,0,0)$. $[(0,1,1)] = [1] \neq [0] = [(j,0,0)]$ and thus $(0,1,1)$ is a unique factorization.

Consider $(0,2,0)$ and suppose its longest factorization is $(j,0,l)$. $[(0,2,0)] = [0] = [l] = [(j,0,l)]$. Either $l = 0$ (impossible since $b \geq 3$) or $l \geq 2$. If $l \geq 2$, then $(0,2,0) = 4c < l(2c+b) \leq (j,0,l)$. Thus $(0,2,0)$ is a unique factorization.

Consider $(0,0,2)$ and suppose its longest factorization is $(j,k,0)$. $[(0,0,2)] = [0] = [0] = [(j,k,0)]$. Hence we have $2(2c+b) = j(2b) + k(2c)$, which simplifies to $2c+b = jb+kc$. If $k = 0$, then $2c \equiv 0 \pmod b$ but $b \geq 3$. If $k = 1$, then $c \equiv 0 \pmod b$ but $b \geq 3$. If $k \geq 3$, then $(0,0,2) = 2(2c+b) < 2(ck) \leq (j,k,0)$. If $k = 2$, then $2c+b = jb+2c$, so $j = 1$.
Therefore $\rho_2(S) = 3$.

Case 2: $S = \langle 2b, 2c-b, 2c \rangle$
This implies that b must be odd. By Proposition 3.1, we only need to check four factorizations.

Consider $(1,0,1)$ and suppose its longest factorization is $(0,k,0)$. $[(1,0,1)] = [0] = [k] = [(0,k,0)]$. Either $k = 0$ (impossible) or $k \geq 2$. If $k = 2$, then $2(2b) = 2c$, a contradiction since $2c$ is an atom. Suppose $k \geq 4$. Since $c - b > 0$, $2c - b > c$. Hence $(1,0,1) = 2b + 2c < 4c \leq k(2c-b) = (0,k,0)$. Thus $(1,0,1)$ is a unique factorization.

Consider $(0,1,1)$ and suppose its longest factorization is $(j,0,0)$. $[(0,1,1)] = [1] \neq [0] = [(j,0,0)]$ and thus $(0,1,1)$ is a unique factorization.

Consider $(0,2,0)$ and suppose its longest factorization is $(j,0,l)$. $[(0,2,0)] = [0] = [0] = [(j,0,l)]$. Whence we have $2(2c-b) = j(2b) + l(2c)$, which simplifies to $2c - b = jb + lc$. If $l = 0$, then $2c \equiv 0 \pmod b$ but $b \geq 3$. If $l = 1$, then $c \equiv 0 \pmod b$ but $b \geq 3$. If $l \geq 2$, then $(0,2,0) = 2(2c-b) < l(2c) \leq (j,0,l)$. Thus $(0,2,0)$ is a unique factorization.

Consider $(0,0,2)$ and suppose its longest factorization is $(j,k,0)$. $[(0,0,2)] = [0] = [k] = [(j,k,0)]$. We have $2(2c) = j(2b) + k(2c-b)$. $l = 0,1,3$ are impossible since $b \geq 3$. If $l \geq 4$, then $(0,0,2) = 4c < k(2c-b) \leq (j,k,0)$. If $l = 2$, then we get $4c = j(2b) + 4c - 2b$. Thus $j = 1$.
Therefore $\rho_2(S) = 3$.

Case 3: $S = \langle 2b, 2c, 2b+c \rangle$
This implies that c must be odd. In this case we do not know whether $2b < c$ or $c < 2b$, so we must consider all 6 factorizations.

Consider $(1,1,0)$ and suppose its longest factorization is $(0,0,l)$. $[(1,1,0)] = [0] = [l] = [(0,0,l)]$. Either $l = 0$ (impossible) or $l \geq 2$. If $l \geq 2$, then $(1,1,0) = 2(b+c) < l(2b+c) = (0,0,l)$. Thus $(1,1,0)$ is a unique factorization.

Consider $(1,0,1)$ and suppose its longest factorization is $(0,k,0)$. $[(1,0,1)] = [1] \neq [0] = [(0,k,0)]$ and thus $(1,0,1)$ is a unique factorization.

Consider $(0,1,1)$ and suppose its longest factorization is $(j,0,0)$. $[(0,1,1)] = [1] \neq [0] = [(j,0,0)]$ and thus $(0,1,1)$ is a unique factorization.

Consider $(2,0,0)$ and suppose its longest factorization is $(0,k,l)$. $[(2,0,0)] = [0] = [l] = [(0,k,l)]$. Either $l = 0$ (impossible since $4 \leq c$) or $l \geq 2$. If $l \geq 2$, then $(2,0,0) = 2(2c) < l(2b+c) \leq (0,k,l)$. Thus $(2,0,0)$ is a unique factorization.

Consider $(0,2,0)$ and suppose its longest factorization is $(j,0,l)$. $[(0,2,0)] = [0] = [l] = [(j,0,l)]$. Either $l = 0,2$ (impossible since $b \geq 3$) or $l \geq 4$. If $l \geq 4$, then $(0,2,0) = 2(2c) < l(2b+c) \leq (j,0,l)$. Thus $(0,2,0)$ is a unique factorization.

Consider $(0,0,2)$ and suppose its longest factorization is $(j,k,0)$. $[(0,0,2)] = [0] = [0] = [(j,k,0)]$. If $k = 0,2$ is impossible since $b \geq 3$. If $k \geq 3$, then $(0,0,2) = 2(2b+c) < 2(kc) \leq (j,k,0)$. If $k = 1$, then $2(2b+c) = j(2b) + 2c$, which simplifies to give us $j = 2$.
Therefore $\rho_2(S) = 3$.

Case 4: $S = \langle 2b-c, 2b, 2c \rangle$
This case is equivalent to the numerical semigroup $S = \langle x, 2x+d, 2x+2d \rangle$ where $x = 2b-c$ and $d = 2(c-b)$. By Corollary 3.5, $\rho_2(S) = 3$.

□

Example 3.19. Consider $\langle 12, 14, n \rangle$. This is a semigroups of the above form where $b = 6$ and $c = 7$. This means that $n \in \{5, 19\}$ we have excluded 8 and 20 from the set of n values because they would not give a minimally generated numerical semigroup. Therefore $\rho_2(\langle 12, 14, n\rangle) = 3$.

For our next lemma we will use the fact that for a semigroup of the form $\langle b+c, ab, ac\rangle$ where $\gcd(b, c) = 1$, $\rho_2 \geq a$. The factorization $(0, 1, 1) = ab + ac = a(b + c) = (a, 0, 0)$. We have characterized all values when $a = 2$, so we will set $a = 3$. Now we have $\langle b + c, 3b, 3c\rangle$. But when $b = 2$ we can write $(0, 0, 2) = 2(3c) = c(2 * 3) = (0, c, 0)$. This restricts $c = 3$ in this case.

Lemma 3.20. *Let $S = \langle n, 3b, 3c\rangle$ be a minimally generated numerical semigroup where 3 is the largest common factor between any two atoms and $n \leq 2c$. $\rho_2(S) = 3$ if and only if $n = b + c$ and $b = 2$ implies $c = 3$.*

Proof. Let $S = \langle n, 3b, 3c\rangle$ be a minimally generated numerical semigroup where 3 is the largest common factor between any two atoms and $n \leq 2c$. We can use the division algorithm to get $n = 3q + r$. Let (j, k, l) be the semigroup element $j(3q + r) + k(3b) + l(3c)$, and let $[(j, k, l)] = [jr]$ be the residue modulo 3. To begin this proof we will eliminate some factorizations based on their residue modulo 3.

From Proposition 3.1 and since $3c$ is the largest atom we can ignore the factorization $(1, 1, 0)$.

Consider $(1, 0, 1)$ and suppose its longest factorization is $(0, k, 0)$. $[(1, 0, 1)] = [r] \neq [0] = [(0, k, 0)]$ and thus $(1, 0, 1)$ is a unique factorization.

Consider $(2, 0, 0)$ and suppose its longest factorization is $(0, k, l)$. $[(2, 0, 0)] = [2r] \neq [0] = [(0, k, l)]$ and thus $(2, 0, 0)$ is a unique factorization.

Consider $(0, 2, 0)$ and suppose its longest factorization is $(j, 0, l)$. $[(0, 2, 0)] = [0] = [jr] = [(j, 0, l)]$. This factorization is not necessarily unique.

Consider $(0, 0, 2)$ and suppose its longest factorization is $(j, k, 0)$. $[(0, 0, 2)] = [0] = [jr] = [(j, k, 0)]$. This factorization is not necessarily unique.

Consider $(0, 1, 1)$ and suppose its longest factorization is $(j, 0, 0)$. $[(0, 1, 1)] = [0] = [jr] = [(j, 0, 0)]$. This factorization is not necessarily unique.

($\Rightarrow$) Suppose $\rho_2(S) = 3$.

Consider $(0, 2, 0)$, from above we know $[(0, 2, 0)] = [0] = [jr] = [(j, 0, l)]$. So either $j = 0$ and $l = 3$ which yields $2b \equiv 0 \pmod{c}$ (impossible) or $j = 3$ and $l = 0$. Hence $2(3b) = 3(n)$ which simplifies to $n = 2b$. Thus $\gcd(n, 3b) = b < 3$, so $b = 2$. But by Lemma 3.17 $\rho_2(\langle 4, 6, 3c\rangle) \neq 3$. Therefore $(0, 2, 0)$ is a unique factorization.

Consider $(0, 0, 2)$, from above we know $[(0, 0, 2)] = [0] = [jr] = [(j, k, 0)]$. So either $j = 0$ and $k = 3$ or $j = 3$ and $k = 0$. The first gives $6c = 9b$ and yields $2c \equiv 0 \pmod{b}$. In other words $b = 2$ implies that $c = 3$. Hence $2(3c) = 3(n)$ which simplifies to $n = 2c$. Thus $\gcd(n, 3c) = c < 3$, which is a contradiction.

Consider $(0, 1, 1)$ from above we know $[(0, 1, 1)] = [0] = [jr] = [(j, 0, 0)]$. Hence $j = 3$ and $(0, 1, 1) = 3b + 3c = 3n = (3, 0, 0)$. Thus $n = b + c$.

($\Leftarrow$) Now suppose that $S = \langle b + c, 3b, 3c\rangle$.

Consider $(0, 2, 0)$ from above we know $[(0, 2, 0)] = [0] = [jr] = [(j, 0, l)]$. So either $j = 0$ and $l \geq 3$ which yields $2b \equiv 0 \pmod{c}$ (impossible) or $j \geq 3$. Hence $(0, 2, 0) = 6b < j(b + c) \leq (j, 0, l)$. Therefore $(0, 2, 0)$ is a unique factorization.

Consider $(0, 0, 2)$ from above we know $[(0, 0, 2)] = [0] = [jr] = [(j, k, 0)]$. So either $j = 0$ and $k \geq 3$ or $j \geq 3$. The first of which yields $2 \equiv 0 \pmod{b}$, in other words $b = 2$ and $c = 3$ (which gives $rho_2 = 3$). If $j = 3$, then we have $3(b + c) + k(3b) = 6c$. It can be simplified to $c \equiv 0 \pmod{b}$ (impossible). If $j \geq 6$, then $(0, 0, 2) = 6c < j(b + c) \leq (j, k, 0)$. Thus $(0, 0, 2)$ gives $\rho_2 = 3$ when $b = 2$.

Consider $(0, 1, 1)$ from above we know $[(0, 1, 1)] = [0] = [jr] = [(j, 0, 0)]$. Hence $j \geq 3$ and $(0, 1, 1) = 3b + 3c = 3(b + c) = j(b + c) = (j, 0, 0)$. Thus $j = 3$ and $\rho_2 = 3$.

□

Example 3.21. Consider $\langle 9, 14, 24\rangle$. This is a semigroup of the form above where $b = 3$, $c = 8$, and $n = 14 \leq 16$. We notice that $\gcd(n, 3c) = 2 < 3$ which meets our requirements. Hence $\rho_2(\langle 9, 14, 24\rangle) = 3$.

Next we have the monoid $S = \langle 2a, ac, n\rangle$ where there is no necessary order. In this form we can see that $2(ac)$ and always be written as $c(2a)$ where $2 < c$. This leaves us with $\rho_2(S) \geq c$. Hence the only possibility for c that may keep $\rho_2(S) \leq 3$ is $c = 3$. Thus the following lemma will characterize $\rho_2(S) = 3$ when $b = 2$ and $c = 3$.

Lemma 3.22. *Let $S = \langle 2a, 3a, n\rangle$ be a minimally generated numerical semigroup where $a \geq 3$ is the largest common factor between any two atoms and $\gcd(a, n) = 1$, then $\rho_2(S) = 3$. Except for the special cases when $n \in \{4, 6\}$ and when $n = 5$, then only $a = 3$ gives $\rho_2(S) = 3$.*

Proof. Let $S = \langle 2a, 3a, n\rangle$ where $a \geq 3$ and $\gcd(a, n) = 1$. Note that the only values possible for n are those that are coprime with a. We can use the division algorithm to get $n = qa + r$. Let (j, k, l) be the semigroup element $j(2a) + k(3a) + l(qa + r)$, and let $[(j, k, l)] = [lr]$ be the residue modulo a. Note we have not designated any order between generators.

Consider $(1, 1, 0)$ and suppose its longest factorization is $(0, 0, l)$. $[(1, 1, 0)] = [0] = [rl] = [(0, 0, l)]$. But since $\gcd(r, a) = 1$ we must have $l \geq a$. Hence $2a + 3a = l(n) \geq an$, so $n \leq 5$.

Consider $(1, 0, 1)$ and suppose its longest factorization is $(0, k, 0)$. $[(1, 0, 1)] = [r] \neq [0] = [(0, k, 0)]$ and thus $(1, 0, 1)$ is a unique factorization.

Consider $(0, 1, 1)$ and suppose its longest factorization is $(j, 0, 0)$. $[(0, 1, 1)] = [r] \neq [0] = [(j, 0, 0)]$ and thus $(0, 1, 1)$ is a unique factorization.

Consider $(2, 0, 0)$ and suppose its longest factorization is $(0, k, l)$. $[(2, 0, 0)] = [0] = [rl] = [(0, k, l)]$. Either $l = 0$ (impossible since $2a < 3a$) or $l \geq a$. Hence $2(2a) = k(3a) + l(n) \geq k(3a) + an$, so $n \leq 4$.

Consider $(0, 2, 0)$ and suppose its longest factorization is $(j, 0, l)$. $[(0, 2, 0)] = [0] = [rl] = [(j, 0, l)]$. Either $l = 0$ (which gives $\rho_2 = 3$) or $l \geq a$. Hence $2(3a) = j(2a) + l(n) \geq j(2a) + an$, so $n \leq 6$.

Consider $(0, 0, 2)$ and suppose its longest factorization is $(j, k, 0)$. $[(0, 0, 2)] = [2r] \neq [0] = [(j, k, 0)]$ and thus $(0, 0, 2)$ is a unique factorization.

Thus all the values for $n > 6$ that are coprime with a give $\rho_2(S) = 3$. Now consider the values for $n \leq 6$, namely $n \in \{4, 5, 6\}$.

For $n = 4$, we have the semigroup $\langle 4, 2a, 3a\rangle$. But $\rho_2(\langle 4, 2a, 3a\rangle) \geq a + 1$, because $2(3a) = a(4) + (2a)$. So for $a \geq 3$, $\rho_2(S) > 3$.

For $n = 5$, we have the semigroup $\langle 5, 2a, 3a\rangle$. But $\rho_2(\langle 5, 2a, 3a\rangle) \geq a$, because $2a + 3a = a(5)$. Hence this is only possible when $a = 3$. By a quick computation we can see that $\rho_2(\langle 5, 6, 9\rangle) = 3$.

For $n = 6$, we have the semigroup $\langle 6, 2a, 3a\rangle$. But $\rho_2(\langle 6, 2a, 3a\rangle) \geq a$, because $2(3a) = a(6)$. When $a = 3$, S isn't minimal. Thus $\rho_2(S) > 3$.

Since we have shown that for all values that of $n > 2c$ give $\rho_2(S) = 3$ and exhausted all other cases ($n \in \{4, 5, 6\}$), we have fully characterized all semigroups of the form $\langle 2a, 3a, n\rangle$. □

Theorem 3.23. *Suppose $S = \langle ab, ac, n\rangle$ is a minimally generated numerical semigroup where a is the greatest common factor between any two atoms and $b < c$. Then $\rho_2(S) = 3$ if and only if one of the following cases hold*

(1) $a = 2$, $b = 2$, $c = 3$, *and* $n \in \{5, 7\}$, *or*

(2) $a = 2$, $2 < b$, *and* $n \in \{2c \pm b, 2b \pm c\}$, *or*

(3) $a = 3$, $2 < b$, $n \leq 2c$, $n = b + c$, *and* $b = 2$ *implies* $c = 3$, *or*

(4) $2 < a$, $b = 2$, $c = 3$. *Except for the special cases when $n \in \{4, 6\}$ and when $n = 5$, then only $a = 3$ gives $\rho_2(S) = 3$.*

Proof. (1) follows from Lemma 3.17. (2) follows from Proposition 3.18. (3) follows from Lemma 3.20. (4) follows from Lemma 3.22. By Proposition 3.12, we can say that for $2 < a$, $2 < b$, and $2c < n$ there are no numerical semigroups that have $\rho_2 = 3$. □

Notice that this theorem also characterizes $\rho_2 = 3$ for gluings of $\langle b, c\rangle$.

3.4 Non-Symmetric

Again suppose $S = \langle x, y, z\rangle$ is a minimally generated numerical semigroup. Now we will look at numerical semigroups where the atoms are pairwise coprime. Note we have not designated any order. Lemma 3.25 shows that there must be exactly one even generator. Lemma 3.26 will characterize all semigroups where the even generator is 4. Theorem 3.27 will characterize all semigroups where the even generator is large than 4.

We begin by narrowing our search for non-symmetric semigroups with $\rho_2(S) = 3$ by eliminating the places in which we can show that they cannot occur.

Theorem 3.24. *Let $S = \langle a, b\rangle$ be a numerical semigroup with $gcd(a,b) = 1$, $5 \leq a < b < g(S)$, and $g(S) = \dfrac{ab-a-b+1}{2}$. If $T(c) = \langle a,b,c\rangle$ then $\rho_2(T) > 3$ for all $g(S) \leq c \leq ab-a-b$.*

Proof. To show this, we must only show that there will always be at least one factorization of some combination of two generators n which always has $\rho_2(n) \geq 4$. We now break the proof into cases. Because $gcd(a,b) = 1$ we know that both values are either both odd or of differing parities. We now consider the case where both a and b are odd:
Because $2g(S) = 2\left(\dfrac{ab-a-b+1}{2}\right) = ab-a-b+1$, we know that $2g(S) \in \langle a,b\rangle$, and therefore has the factorization $2g(S) = Aa + Bb$, for $A, B \in \mathbb{Z}^+$. Since $\langle a,b\rangle$ is in embedding dimension 2, we know that $\langle a,b\rangle$ is symmetric, so $ab-a-b+1$ will always be even. Knowing that a and b are both odd, we know that the $A+B$ must be even, and since $a+b < 2b < 2g(S)$, we can exclude $A = B = 1$. Because $A = 2, B = 1$ or $A = 1, B = 2$ produce odd values, we can also eliminate those cases as well. From this we can conclude that if $2g(S) = Aa + Bb$, then $A + B \geq 4$ and so our $\rho_2(T) \geq 4$.
In the case in which a and b are of differing parities, we can observe that there will be two cases. In the first case we consider when a is odd and b is even.
First we eliminate the possibility that $2g(S)$ could be a multiple of a or b with a length less than 4.
If $2g(S) = Aa$, then $A = 2k$ because a is odd. If A is even, we reach a contradiction since $2g(S)$ cannot be a multiple of another generator. If $2g(S) = Bb$, we know that B cannot be even or else we reach a contradiction. Since we know $2b < 2g(S)$ we need to show that $B = 3$ is not a possibility.

$$\begin{aligned}
3b &= ab-a-b+1\\
3b-ab+b &= 1-a\\
0 &\equiv 1-a \bmod b\\
a &\equiv 1 \bmod b
\end{aligned}$$

Because $5 \leq a < b$ we can see that this equation has no solutions and thus this factorization cannot occur.
We now look at the case in which $2g(S) = Aa + Bb$. Because $a + b < 2g(S)$, we know that $A = B = 1$ is not a possible factorization. Because a is odd, we know that $A = 1, B = 2$ is not a solution, so we need to eliminate $A = 2, B = 1$. This case can be eliminated by showing that for $5 \leq a < b$, $2a + b < 2g(S)$.

$$\begin{aligned}
2a+b &< ab-a-b+1\\
2a+2b &< a(b-1)+1
\end{aligned}$$

Since a is fixed and b can grow large,we show that this statement holds by performing induction on b.
Base Case: for $a = 5$ and $b = 6$ we can see that the inequality holds as $22 < 26$. For $b+1$ we get:

$$\begin{aligned}
2a+2b+2 &< ab+1\\
2a+2b+2 &< a(b-1)+1+a\\
2 &< a \quad \text{Inductive Hypothesis}
\end{aligned}$$

Because $5 \leq a$, we can see the LHS will always be smaller than the RHS, and so the factorization cannot occur.
If a is even and b is odd, we again begin by eliminating the cases with $2g(S)$ being a multiple of a single generator.
For $2g(S) = Bb$, we reach a contradiction since B has to be even which would imply that $2g(S)$ is a multiple of b.
For $2g(S) = Aa$ we can use our previous result to infer that $3a < 2a+b < 2g(S)$, so this factorization cannot occur and so $A \geq 4$.
Lastly we know that $A + B > 2$,so we need to demonstrate that $A = 2, B = 1$ and $A = 1, B = 2$ are not a possibility. Since b is odd we know that $A = 2, B = 1$ is not a possible factorization, and we can use induction in the following way to eliminate $A = 2, B = 1$:

$$\begin{aligned}
2b+a &< ab-a-b+1\\
2b+2a &< b(a-1)+1
\end{aligned}$$

Base Case: For $a = 6$ and $b = 7$ we get $26 < 36$, and for $b + 1$:

$$\begin{aligned} 2b + 2a + 2 &< (b + 1)(a - 1) + 1 \\ 2b + 2a + 2 &< b(a - 1) + 1 + (a - 1) \\ 3 &< a \quad \text{Inductive Hypothesis} \end{aligned}$$

Because $5 \leq a$ we can see that the inequality holds and $A = 2, B = 1$ is not a valid factorization.
We can now conclude that $\langle a, b, g(S)\rangle$ will always have $\rho_2(S) > 3$, but what about the values of $g(S) < c \leq ab - a - b$? To show this is not possible we need only to show that the factorizations $a + b, 2a + b, 2b + a$ cannot produce the max length of 3 for our given interval of c (other factorizations would not produce a valid semigroup).
For $a + b$ we can assume that at some values of a and b that $c = a + b$, and using the following inequality we can check what those values are.

$$\begin{aligned} \frac{ab - a - b + 1}{2} &\leq a + b \\ ab - a - b + 1 &\leq 2a + 2b \\ ab - 3b &\leq 3a - 1 \\ b &\leq \frac{3a - 1}{a - 3} \end{aligned}$$

We have the condition that $a \geq 5$, and plugging in 5 we get that $b \leq 7$, and for $a = 6$, b does not have any valid values.
For $2a + b$ we can make the same assumption that $c = 2a + b$, so we check the possible values of b obtained from the following inequality:

$$\begin{aligned} \frac{ab - a - b + 1}{2} &\leq 2a + b \\ b &\leq \frac{5a - 1}{a - 3} \end{aligned}$$

For $a = 5$, $b \leq 12$. For $a = 6$, $b \leq 9$. $a = 7$, $b \leq 8$. For $a = 8$ no valid values of b occur.
For $2b + a$, we again use the same inequality strategy to determine our valid values for a and b.

$$\begin{aligned} \frac{ab - a - b + 1}{2} &\leq a + 2b \\ b &\leq \frac{3a - 1}{a - 5} \end{aligned}$$

Our inequality tells us that $a \geq 6$, and in plugging in $a = 6$ we get that $b \leq 17$. For $a = 7$ we get that $b \leq 10$. For $a = 8$ we have no valid values for b.
After running a program to check all the possible semigroups with these characteristics, there were none with $\rho_2(S) < 4$. We can now conclude that for any $g(S) \leq c \leq F(S)$ we will find no semigroups with $\rho_2 < 4$. □

Lemma 3.25. *Suppose $S = \langle x, y, z\rangle$ is a minimally generated numerical semigroup where x, y, and z are pairwise coprime. If $\rho_2(S) = 3$, then exactly one atom is even.*

Proof. Suppose $S = \langle x, y, z\rangle$ is a minimally generated numerical semigroup where x, y, and z are pairwise coprime. Also suppose that $\rho_2(S) = 3$. If there are more than two even atoms, then two atoms share a common factor greater than 2. BWOC suppose all atoms are odd. Since the sum of any two atoms is even, $\rho_2(S)$ must be even. Therefore exactly one atom is even. □

By Lemma 3.25 we will write our semigroup as $S = \langle 2x', 2y' + 1, 2z' + 1\rangle$ where y and z are both odd and $y < z$. The following Lemma will characterize all symmetric semigroups of embedding dimension 3 where the even generator is 4.

Lemma 3.26. *Suppose $S = \langle 4, y, z\rangle$ is a minimally generated numerical semigroup where $4, y$, and z are pairwise coprime. $\rho_2(S) = 3$ if and only if $y = 3$ and $z = 5$ or $y = 5$ and $z = 7$.*

Proof. Suppose $S = \langle 4, y, z\rangle$ is a minimally generated numerical semigroup where $4, y$, and z are pairwise coprime. By Lemma 3.25 y and z are both odd. Thus $y = 2y' + 1$ and $z = 2z' + 1$. WLOG let $y' < z'$.

($\Rightarrow$) Suppose $\rho_2(S) = 3$. Since $3 \le y < z$ ($y = 1$ is impossible), then $x < z$. We do not need to check the factorization $(1, 1, 0)$. This leaves 5 factorizations. Let (j, k, l) be the semigroup element $j(x) + k(y) + l(z)$, and let $[(j, k, l)] = [k + l]$ be the residue modulo 2.

$$[(1,0,1)] = [1] = [k] = [(0,k,0)] \text{ which implies that } k = 3. \tag{3}$$

$$[(0,1,1)] = [0] = [0] = [(j,0,0)] \text{ which implies that } j = 2 \text{ or } 3. \tag{4}$$

$$[(2,0,0)] = [0] = [k+l] = [(0,k,l)] \text{ which implies that } k = l = 1 \text{ (covered in case above).}$$

$$[(0,2,0)] = [0] = [l] = [(j,0,l)] \text{ which implies that } l = 2 \text{ and } j = 1. \text{ This is impossible since } 2y < 2z.$$

$$[(0,0,2)] = [0] = [k] = [(j,k,0)] \text{ which implies that } k = 2 \text{ and } j = 1. \tag{5}$$

From (3), we get $4 + z = 3y$. Simplifying, we get $3y' = z' + 1$. This means that our semigroup is $\langle 4, 2y' + 1, 6y' - 1\rangle$. We can see that $(0, 0, 2) = 2(6y' - 1) = 4(2y' - 1) + (2y' + 1)2 = (2y' - 1, 2, 0)$. Hence $\rho_2 \ge 2y' + 1$, so the only possible values are $y = 3$ and $z = 5$

From (4), we get the two equations $y + z = 8$ and $y + z = 12$. The first equation can be ignored since it gives back the possibility for $\rho_2 = 2$. For the second equation, we have the possibilities $y = 3$ and $z = 11$ and $y = 5$ and $z = 7$. The semigroup $\langle 3, 4, 11\rangle$ has $\rho_2 = 7$, $2 \cdot 11 = 4 + 6 \cdot 3$.

From (5), we get $2z = 2y + x$. Simplifying, we get $z' = y' + 1$. This means that our semigroup is $\langle 4, 2y' + 1, 2y' + 3\rangle$. We can see that $(0, 1, 1) = 4y' + 4 = 4(y' + 1)$. Hence $\rho_2 \ge y' + 1$, so the possible values are $y = 3$ and $z = 5$ and also $y = 5$ and $z = 7$.

($\Leftarrow$) Suppose $S = \langle 3, 4, 5\rangle$. By Corollary 3.3, $\rho_2(S) = 3$. Now suppose $S = \langle 4, 5, 7\rangle$. By Lemma 3.6, $\rho_2(S) \le 3$ but $(0, 1, 1) = 5 + 7 = 3 * 4 = (3, 0, 0)$. Therefore $\rho_2(S) = 3$. ◻

The following theorem will characterize all numerical semigroups for dimension 3 that are pairwise coprime such that the even generator is at least 6. Combined with the last Lemma 3.26 we have completed the characterization.

Theorem 3.27. *Suppose $S = \langle x, y, z\rangle$ is a minimally generated numerical semigroup where x, y, and z are pairwise coprime, $x = 2x'$ is even, and $x' \ge 3$. $\rho_2(S) = 3$ if and only if S is of one of the following forms*

1. $y = 6\lambda + 1$ *and* $z = \frac{2x'-2}{3} + 2\lambda + 1$ *where* $\lambda \in \{1, 2, \ldots, \lfloor \frac{x'-4}{6} \rfloor\}$, $\gcd(x', 6\lambda + 1) = 1$, *and* $x' \equiv 1 \pmod 3$, *or*

2. $y = 6\lambda + 5$ *and* $z = \frac{2x'+2}{3} + 2\lambda + 1$ *where* $\lambda \in \{1, 2, \ldots, \lfloor \frac{x'-6}{6} \rfloor\}$, $\gcd(x', 6\lambda + 5) = 1$, *and* $x' \equiv 2 \pmod 3$, *or*

3. $y = \frac{2x'-2}{3} + 2\lambda + 1$ *and* $z = 6\lambda + 1$ *where* $\gcd(x', 6\lambda + 1) = 1$, $x' \equiv 1 \pmod 3$, *and* $\lambda \ge (\lfloor \frac{x'}{6} \rfloor + 1)$, *or*

4. $y = \frac{2x'+2}{3} + 2\lambda + 1$ *and* $z = 6\lambda + 5$ *where* $\gcd(x', 6\lambda + 5) = 1$, $x' \equiv 2 \pmod 3$, *and* $\lambda \ge \lfloor \frac{x'+1}{6} \rfloor$, *or*

5. $y = 2\lambda + 7$ *and* $z = 6x' - (2\lambda + 7)$ *where* $\lambda \in \{0, \ldots, (\lfloor \frac{3x'-4}{2} \rfloor - 2)\}$, $3 \nmid (\lambda + 2)$, *and* $\gcd(x', 2\lambda + 7) = 1$, *or*

6. $y = 2\lambda + 5$ *and* $z = x' + 2\lambda + 5$ *where* $\lambda \ge 0$ *and* $\gcd(x', 2\lambda + 5) = 1$.

Proof. Suppose $S = \langle x, y, z\rangle$ is a minimally generated numerical semigroup where x, y, and z are pairwise coprime and $x \ge 6$. From Lemma 3.25 we can write $x = 2x'$, $y = 2y' + 1$, and $z = 2z' + 1$. Also, WLOG let $y' < z'$.

($\Rightarrow$) Suppose that $\rho_2(S) = 3$. We must check all six pairwise factorizations. Let (j, k, l) be the semigroup element $j(x) + k(y) + l(z)$, and let $[(j, k, l)] = [k + l]$ be the residue modulo 2. We will consider each pairwise sum of atoms and its supposed longest factorization.

$[(1,1,0)] = [1] = [l] = [(0,0,l)]$ which implies that $l = 3$. (6)

$[(1,0,1)] = [1] = [k] = [(0,k,0)]$ which implies that $k = 3$. (7)

$[(0,1,1)] = [0] = [0] = [(j,0,0)]$ which implies that $j = 2$ or 3. (8)

$[(2,0,0)] = [0] = [k+l] = [(0,k,l)]$ which implies that $k = l = 1$ (covered in case above).

$[(0,2,0)] = [0] = [l] = [(j,0,l)]$ which implies that $l = 2$ and $j = 1$. This is impossible since $2y < 2z$.

$[(0,0,2)] = [0] = [k] = [(j,k,0)]$ which implies that $k = 2$ and $j = 1$. (9)

From (6), we get the equation $x + y = 3z$. Simplifying, we get $x' + y' = 3z' + 1$. Note this is only possible when $z' < x'$. Since $3(z) \equiv 0 \pmod 3$, $x + y \equiv 0 \pmod 3$. If $x \equiv 0 \pmod 3$, then $y \equiv 0 \pmod 3$. This cannot happen or else $\gcd(x, y) > 1$. So now we have two cases.

Case 1: $x = 2$ and $y = 1$ modulo 3

Then we have $x' = 1$ and $y' = 0$ modulo 3. This is also the same as saying $\frac{x'-1}{3}$ is an integer and $y' = 3\lambda$ for some $\lambda \geq 0$. We substitute y' into our z' equation above and get $z' = \frac{x'-1}{3} + \lambda$. We know that $\lambda \geq 0$, but we want to show that $1 \leq \lambda \leq \lfloor \frac{x'-4}{6} \rfloor$.

BWOC Suppose that $\lambda = 0$, then $y = 1$ which contradicts our minimality restriction. So $1 \leq \lambda$.

BWOC Suppose that $\lambda = \lfloor \frac{x'-4}{6} \rfloor + k$ for some $k \geq 1$. Choose c to be the smallest non-negative integer such that $6|(x' - 4 - c)$. This bounds $c \in \{0, 3\}$. Then we can say that $\lfloor \frac{x'-4}{6} \rfloor = \frac{x'-4-c}{6}$.

$$y' = 3\lambda + 2 < \frac{x'+1}{3} + \lambda = z'$$
$$2\left(\frac{x'-6-c}{6} + k\right) + 2 < \frac{x'+1}{3}$$
$$6k \not< 1 + c$$

Therefore $\lambda \leq \lfloor \frac{x'-4}{6} \rfloor$. Also, since $y = 2(3\lambda + 2) + 1 = 6\lambda + 5$, $x' \nmid 6\lambda + 5$.

Case 2: $x = 1$ and $y = 2$ modulo 3

Then we have $x' = 2$ and $y' = 2$ modulo 3. This is also the same as saying $\frac{x'+1}{3}$ is an integer and $y' = 2 + 3\lambda$ for some $\lambda \geq 0$. We substitute y' into our z' equation above and get $z' = \frac{x'+1}{3} + \lambda$. We know that $\lambda \geq 0$, but now we want to show that $1 \leq \lambda \leq \lfloor \frac{x'-6}{6} \rfloor$.

BWOC Suppose that $\lambda = 0$, then $y = 5$ and $z = \frac{2x'+5}{3}$. Consider the factorization

$$(2,0,0) = 4x' = \frac{2x'+5}{3} + 5\left(\frac{2x'+5}{3} - 2\right) = (0, z-2, 1).$$

Since $z > 5$, $\rho_2(S) = z - 1 > 4$ which is a contradiction.

BWOC Suppose that $\lambda = \lfloor \frac{x'-6}{6} \rfloor + k$ for some $k \geq 1$. Choose c to be the smallest non-negative integer such that $6|(x' - c)$. This bounds $c \in \{2, 5\}$. Then we can say that $\lfloor \frac{x'-6}{6} \rfloor = \frac{x'-6-c}{6}$.

$$y' = 3\lambda + 2 < \frac{x'+1}{3} + \lambda = z'$$
$$2\left(\frac{x'-6-c}{6} + k\right) + 2 < \frac{x'+1}{3}$$
$$6k \not< 1 + c$$

Therefore $\lambda \leq \lfloor \frac{x'-4}{6} \rfloor$. Also, since $y = 2(3\lambda + 2) + 1 = 6\lambda + 5$, $x' \nmid 6\lambda + 5$.

From (7), we get the equation $x + z = 3y$. Simplified we get $x' + z' = 3y' + 1$. Since $3(y) \equiv 0 \pmod 3$, $x + z \equiv 0 \pmod 3$. If $x \equiv 0 \pmod 3$, then $z \equiv 0 \pmod 3$. This cannot happen or else $\gcd(x, z) > 1$. So now we have two cases.

Case 1: $x = 2$ and $z = 1$ modulo 3

Then we have $x' = 1$ and $z' = 0$ modulo 3. This is also the same as saying $\frac{x'-1}{3}$ is an integer and $z' = 3\lambda$

for some $\lambda \geq 0$. We substitute z' into our y' equation above and get $y' = \frac{x'-1}{3} + \lambda$. Now we will show that $\lambda \geq (\lfloor \frac{x'}{6} \rfloor + 1)$. Choose c to be the smallest non-negative integer such that $6|(x'-c)$. This bounds $c \in \{1, 4\}$. Then we can say that $\lfloor \frac{x'}{6} \rfloor = \frac{x'-c}{6}$.

BWOC Suppose $\lambda = \lfloor \frac{x'}{6} \rfloor + 1 - k = \frac{x'-c}{6} + 1 - k$ for some $k > 0$. We find that

$$\begin{aligned} z' = 3\left(\frac{x'-c}{6} + 1 - k\right) &> \frac{x'+1}{3} + \left(\frac{x'-c}{6} + 1 - k\right) = y' \\ 2\left(\frac{x'-c}{6} + 1 - k\right) &> \frac{x'+1}{3} \\ x' + 6 - c - 6k &> x' + 1 \\ 5 &\not> 6k + c \end{aligned}$$

Therefore, $\lambda \geq \lfloor \frac{x'+1}{6} \rfloor$. Since $z = 2(x' + 3\lambda) + 1 = 2x' + 3(2\lambda + 1)$, $x' \nmid 2\lambda + 1$.

Case 2: $x = 1$ and $z = 2$ modulo 3

Then we have $x' = 2$ and $z' = 2$ modulo 3. This is also the same as saying $\frac{2x'-1}{3}$ is an integer and $z' = 3\lambda + 2$ for some $\lambda \geq 0$. We substitute z' into our y' equation above and get $y' = \frac{x'+1}{3} + \lambda$. Now we will show that $\lambda \geq \lfloor \frac{x'+1}{6} \rfloor$. Choose c to be the smallest non-negative integer such that $6|(x'+1-c)$. This bounds $c \in \{0, 3\}$. Then we can say that $\lfloor \frac{x'+1}{6} \rfloor = \frac{x'+1-c}{6}$.

BWOC Suppose $\lambda = \lfloor \frac{x'+1}{6} \rfloor - k = \frac{x'+1-c}{6} - k$ for some $k > 0$. We find that

$$\begin{aligned} z' = 3\left(\frac{x'+1-c}{6} - k\right) + 2 &> \frac{x'+1}{3} + \left(\frac{x'+1-c}{6} - k\right) = y' \\ 2\left(\frac{x'+1-c}{6} - k\right) + 2 &> \frac{x'+1}{3} \\ x' + 1 - c - 6k + 6 &> x' + 1 \\ 6 &\not> 6k + c \end{aligned}$$

Therefore, $\lambda \geq \lfloor \frac{x'+1}{6} \rfloor$. Since $z = 2(x' + 3\lambda) + 1 = 2x' + 3(2\lambda + 1)$, $x' \nmid 2\lambda + 1$.

From (8), we get the two equations $y + z = 2x$ and $y + z = 3x$. For the first equation, $y < x < z$ must hold and we must have an arithmetic sequence of the form $(y, y + d, y + 2d)$ by Corollary 3.11. The only arithmetic sequence with one even number and $\rho_2 = 3$ is $\langle 3, 4, 5 \rangle$ by Corollary 3.3.

For the second equation, since $3(x) \equiv 0 \pmod 3$, $z + y \equiv 0 \pmod 3$. If $y \equiv 0 \pmod 3$, then $z \equiv 0 \pmod 3$. This cannot happen or else $\gcd(y, z) > 1$, thus $y' \neq 1$. Suppose $y' = 2$, so $y = 5$. We then have the semigroup $\langle 5, x, 3x - 5 \rangle$. I claim that this semigroup has $\rho_2 > 3$. Consider $(0, 0, 2) = 2(3x - 5) = x + 5(x - 2) = (x - 2, 1, 0)$. Since $x > 6$, $\rho_2 > 3$. Therefore $y' \geq 3$. We can write $y' = 3 + \lambda$ and $z' = 3x' - 4 - \lambda$ for some $\lambda \geq 0$. Now we must show that $\lambda \leq \lfloor \frac{3x'-4}{2} \rfloor - 2$. BWOC Suppose $\lambda = \lfloor \frac{3x'-4}{2} \rfloor - 2 + k$ for some $k \geq 1$. Choose c to be the smallest non-negative integer such that $2|(3x' - 4 - c)$. This bounds $c \in \{0, 1\}$. Then we can say that $\lfloor \frac{3x'-4}{2} \rfloor = \frac{3x'-4-c}{2}$.

$$\begin{aligned} y' = 3 - 2 + k + \frac{3x'-4-c}{2} &< 3x' - 4 + 2 - k - \frac{3x'-4-c}{2} = z' \\ y = (2 + 2k + 3x' - 4 - c) + 1 &< (6x' - 4 - 2k - 3x' + 4 + c) + 1 = z \\ 2k + 3x' - c - 1 &< -2k + 3x' + c + 1 \\ 4k &\not< 2c + 2 \end{aligned}$$

Therefore $\lambda \leq \lfloor \frac{3x'-4}{2} \rfloor$. Also, since $y = 2(3 + \lambda) + 1 = 2\lambda + 7 = 2(\lambda + 2) + 3$ and $z = 2(3x' - 4 - \lambda) + 1 = 3(2x') - 2(\lambda + 2) - 3$, $x' \nmid 2\lambda + 7$ and $3 \nmid \lambda + 2$.

From (9), we get the equation $2z = 2y + x$. This simplifies to $2z' = 2y' + x'$, thus x' must be even. BWOC Suppose $y' = 1$. Then we have the semigroup $\langle 3, 2x', 3 + x' \rangle$. I claim that $\rho_2 > 3$. Consider $(0, 1, 1) = 2x' + 3 + x' = 3(1 + x') = (x' + 1, 0, 0)$. Since $x' \geq 3$, $\rho_2 > 3$. Hence $y' \geq 2$. We can write $y' = 2 + \lambda$ and $z' = \frac{x'}{2} + 2 + \lambda$ for some $\lambda \geq 0$. Since $y = 2\lambda + 5$, $x' \nmid 2\lambda + 5$.

($\Leftarrow$) For the backwards direction, we will take each of the six cases and show that the numerical semigroup has $\rho_2 = 3$. This will be done by showing at least one factorization has a length of 3, and that all other factorizations are unique.

Case I:

Suppose $y = 6\lambda + 1$ and $z = \frac{2x'-2}{3} + 2\lambda + 1$ where $\lambda \in \{1, 2, \ldots, \lfloor \frac{x'-4}{6} \rfloor\}$, $\gcd(x', 6\lambda + 1) = 1$, and $x' \equiv 1 \pmod 3$. So our semigroup is $\langle 6\lambda + 1, \frac{2x'+6\lambda+1}{3}, 2x' \rangle$. Choose c to be the smallest non-negative integer such that $6|(x'-4-c)$. This bounds $c \in \{0, 3\}$. Then we can say that $\lfloor \frac{x'-4}{6} \rfloor = \frac{x'-4-c}{6}$ and $6\lambda+1 \leq x'-3-c < x'$. We can eliminate $(1, 1, 0)$ and $(2, 0, 0)$ because $\frac{2x'+6\lambda+1}{3} < \frac{2x'+x'}{3} < 2x'$. Now we will show that a factorization has length three, $(1, 0, 1) = (0, 3, 0)$.

$$2x' + 6\lambda + 1 = 3\left(\frac{2x' + 6\lambda + 1}{3}\right)$$

It will be shown that the other 3 factorizations are unique. Let (j, k, l) be the semigroup element $j(x) + k(y) + l(z)$, and let $[(j, k, l)] = [k + l]$ be the residue modulo 2.

Consider $(0, 1, 1)$ and suppose its longest factorization is $(j, 0, 0)$. $[(0, 1, 1)] = [1] = [j] = [(j, 0, 0)]$. We must have $j \geq 3$. Hence $(0, 1, 1) = 2x' + \frac{2x'+6\lambda+1}{3} = j(6\lambda + 1) = (j, 0, 0)$. The equation can be simplified to $2^3x' = (3j - 1)(6\lambda + 1)$. Since $6\lambda + 1$ is odd, $2^3 \mid 3j - 1$. This means that $x' \mid 6\lambda + 1$, a contradiction. Thus $(1, 1, 0)$ is a unique factorization.

Consider $(0, 2, 0)$ and suppose its longest factorization is $(j, 0, l)$. $[(0, 2, 0)] = [0] = [j] = [(j, 0, l)]$. We must have $j \geq 2$ and $l = 1$. Hence $(0, 2, 0) = \frac{2(2x'+6\lambda+1)}{3} = j(6\lambda + 1) = 2x' = (j, 0, 1)$. This simplifies to $4x' + 2(6\lambda + 1) = 3j(6\lambda + 1) + 6x'$. With $j \geq 2$ it is easy to tell that the $(0, 2, 0) < (j, 0, l)$. Thus $(0, 2, 0)$ is a unique factorization.

Consider $(0, 0, 2)$ and suppose its longest factorization is $(j, k, 0)$. $[(0, 0, 2)] = [0] = [k] = [(j, k, 0)]$. We must have $k \geq 2$ and $j \geq 1$ be odd. Hence

$$\begin{aligned}(0, 0, 2) = 4x' = j(6\lambda + 1) + \frac{k(2x' + 6\lambda + 1)}{3} = (j, k, 0)\\ (12 - 2k)x' = (3j' + k)(6\lambda + 1)\end{aligned}$$

Remembering that $6\lambda + 1 < x'$ we get $(12 - 2k)x' < (3j' + k)x'$, so $4 < j + k$. This means that the smallest values are $j = 1$ and $k = 5$. When $k = 5$, we get that $x \mid 6\lambda+1$. When $k \geq 6$, we find that $(0, 0, 2) < (j, k, 0)$. Thus $(0, 0, 2)$ is a unique factorization.

Therefore $\rho_2(\langle 6\lambda + 1, \frac{2x'+6\lambda+1}{3}, 2x' \rangle) = 3$.

Case II:

Suppose $y = 6\lambda + 5$ and $z = \frac{2x'+2}{3} + 2\lambda + 1$ where $\lambda \in \{1, 2, \ldots, \lfloor \frac{x'-6}{6} \rfloor\}$, $\gcd(x', 6\lambda + 5) = 1$, and $x' \equiv 2 \pmod 3$. So our semigroup is $\langle 6\lambda + 5, \frac{2x'+6\lambda+5}{3}, 2x' \rangle$. Choose c to be the smallest non-negative integer such that $6|(x'-c)$. This bounds $c \in \{2, 5\}$. Then we can say that $\lfloor \frac{x'-6}{6} \rfloor = \frac{x'-6-c}{6}$ and $6\lambda+5 \leq x'-1-c < x'$. We can eliminate $(1, 1, 0)$ and $(2, 0, 0)$ because $\frac{2x'+6\lambda+5}{3} < \frac{2x'+x'}{3} < 2x'$. Now we can show that a factorization has length three, $(1, 0, 1) = (0, 3, 0)$.

$$2x' + 6\lambda + 5 = 3\left(\frac{2x' + 6\lambda + 5}{3}\right)$$

It will be shown that the other 3 factorizations are unique. Let (j, k, l) be the semigroup element $j(x) + k(y) + l(z)$, and let $[(j, k, l)] = [k + l]$ be the residue modulo 2.

Consider $(0, 1, 1)$ and suppose its longest factorization is $(j, 0, 0)$. $[(0, 1, 1)] = [1] = [j] = [(j, 0, 0)]$. We must have $j \geq 3$. Hence $(0, 1, 1) = 2x' + \frac{2x'+6\lambda+5}{3} = j(6\lambda + 5) = (j, 0, 0)$. The equation can be simplified to $2^3x' = (3j - 1)(6\lambda + 5)$. Since $6\lambda + 5$ is odd, $2^3 \mid 3j - 1$. This means that $x' \mid 6\lambda + 5$, a contradiction. Thus $(1, 1, 0)$ is a unique factorization.

Consider $(0, 2, 0)$ and suppose its longest factorization is $(j, 0, l)$. $[(0, 2, 0)] = [0] = [j] = [(j, 0, l)]$. We must have $j \geq 2$ and $l = 1$. Hence $(0, 2, 0) = \frac{2(2x'+6\lambda+5)}{3} = j(6\lambda + 5) = 2x' = (j, 0, 1)$. This simplifies to $4x' + 2(6\lambda + 5) = 3j(6\lambda + 5) + 6x'$. With $j \geq 2$ it is easy to tell that the $(0, 2, 0) < (j, 0, l)$. Thus $(0, 2, 0)$ is a unique factorization.

Consider $(0,0,2)$ and suppose its longest factorization is $(j,k,0)$. $[(0,0,2)] = [0] = [k] = [(j,k,0)]$. We must have $k \geq 2$ and $j \geq 1$ be odd. Hence

$$\begin{aligned}(0,0,2) = 4x' = j(6\lambda+5) + \frac{k(2x'+6\lambda+5)}{3} = (j,k,0)\\ (12-2k)x' = (3j'+k)(6\lambda+5)\end{aligned}$$

Remembering that $6\lambda+5 < x'$ we get $(12-2k)x' < (3j'+k)x'$, so $4 < j+k$. This means that the smallest values are $j=1$ and $k=5$. When $k=5$, we get that $x \mid 6\lambda+5$. When $k \geq 6$, we find that $(0,0,2) < (j,k,0)$. Thus $(0,0,2)$ is a unique factorization.
Therefore $\rho_2(\langle 6\lambda+5, \frac{2x'+6\lambda+5}{3}, 2x'\rangle) = 3$.

Case III:
Suppose $y = \frac{2x'-2}{3} + 2\lambda + 1$ and $z = 6\lambda+1$ where $\gcd(x', 6\lambda+1) = 1$, $x' \equiv 1 \pmod 3$, and $\lambda \geq \lfloor \frac{x'}{6} + 1 \rfloor$. Thus our numerical semigroup is $\langle 2x', \frac{2x'-2}{3} + 2\lambda + 1, 6\lambda+1\rangle$. First, we can show that a factorization has length three, $(1,0,1) = (0,3,0)$.

$$2x' + 6\lambda + 1 = 3\left(\frac{2x'-2}{3} + 2\lambda + 1\right)$$

Now it will be shown that the other 5 factorizations are unique. Let (j,k,l) be the semigroup element $j(x) + k(y) + l(z)$, and let $[(j,k,l)] = [k+l]$ be the residue modulo 2. Choose c to be the smallest non-negative integer such that $6|(x'-c)$. This bounds $c \in \{1,4\}$. Then we can say that $\lfloor \frac{x'}{6} \rfloor = \frac{x'-c}{6}$.

Consider $(1,1,0)$ and suppose its longest factorization is $(0,0,l)$. $[(1,1,0)] = [1] = [l] = [(0,0,l)]$. We must have $l \geq 3$. Hence $(1,1,0) = 2x' + \frac{2x'-2}{3} + 2\lambda + 1 = l(6\lambda+1) = (0,0,l)$. Substituting in the smallest values for l and λ, we will show that $(0,0,l)$ is still greater than $(1,1,0)$.

$$\begin{aligned}2x' + \frac{2x'-2}{3} + 2\lambda + 1 &\geq l(6\lambda+1)\\ 8x' &\geq (3l-1)(6\lambda+5)\\ 8x' &\not\geq 8(x'+6-c)\end{aligned}$$

Thus $(1,1,0)$ is a unique factorization.

Consider $(0,1,1)$ and suppose its longest factorization is $(j,0,0)$. $[(0,1,1)] = [1] = [2j] = [(j,0,0)]$. We must have $j \geq 3$. Hence $(0,1,1) = \frac{2x'-2}{3} + 2\lambda + 1 + 6\lambda + 1 = j2x' \leq (j,0,0)$. We will show that $x' \mid 6\lambda+1$ for this to be true.

$$\begin{aligned}\frac{2x'-2}{3} + 2\lambda + 1 + 6\lambda + 1 &= j2x'\\ 2x' + 24\lambda + 4 &= 6x'\\ 6\lambda + 1 &\neq jx'\end{aligned}$$

Thus $(0,1,1)$ is a unique factorization.

Consider $(2,0,0)$ and suppose its longest factorization is $(0,k,l)$. $[(2,0,0)] = [0] = [k+l] = [(0,k,l)]$. We must have $k+l \geq 2$ and both be odd numbers, or else the atoms are not minimal. Hence $(2,0,0) = 4x' = \frac{k}{3}(2x'-2) + k(2\lambda+1) + l(6\lambda+1) = (0,k,l)$. We do not need to consider $k=1$ and $l=1$, since the length would be less than 3. Thus we will show and that $(2,0,0) < (0,k,l)$ in the second smallest case (and every other one). Suppose $l=1$ and $k=3$ and consider the smallest value for λ,

$$\begin{aligned}4x' &= 2x' + 2(6\lambda+1)\\ x &\not\geq x' + 6 - c\end{aligned}$$

Thus $(2,0,0)$ is a unique factorization.

Consider $(0,2,0)$ and suppose its longest factorization is $(j,0,l)$. $[(0,2,0)] = [0] = [l] = [(j,0,l)]$. We must have $l \geq 2$. Since we know $2y < 2z$, then $(0,2,0) < (j,0,l)$. Thus $(0,2,0)$ is a unique factorization.

Consider $(0,0,2)$ and suppose its longest factorization is $(j,k,0)$. $[(0,0,2)]=[0]=[k]=[(j,k,0)]$. We must have $k\geq 2$ and $j\geq 1$ be odd. Hence

$$\begin{aligned}(0,0,2)=2(6\lambda+1)&=j2x'+\frac{k}{3}(2x'+6\lambda+1)=(j,k,0)\\ 6(6\lambda+1)&=2(3j+k)x'+k(6\lambda+1)\\ \left(3-\frac{k}{2}\right)(6\lambda+1)&=(3j+k)x'\end{aligned}$$

We notice that when $k\in\{2,4\}$, then $\gcd(x',6\lambda+1)>1$. Also when $k>4$, then the equality doesn't hold. Thus $(0,0,2)$ is a unique factorization.
Therefore $\rho_2(\langle 2x',\frac{2x'-2}{3}+2\lambda+1,6\lambda+1\rangle)=3$.

Case IV:
Suppose $y=\frac{2x'+2}{3}+2\lambda+1$ and $z=6\lambda+5$ where $\gcd(x',6\lambda+5)=1$, $x'\equiv 2 \pmod 3$, and $\lambda\geq\lfloor\frac{x'+1}{6}\rfloor$. Thus our numerical semigroup is $\langle 2x',\frac{2x'+2}{3}+2\lambda+1,6\lambda+5\rangle$. First, we can show that the factorization $(1,0,1)=(0,3,0)$.

$$2x'+6\lambda+5=3\left(\frac{2x'+2}{3}+2\lambda+1\right)$$

Now it will be shown that the other 5 factorizations are unique. Let (j,k,l) be the semigroup element $j(x)+k(y)+l(z)$, and let $[(j,k,l)]=[k+l]$ be the residue modulo 2. Choose c to be the smallest non-negative integer such that $6|(x'+1-c)$. This bounds $c\in\{0,3\}$. Then we can say that $\lfloor\frac{x'+1}{6}\rfloor=\frac{x'+1-c}{6}$.

Consider $(1,1,0)$ and suppose its longest factorization is $(0,0,l)$. $[(1,1,0)]=[1]=[l]=[(0,0,l)]$. We must have $l\geq 3$. Hence $(1,1,0)=2x'+\frac{2x'+2}{3}+2\lambda+1=l(6\lambda+5)=(0,0,l)$. Substituting in the smallest values for l and λ, we will show that $(0,0,l)$ is still greater than $(1,1,0)$.

$$\begin{aligned}2x'+\frac{2x'+2}{3}+2\lambda+1&\geq l(6\lambda+5)\\ 8x'&\geq(3l-1)(6\lambda+5)\\ 8x'&\ngeq 8(x'+6-c)\end{aligned}$$

Thus $(1,1,0)$ is a unique factorization.

Consider $(0,1,1)$ and suppose its longest factorization is $(j,0,0)$. $[(0,1,1)]=[1]=[2j]=[(j,0,0)]$. We must have $j\geq 3$. Hence $(0,1,1)=\frac{2x'+2}{3}+2\lambda+1+6\lambda+5=j2x'\leq(j,0,0)$. We will show that $x'\mid 6\lambda+5$ for this to be true.

$$\begin{aligned}\frac{2x'+2}{3}+2\lambda+1+6\lambda+5&=j2x'\\ 2x'+24\lambda+20&=6x'\\ 6\lambda+5&\neq jx'\end{aligned}$$

Thus $(0,1,1)$ is a unique factorization.

Consider $(2,0,0)$ and suppose its longest factorization is $(0,k,l)$. $[(2,0,0)]=[0]=[k+l]=[(0,k,l)]$. We must have $k+l\geq 2$ and both be odd numbers, or else the atoms are not minimal. Hence $(2,0,0)=4x'=\frac{k}{3}(2x'+2)+k(2\lambda+1)+l(6\lambda+5)=(0,k,l)$. We do not need to consider $k=1$ and $l=1$, since the length would be less than 3. Thus we will show and that $(2,0,0)<(0,k,l)$ in the second smallest case (and every other one). Suppose $l=1$ and $k=3$ and consider the smallest value for λ,

$$\begin{aligned}4x'&=2x'+2(6\lambda+5)\\ x&\ngeq x'+6-c\end{aligned}$$

Thus $(2,0,0)$ is a unique factorization.

Consider $(0,2,0)$ and suppose its longest factorization is $(j,0,l)$. $[(0,2,0)]=[0]=[l]=[(j,0,l)]$. We must have $l\geq 2$. Since we know $2y<2z$, then $(0,2,0)<(j,0,l)$. Thus $(0,2,0)$ is a unique factorization.

Consider $(0,0,2)$ and suppose its longest factorization is $(j,k,0)$. $[(0,0,2)] = [0] = [k] = [(j,k,0)]$. We must have $k \geq 2$ and $j \geq 1$ be odd. Hence

$$\begin{aligned}(0,0,2) = 2(6\lambda+5) = j2x' + \frac{k}{3}(2x' + 6\lambda + 5) = (j,k,0)\\ 6(6\lambda+5) = 2(3j+k)x' + k(6\lambda+5)\\ \left(3 - \frac{k}{2}\right)(6\lambda+5) = (3j+k)x'\end{aligned}$$

We notice that when $k \in \{2,4\}$, then $\gcd(x', 6\lambda+5) > 1$. Also when $k > 4$, then the equality doesn't hold. Thus $(0,0,2)$ is a unique factorization.
Therefore $\rho_2(\langle 2x', \frac{2x'+2}{3} + 2\lambda + 1, 6\lambda+5\rangle) = 3$.

Case V:
Suppose $y = 2\lambda+7$ and $z = 6x' - (2\lambda+7)$ where $\lambda \in \{0, \ldots, (\lfloor \frac{3x'-4}{2} \rfloor - 2)\}$, $3 \nmid (\lambda+2)$, and $\gcd(x', 2\lambda+7) = 1$. Thus our numerical semigroup is $\langle 2x', 2\lambda+7, 6x' - (2\lambda+7)\rangle$.

I claim that $z > x$. We must show that $4x' > 2\lambda + 7$ for the largest value of λ. Choose c to be the smallest non-negative integer such that $2|(3x' - 4 - c)$. This bounds $c \in \{0,1\}$. Then we can say that $\lfloor \frac{3x'-4}{2} \rfloor = \frac{3x'-4-c}{2}$. Then $4x' > 2\lambda + 7$ simplifies to $4x' > 3x' - (1+c)$ and since $-(1+c) < 0$, $z > x$. So we can eliminate the factorization $(1,1,0)$. Now, we can show that the factorization $(0,1,1) = (3,0,0)$.

$$2\lambda + 5 + 6x' - (2\lambda+5) = 3(2x')$$

Now it will be shown that the other 4 factorizations are unique. Let (j,k,l) be the semigroup element $j(x) + k(y) + l(z)$, and let $[(j,k,l)] = [k+l]$ be the residue modulo $2\lambda+7$.

Consider $(1,0,1)$ and suppose its longest factorization is $(0,k,0)$. $[(1,0,1)] = [8] \neq [0] = [(0,k,0)]$ since $2\lambda+7$ is odd. Thus $(1,0,1)$ is a unique factorization.

Consider $(2,0,0)$ and suppose its longest factorization is $(0,k,l)$. $[(2,0,0)] = [4] = [6l] = [(0,k,l)]$. We must have $3l \geq (2\lambda+7+2)$, so $l \geq 3$. Since $x < z$, $2x < 3z$ and hence $(2,0,0) < (0,0,3) \leq (0,k,l)$. Thus $(2,0,0)$ is a unique factorization.

Consider $(0,2,0)$ and suppose its longest factorization is $(j,0,l)$. $[(0,2,0)] = [0] = [2(j+3l)] = [(j,0,l)]$. We must have $j + 3l \geq 2\lambda+7$. Consider $j = 0$, then $l \geq 3$ and $(0,2,0) < (0,0,3) \leq (0,0,l)$. Now consider $l = 0$, then $j \geq 2\lambda+7$ and $(0,2,0) = 2(2\lambda+7) < x'(2\lambda+7) \leq (j,0,0)$. Hence $(0,2,0) < (j,0,l)$, a contradiction. Thus $(0,2,0)$ is a unique factorization.

Consider $(0,0,2)$ and suppose its longest factorization is $(j,k,0)$. $[(0,0,2)] = [12] = [2j] = [(j,k,0)]$. We must have $j \geq 2\lambda+7+6$, so $j \geq 13$. Hence $(0,0,2) = 12x' - 2(2\lambda+7) < 26x' \leq (j,k,0)$. Thus the factorization $(0,0,2)$ is unique.
Therefore $\rho_2(\langle 2x', 2\lambda+7, 6x' - (2\lambda+7)\rangle) = 3$.

Case VI:
Suppose $y = 2\lambda+5$ and $z = x' + 2\lambda + 5$ where $\gcd(x', 2\lambda+5) = 1$. Thus our numerical semigroup is $\langle 2x', 2\lambda+5, x'+2\lambda+5\rangle$. Note that if x' must be even, or else z would be even. First, we can show that the factorization $(0,0,2) = (1,2,0)$.

$$2(x' + 2\lambda + 5) = 2x' + 2(2\lambda+5)$$

Now it will be shown that the other 5 factorizations are unique. Let (j,k,l) be the semigroup element $j(x) + k(y) + l(z)$, and let $[(j,k,l)] = [k+l]$ be the residue modulo $2\lambda+5$.

Consider $(1,1,0)$ and suppose its longest factorization is $(0,0,l)$. $[(1,1,0)] = [2] = [l] = [(0,0,l)]$. We must have $l \geq (2\lambda+5+2)$. Hence $(1,1,0) = 2x' + 2\lambda + 5 < 2x' + (2\lambda+5)(x' + 2\lambda + 7) \leq (0,0,l)$, a contradiction. Thus $(1,1,0)$ is a unique factorization.

Consider $(1,0,1)$ and suppose its longest factorization is $(0,k,0)$. $[(1,0,1)] = [3] \neq [0] = [(0,k,0)]$ since $2\lambda+5 \geq 5$. Thus $(1,0,1)$ is a unique factorization.

Consider $(0,1,1)$ and suppose its longest factorization is $(j,0,0)$. $[(0,1,1)] = [1] = [2j] = [(j,0,0)]$. We must have $2j \geq (2\lambda+5+1)$. Hence $(0,1,1) = x' + 2(2\lambda+5) < x' + x'(2\lambda+5) \leq (j,0,0)$. Thus $(0,1,1)$ is a unique factorization.

Consider $(2,0,0)$ and suppose its longest factorization is $(0,k,l)$. $[(2,0,0)] = [4] = [l] = [(0,k,l)]$. We must have $l \geq (2\lambda+5+4)$. Hence $(2,0,0) = 4x' < 4x' + (2\lambda+5)x' \leq (0,k,l)$. Thus $(2,0,0)$ is a unique factorization.

Consider $(0,2,0)$ and suppose its longest factorization is $(j,0,l)$. $[(0,2,0)] = [0] = [2j+l] = [(j,0,l)]$. We must have $2j+l \geq (2\lambda+5)$. Consider $j=0$, then $(0,2,0) = 2(2\lambda+5) < (2\lambda+5)(x'+2\lambda+5) \leq (0,0,l)$. Now consider $l=0$, then $(0,2,0) = 2(2\lambda+5) < x'(2\lambda+5) \leq (j,0,0)$. Hence $(0,2,0) < (j,0,l)$, a contradiction. Thus $(0,2,0)$ is a unique factorization.
Therefore $\rho_2(\langle 2x', 2\lambda+5, x'+2\lambda+5\rangle) = 3$. □

Corollary 3.28. *For case 1, given some $x' = 4+3n$, the number of possible semigroups will be given by the formula $\lfloor \frac{n}{2} \rfloor$*

Corollary 3.29. *For case 2, given some $x' = 5+3n$, the number of possible semigroups will be given by the formula $\lfloor \frac{n-1}{2} \rfloor$*

Corollary 3.30. *For case 5, given an x', the number of possible semigroups is given by the formula $\lfloor \frac{3(x'-2)}{2} \rfloor$*

3.4.1 Maximizing $\rho_2(S)$ for Non-symmetric semigroups in embdedding dimension 3

Here we analyze non-symmetric semigroups of embedding dimension 3, and show where the maximum $\rho_2(S)$ occurs, and provide a simple formula to calculate the value.

Lemma 3.31. *For numerical semigroups of the form $S = \langle a, b, ab-a-b\rangle$, where $3 \leq a < b$ and $gcd(a,b) = 1$, $\rho_2(S) = a+b-4$.*

Proof. To show that $\rho_2(S) = a+b-4$, we must first determine all of the possible factorizations for combinations of two generators of numerical semigroups meeting the above conditions, and then show that $a+b-4$ is the largest possible factorization. We begin by noting that the factorizations $(2,0,0)$ and $(1,1,0)$ are unique for any numerical semigroup, and will not be considered since they give the smallest possible factorization of 2.
Let (r,s,t) denote the factorization for the semigroup element $n = r(a) + s(b) + t(ab-a-b)$, and let $[(r,s,t)] = [ra - ta]$ denote the equivalence class of any factorization modulo b.
Looking at the four remaining combinations of two generators, we determine what equivalence class they belong to in order to find out whether any different combinations can produce the same n as listed below:
$[(0,2,0)] = [0]$
$[(0,1,1)] = [-a]$
$[(1,0,1)] = [0]$
$[(0,0,2)] = [-2a]$
Because $[(0,2,0)] = [0] = [(1,0,1)]$, we can set each factorization equal to each other to determine whether the factorization is in fact possible.

$$\begin{aligned} 2b &\equiv (a + ab - a - b) \bmod b \\ 2b &\equiv (ab - b) \bmod b \\ 3b &\equiv ab \bmod b \\ 0 &\equiv 0 \bmod b \end{aligned}$$

So now we can infer that the two different combinations can produce the same n in some instances.
We will now look at each of the four factorizations listed above and determine their uniqueness and/or max lengths.
Factorization 1: $(0,2,0) = (r,0,0), (r,0,1)$
In the case of $(0,2,0) = (r,0,0)$, we can solve the congruence involving their equivalence classes to determine whether the factorization is possible as seen below:

$$\begin{aligned} 2b &\equiv ra \bmod b \\ 0 &\equiv ra \bmod b \\ 0 &\equiv r \bmod b \quad gcd(a,b) = 1 \end{aligned}$$

This implies that for $k \in \mathbb{Z}^+$ there should be a solution when $r = bk$. Upon checking $k = 1$ we can see that the result does not hold as we get the equation:

$$2b = ba$$

We can observe that $2b < ab$ given $a \geq 3$, so this factorization is not possible.
For $(0, 2, 0) = (r, 0, 1)$, we can again solve the congruence involving their equivalence classes as seen below:

$$\begin{aligned}
&0 \equiv (ra - a) \bmod b \\
&0 \equiv a(r - 1) \bmod b \\
&0 \equiv (r - 1) \bmod b \\
&r \equiv 1 \bmod b \Longrightarrow r = bk + 1, k \in \mathbb{Z}^+
\end{aligned}$$

Plugging in the first possible value for r into our equation yields:

$$\begin{aligned}
&2b = (b + 1)a + ab - a - b \\
&3b = 2ab
\end{aligned}$$

Since $a \geq 3$ we can see the equation has no solutions and thus the factorization of $(0, 2, 0)$ is unique.
Factorization 2: $(0, 1, 1) = (r, 0, 0)$
For $(0, 1, 1) = (r, 0, 0)$ we obtain the equation:

$$\begin{aligned}
b + ab - a - b &= ar \\
a(b - 1) &= ar \\
r &= b - 1
\end{aligned}$$

We can see that this factorization is possible for $b - 1$ multiples of the first generator.
Factorization 3: $(1, 0, 1) = (0, s, 0)$
Equating the two factorizations we get:

$$\begin{aligned}
a + ab - a - b &= sb \\
ab - b &= sb \\
(a - 1) &= s
\end{aligned}$$

So when $s = (a - 1)$ we have a possible factorization of length $(a - 1)$.
Factorization 4: $(0, 0, 2) = (r, 0, 0)$
Using the equivalence classes mod b for the factorization we can set up a congruence to obtain:

$$\begin{aligned}
-2a &\equiv ra \bmod b \\
-2 &\equiv r \bmod b \\
r + 2 &\equiv 0 \bmod b
\end{aligned}$$

We can see that $r = bk + 2, k \in \mathbb{Z}^+$, and we will try plugging in the first possible value into the factorization and equate the two to get:

$$\begin{aligned}
2ab - 2a - 2b &= (bk - 2)a \\
2ab - 2b &= abk \\
ab(2 - k) &= 2b
\end{aligned}$$

We see that the only possible value of k is $k = 1$, and since $a \geq 3$ the equation has no solution and thus this is not a factorization of $(0, 0, 2)$.

For $(0,0,2) = (0,s,0)$, we equate the two factorizations and then compare their equivalence classes modulo a as seen below:

$$\begin{aligned} 2ab - 2a - 2b &= sb \\ 2ab - 2a &= sb + 2b \\ 2a(b-1) &\equiv b(s+2) \bmod a \\ 0 &\equiv b(s+2) \bmod a \\ 0 &\equiv (s+2) \bmod a \end{aligned}$$

This implies that $s = ak - 2, k \in \mathbb{Z}^+$, and by plugging in $s = ak - 2$ when we equate factorizations we get:

$$\begin{aligned} 2ab - 2a - 2b &= (ak-2)b \\ 2ab - 2a &= abk \\ ab(2-k) &= 2a \end{aligned}$$

It is obvious that the only possible value of k is $k = 1$, and since $3 \leq a < b$, we can see that the equation has no solutions and so this factorization cannot occur.
For $(0,0,2) = (r,s,0)$ we once again compare the equivalence classes of the factorizations modulo b to obtain:

$$\begin{aligned} -2a &\equiv ra \bmod b \\ (r+2) &\equiv 0 \bmod b \end{aligned}$$

This implies that $r = bk - 2, k \in \mathbb{Z}^+$, and plugging our value for r into the factorization yields the equation:

$$\begin{aligned} 2ab - 2a - 2b &= (bk-2)a + sb \\ 2ab - 2b &= abk + sb \\ ab(2-k) &= b(s+2) \end{aligned}$$

Because k can only equal 1, we see that our $s + 2$ must equal a, and therefore $s = a - 2$. So in order for our factorization to occur, we need $r = b - 2$ and $s = a - 2$, which gives us a length of $a + b - 4$.
To show that this is the longest length we merely need to show that it is greater than or equal to $b - 1$, which can be demonstrated by the following inequality:

$$\begin{aligned} b + a - 4 &\geq b - 1 \\ a &\geq 3 \end{aligned}$$

We can see this is consistent with our condition stated earlier in the proof, and therefore $\rho_2(S) = a + b - 4$ For all $S = \langle a, b, ab - a - b\rangle$. $\square$

Theorem 3.32. *Let $S = \langle a, b\rangle$ be a numerical semigroup such that a and b are coprime and $a \geq 3$. Let c be a natural such that $c \notin \langle a, b\rangle$ (with $c > b$). Let $T(S) = \langle a, b, c\rangle$, then we have that $\rho_2(S) \leq a + b - 4$, and equality can be achieved when c is the Frobenius number of $\langle a, b\rangle$.*

Proof. Let t be one of the following $2a, a+b, a+c, 2b, b+c$. Then we have $L(t) \leq a + b - 4$. To see this we use the bound $L(t) \leq \frac{t}{a}$.

$t = 2a$: $L(t) \leq 2$.

$t = a + b$: $L(t) \leq 1 + b/a$.

$t = a + c$: $L(t) \leq (a + c)/a$, but note that $c \notin \langle a, b\rangle$ implies that $c \leq ab - a - b$, so this implies $L(t) \leq b - \frac{b}{a}$, but this is less than $a + b - 4$.

$t = 2b$: $L(t) \leq \frac{2b}{a} \leq \frac{2b}{3}$ but this is less or equal to $a + b - 4$.

$t = b + c$: $L(t) \leq \frac{b+c}{a} \leq \frac{b+ab-a-b}{a} = b - 1$ which is less or equal to $a + b - 4$.

Hence, it remains to prove that $L(2c) \leq a+b-4$ and that equality can be obtained. First of all note that if we let $c = ab-a-b$ (the Frobenius number of $\langle a,b\rangle$), then we have $2c = (b-2)a + (a-2)b$, so we see that $a+b-4$ can be obtained. To show this is the maximum let c be any natural not in $\langle a,b\rangle$. Since $\langle a,b\rangle$ is a symmetric numerical semigroup, we have that c is of the from $ab-a-b-(t_1a+t_2b)$ with t_1,t_2 natural numbers (i.e, the Frobenius number minus an element in the monoid). Now say $2c = Aa+Bb$ where $A+B = L(2c)$. We have the following:

$$2(ab-a-b-t_1a-t_2b) = Aa+Bb \tag{10}$$

looking at the equation modulo a:

$$-2b-2t_2b \equiv Bb \pmod{a} \implies -2-2t_2 \equiv B \pmod{a}$$

ergo, $B+2+2t_2 = ma$ for m a natural number. Multiplying by b and moving terms around we obtain $Bb+2t_2b = mab-2b$. Plugging into (1):

$$2ab-2a-2b-2t_1a = Aa+Bb+2t_2b \implies 2ab-2a-2b-2t_1a = Aa+mab-2b \implies 2ab-2a-2t_1a = Aa+mab$$

hence, $-2-2t_1 = A+b(m-2)$. If $m \geq 2$ we see that the RHS is positive and the LHS is negative, a contradiction. Also, $m=0$ is not possible. Note $m=1$ would give: $B+2t_2 = a-2$. Plugging back into (1) and solving we get $A = b-2-2t_1$. Hence, $A+B = a+b-4-2t_1-2t_2 \leq a+b-4$, just as desired. □

In analyzing data for embedding dimensions 4 and 5, the addition of the frobenius number as the last generator of the semigroup continued to produce the maximum value for $\rho_2(S)$, although the addition of more generators seems to produce less of an obvious pattern. Because the frobenius number plus any generator in the semigroup produces a value which can be expressed in terms of the other generators in the semigroup, I conjecture that this will always be the case for Non-Symmetric semigroups regardless of embedding dimension. Note that the maximum value of $\rho_2(S)$ is not necessarily unique.

Conjecture 3.33. *Let $S = \langle n_1, n_2, \ldots, n_{x-1}\rangle$, where $n_1 < n_2 < \ldots < n_{x-1} < n_x$ and $gcd(n_i,n_j) = 1, i \neq j, 1 \leq i,j \leq x$. Then for $T = \langle n_1, n_2, \ldots, n_{x-1}, n_x\rangle$, $\rho_2(T)$ is maximal when $n_x = F(S)$.*

3.5 ρ_k for Modified General Arithmetic sequences

Now we will work on the numerical semigroup $S' \subseteq S$ where S is generated by a generalized arithmetic sequence $a, ah+d, ..., ah+xd$ where $1 \leq n \leq x-1$. Notice that if you were to remove either the first or the last generators, then we would just be changing arithmetic sequences which is not very interesting. S' is the semigroup formed by removing one of the middle generators, i.e. $S' = \langle a, ah+d, \ldots, ah+(n-1)d, ah+(n+1)d, \ldots, ah+xd\rangle$ where $1 \leq n \leq x-1$. In this section we will show that when we remove a middle generator $\rho_k(S')$ will either be the same or change in a simple manner. Note that we impose the condition $1 < x < a$, otherwise we get relations on the generators, also we suppose that $\gcd(a,d) = 1$, otherwise we can factor their common divisor.

Say we have an element t that can be written as k atoms. Say we have N copies of a, and $(k-N)$ copies of the other atoms. Hence, we can write

$$t = Na + (k-N)ah + \left(\sum_{i=1}^{k-N} \beta_i\right) d$$

Let $(A_0, A_1, ..., A_x)$ be it's longest factorization. Then we have:

$$t = A_0a + A_1(ah+d) + ... + A_x(ah+xd)$$

let $k_1 = A_0 + h\sum_{i=1}^{x} A_i$ and let $k_2 = \sum_{i=1}^{x} iA_i$. Then we have,

$$t = k_1a + k_2d$$

Looking at the equation modulo d, we have that

$$Na + (k-N)ah \equiv k_1a \pmod{d}$$

canceling a, we obtain that $k_1 = N + (k-N)h + sd$ for some $s \in \mathbb{Z}$. Then note that

$$k_1 = L(t) + (h-1)\sum_{i=1}^{x} A_i$$

this implies that

$$L(t) = k_1 - (h-1)\sum_{i=1}^{x} A_i = N + (k-N)h + sd - (h-1)\sum_{i=1}^{x} A_i = kh + sd + (1-h)(N + \sum_{i=1}^{x} A_i) \quad (11)$$

Lemma 3.34. *Let $a \leq kx$ and define $n = a\left\lfloor \frac{kx}{a} \right\rfloor$. Then $k \leq n$.*

Proof. We do it by cases:

Case $k \leq kx - a$: In this case the conclusion follows trivially since $k < kx - a < a(\frac{kx}{a} - 1) \leq a\left\lfloor \frac{kx}{a} \right\rfloor$, where the last inequality always holds since $kx \leq a\left\lfloor \frac{kx}{a} \right\rfloor + a$ is equivalent to it.

Case $k > kx - a$: Proceed by contradiction and say that $n = a\left\lfloor \frac{kx}{a} \right\rfloor < k$. We have that $a \leq n$ since $a \leq kx$, so we have $k \geq a$. Also we have that $k > kx - a$, so $a > k(x-1) \geq k$. Ergo, $k \geq a > k$ which is a contradiction.

□

Lemma 3.35. *Assume that $a > kx$, then*

$$kh - (h-1)(N + \sum_{i=1}^{x} A_i) \leq (h-1)\left\lfloor \frac{-k}{x} \right\rfloor + kh$$

Proof. It is enough to show

$$x(N + \sum_{i=1}^{x} A_i) \geq k$$

since this is equivalent to $-N - \sum_{i=1}^{x} A_i \leq \left\lfloor \frac{-k}{x} \right\rfloor$ and multiplying by $(h-1)$ and adding kh gives the result. Recall from above that $k_2 = \sum_{i=1}^{x} iA_i$ and it is also equal to $\sum_{i=1}^{k-N} \beta_i - sa$. We also have,

$$k - N \leq \sum_{i=1}^{k-N} \beta_i \implies k - N - sa \leq \sum_{i=1}^{k-N} \beta_i - sa$$

hence,

$$k - N - sa \leq \sum_{i=1}^{x} iA_i \implies k \leq \sum_{i=1}^{x} iA_i + sa + N$$

since $s \leq 0$ we have:

$$k \leq \sum_{i=1}^{x} iA_i + N \leq x(\sum_{i=1}^{x} A_i + N)$$

□

Now we are ready for our theorem:
Theorem 3.2 is used for reference in Theorem 3.36.

Theorem 3.36. *Let S and S' be defined as above. Then $\rho_k(S) = \rho_k(S')$ except if $kx \equiv 1 \pmod{a}$ and $n = x - 1$, then*

$$\rho_k(S) = (k-1)h + \left(\frac{kx-1}{a}\right)d + 1.$$

Proof. Most of the work has already been done. We will use the conditions on k to create bounds for s, and apply them to equation (10).
Let $a \leq kx$. Note that equation (10) can be improved to:

$$L(t) \leq kh + sd$$

we also have that

$$k_2 = \left(\sum_{i=1}^{k-N} \beta_i \right) - sa$$

Note that $k_2 \geq 0$, so we have that:

$$\left(\sum_{i=1}^{k-N} \beta_i \right) \geq sa$$

since $\beta_i \leq x$ we have:

$$sa \leq kx \implies s \leq \left\lfloor \frac{kx}{a} \right\rfloor$$

Hence,

$$L(t) \leq kh + \left\lfloor \frac{kx}{a} \right\rfloor d$$

In our subcases for $a \leq kx$ we will consider the element: $kah + md$ where m is the greatest integer less than or equal to kx which is also a multiple of a (i.e, $m = a\lfloor \frac{kx}{a} \rfloor$). From Lemma 3.34 we know that $k \leq m \leq kx$. Now choose c to be the smallest non-negative integer such that $a \mid kx - c$. Then we can say that

$$\left\lfloor \frac{kx}{a} \right\rfloor = \frac{kx - c}{a}$$

and furthermore $m = kx - c$. We will split this up into 4 cases. $c = 0$, $c = 1$, $2 \leq c \leq k - 2$, and $c = x - 1$.

It is sufficient to show that if we can write m in terms of any k numbers $1 \leq i_j \leq x$ such that $i_j \neq n$ (for $j = 1, 2, \ldots, k$), then ρ_k will not change.

Suppose $\sum i_j = m$. Then let

$$t = (ah + i_1 d) + ... + (ah + i_k d) = kah + md = a(kh + \frac{m}{a}d) = a\left(kh + \left\lfloor \frac{kx}{a} \right\rfloor d\right)$$

Hence, we see that $\rho_k(S) = kh + \lfloor \frac{kx}{a} \rfloor d$.

Case $c = 0$
Then $m = kx$ and $n < x$ so, we are done.

Case $c = 1$
Then $m = kx - 1$, and also $a \mid kx - 1$. One factorization is $m = (k-1)x + (x-1)$. However, I claim that this is the only factorization of m. Suppose $n = x - 1$. If one number is below $x - 1$, we reach $\sum i_j < kx - 1$. If all are above $x - 1$, we get $kx > kx - 1$. Thus consider

$$t = k(ah + xd) = a(kh) + (kx)d = a(k-1)h + (kx-1)d + ah + d = a\left((k-1)h + \frac{d(kx-1)}{a}\right) + (ah + d)$$

Whence, in this case we see that $\rho_2(S) = (k-1)h + \left(\frac{kx-1}{a}\right) d + 1$.

Case $2 \leq c \leq x - 2$
Thus $m = kx - c$. Consider the following factorization

$$m = (x - l_1) + (x - l_2) + \cdots + (x - l_k)$$

where $c = \sum l_j$ and $\lfloor \frac{c}{k} \rfloor \leq l_1 \leq \cdots \leq l_k \leq x - 2$. If $n = x - l_i$, then we will rewrite using

$$(x - (l_i - 2)) + (x + l_i) = 2(x - (l_i - 1)) \text{ or } (x - (l_i + 2)) + (x + l_i) = 2(x - (l_i + 1)).$$

Hence, we have guaranteed a factorization for m, so ρ_k remains unchanged.

Case $c = kx - 1$

Then $m = k$, and also $a \mid k$. So $k = pa$ for some $p \in \mathbb{Z}^+$. We notice that

$$k(ah + xd) = a\left(kh + \frac{kx}{a}d\right).$$

Hence, regardless of n, ρ_k remains unchanged.

Assume now that $a > kx$. From above we see that $s \leq 0$. Then,

$$L(t) = N + (k - N)h + sd - (h-1)\sum_{i=1}^{x} A_i \leq kh - (h-1)(N + \sum_{i=1}^{x} A_i) \leq (h-1)\left\lfloor \frac{-k}{x} \right\rfloor + kh$$

the last inequality follows from Lemma 3.35. Now to show that this is attainable consider the following two constructions which arise from the fact that

$$\left\lfloor \frac{-k}{x} \right\rfloor = \begin{cases} -\left\lfloor \frac{k}{x} \right\rfloor & \text{if } x \mid k \\ -\left\lfloor \frac{k}{x} \right\rfloor - 1 & \text{if } x \nmid k \end{cases}$$

Case $x \mid k$: Let $n = \left\lfloor \frac{k}{x} \right\rfloor$. Then consider

$$t = k(ah + d) = n(ah + xd) + (k - n)ha$$

so $L(t) \geq n + (k-n)h = kh + (h-1)(-n) = (h-1)\left\lfloor \frac{-k}{x} \right\rfloor + kh$, just as desired.

Case $x \nmid k$: Write $k = n_1 x + n_2$ where $0 < n_2 < x$. Note that now $\left\lfloor \frac{-k}{x} \right\rfloor = -n_1 - \left\lfloor \frac{-n_2}{x} \right\rfloor = -n_1 - 1$, so the length of the factorization in this case should be $k + (h-1)(k - n_1 - 1)$. We write $k(ah + d) = n_1(ah + xd) + (ah + n_2 d) + (k - n_1 - 1)ha$. This factorization has length $(n_1 + 1 + (k - n_1 - 1)h = (h-1)\left\lfloor \frac{-k}{x} \right\rfloor + kh$, as desired.

□

Lemma 3.37. *Let A, B be sets of positive integers, with $\gcd(A) = 1 = \gcd(B)$ and $A \subseteq B$. Then $\rho_k(\langle A\rangle) \leq \rho_k(\langle B\rangle)$*

Proof. Let A, B be sets of positive integers, with $\gcd(A) = 1 = \gcd(B)$ and $A \subseteq B$. Consider the two numerical semigroups $A' = \langle A\rangle$ and $B' = \langle B\rangle$. Note that $A' \subseteq B'$. I claim that $\rho_k(A') \leq \rho_k(B')$. Suppose we have $s \in A'$, such that $\ell_A(s) \leq k$ and $L_A(s) = \rho_2(A')$.

Any minimum-length factorization of s in A', is a factorization in B' as well; hence $\ell_B(s) \leq \ell_A(s) \leq k$. Thus if s was considered in computing $\rho_k(A')$, then s is also considered in computing $\rho_k(B')$. Now, consider a factorization of s in A' of maximum length. This is a factorization of s in B' as well; hence $L_B(s) \geq L_A(s) = \rho_2(A')$.

□

Theorem 3.38. *Let $S = \langle a, a+1, \ldots, a+x\rangle$. Suppose $S' = \langle a, a+1, a+x-1, a+x\rangle$, then $\rho_k(S) = \rho_k(S')$ except when $x + 2 \leq a \leq 2x - 6$.*

Proof. Let $S = \langle a, a+1, \ldots, a+x\rangle$ and $S' = \langle a, a+1, a+x-1, a+x\rangle$.

Let $a \leq 2x$. From Theorem 3.2 $\rho_2(S) = 3$ and by Lemma 3.37 $\rho_2(S) \leq \rho_2(S) = 3$. Hence we only need to consider max factorization lengths of 3. Consider the pairwise factorizations $(1,0,1,0), (1,0,0,1), (0,1,1,0), (0,1,0,1), (0,0,1,1), (0,2,0,0), (0,0,2,0)$, and $(0,0,0,2)$. We can see that $(1,0,0,1) = (0,1,1,0)$ because the first two atoms and the last two atoms differ by one. By comparing these pairwise factorizations to factorizations of length three we will find that some factorizations will work for specific values of a. All values for a not given by these factorizations must give $\rho_2(S') = 2$.

Case: $(1,0,1,0) = 2a + x - 1$

The only possible factorization of length three is $(0,3,0,0)$. This leads to $a = x - 4$ which is a contradiction.

Case: $(0,1,0,1) = 2a + x + 1$
Possible factorizations of length three include: $(3,0,0,0), (2,0,1,0), (1,0,2,0)$, and $(0,0,3,0)$. From these factorizations we get $a = (x+1)$, (2), $(3-x)$, and $(4-2x)$, respectively. All but the first are contradictions, so we add $a = x+1$ to our set that gives $\rho_2(S') = 3$.

Case: $(0,0,1,1) = 2a + 2x - 1$
Possible factorizations of length three include: $(3,0,0,0), (2,1,0,0), (1,2,0,0)$, and $(0,3,0,0)$. From these factorizations we get $a = (2x-1)$, $(2x-2)$, $(2x-3)$, and $(2x-4)$, respectively. All values are valid so our set increases to $a \in \{x+1, 2x-4, 2x-3, 2x-2, 2x-1\}$

Case: $(0,2,0,0) = 2a + 2$
Possible factorizations of length three include: $(3,0,0,0)$ and $(2,0,1,0)$. From these factorizations we get $a = 2$ and $(3-x)$. Both are contradictions.

Case: $(0,0,2,0) = 2a + 2x - 2$
Possible factorizations of length three include:

$$(3,0,0,0), (2,1,0,0), (1,2,0,0), (0,3,0,0), (1,1,0,1), (2,0,0,1), \text{ and } (0,2,0,1)$$

. The first four factorizations return the values $a = (2x-2)$, $(2x-3)$, $(2x-4)$, and $(2x-5)$, respectively. These are all valid, but we need only add $a = 2x-5$ to our set. The last three factorizations give contradiction values $a = (x-3)$, $(x-2)$, and $(x-4)$, respectively. Now our set is $a \in \{x+1, 2x-5, 2x-4, 2x-3, 2x-2, 2x-1\}$

Case: $(0,0,0,2) = 2a + 2x$
Possible factorizations of length three include:

$$(3,0,0,0), (2,1,0,0), (1,2,0,0), (0,3,0,0), (2,0,1,0), (1,1,1,0), (1,0,2,0), (0,2,1,0), (0,1,2,0), (0,0,3,0).$$

The first five factorizations return the values $a = (2x)$, $(2x-1)$, $(2x-2)$, $(2x-3)$, and $(x+1)$, respectively. These values are all valid, although we only need to add $a = 2x$ to our set. The last five factorizations return contradiction values $a = (x)$, (2), $(x-1)$, (1), and $(3-x)$, respectively.

Therefore, when $a \in \{x+1\} \cup \{2x-5, \ldots, 2x\}$, $\rho_2(S') = 3$ and stays the same. However, it changes to $\rho_2(S') = 2$ when $x+2 \le a \le 2x-6$.

Now let $a > 2x$. Since $\rho_2(S) = 2$, $\rho_2(S') = 2$ by Lemma 3.37. □

4 Delta Sets of Subsets of Arithmetic Progressions

Let $T = \{a, a+x, \ldots, a+tx\}$ with $a, x, t \ge 1$. We want T to minimally generate a numerical semigroup, so we require $\gcd(a,x) = 1$ and $a > t$. The delta set for $\langle T \rangle$ was computed in the following theorem from "On Delta Sets of Numerical Monoids," Chapman &al. [2, Theorem 3.9]).

Theorem 4.1. *Let $T = \{a, a+x, \ldots, a+tx\}$ with $a, x, t \ge 1$ and $\gcd(a,x) = 1$. Then $\Delta(\langle T \rangle) = \{x\}$.*

Now, for $U \subset T$ with $\#U \ge 2$, we wish to characterize $\Delta(\langle U \rangle)$. This has the potential to be very useful because all generating sets can be written as subsets of arithmetic progressions. In theory, therefore, our work could be extend to classify all semigroups of all embedding dimensions with delta set of size one.

For simplicity, we will always assume that T and U minimally generate $\langle T \rangle$ and $\langle U \rangle$, respectively. If $U = \{a + r_0 x, \ldots, a + r_k x\}$, we assume $r_0 = 0$ and $r_k = t$. If we have $r_0 \ne 0$ or $r_k \ne t$, then we can let $b = a + r_0 x$ and $t' = r_k - r_0$. Write $T' = \{b, b+x, \ldots, b+t'x\}$. Then T' is an arithmetic progression, and we have $U \subset T$ with $b, b+t'x \in U$.

Further, we assume that we have $\gcd(r_1, \ldots, r_k) = 1$. Otherwise, let $\alpha = \gcd(r_1, \ldots, r_k)$ and $T' = \{a, a+(\alpha x), \ldots, a+(t/\alpha)(\alpha x)\}$. Then T' is an arithmetic progression, and we still have $U \subset T'$.

Clearly for $t < 2$, U cannot be a numerical semigroup. The case where $t = 2$ is quite simple— then U is just another arithmetic progression. However, for larger t, characterizing the delta set becomes much more complicated.

We have shown explicitly that removing one generator for $t \ge 3$ or two generators for $t \ge 5$ has no effect on the delta set (Theorems 4.15 and 4.21, respectively). In fact, we can describe exactly which generators we have to remove in order to change the length sets at all, for any t and the removal of any number of generators. However, we do not have a general characterization of the delta set for the general case.

So far, we have not found any set U with $a+x \in U$ such that $\Delta(\langle U \rangle) \neq \{x\}$. Therefore, we have focused on describing the situation when we have $U \cap \{a+x, \ldots, a+(q-1)x\} = \emptyset$ for some $q \leq t$. We looked at it from two perspectives:

(i) $U = \{a, a+qx, a+(q+1)x, \ldots, a+tx\}$, and

(ii) $U = \{a, a+qx, a+tx\}$.

We discuss case (i) in this section and case (ii) in Section 5. Clearly these are the two extremes; we hope to be make some progress on the case in which some of the generators between $a+qx$ and $a+tx$ are present but not all.

4.1 Sliding and Golden Sets

Suppose that we have a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle$ and some element $y \in S$. If we have

$$y = \sum_{i=1}^{k} c_i n_i,$$

then $c = (c_1, \ldots, c_k)$ is a factorization of y. The length of this factorization is

$$|c| = \sum_{i=1}^{k} c_i.$$

We define $\varphi : \mathbb{N}^{t+1} \to S$ as follows:

$$\varphi(c) = \sum_{i=0}^{t} c_i n_i.$$

When we're looking at arithmetic progressions, we can get factorizations of equal lengths by sliding numbers together or apart. For factorization c, we can define a sliding as $c' = c - e_i + e_{i+1} - e_j + e_{j-1}$, which slides one atom from i to $i+1$, and also one atom from j to $j-1$. This satisfies $\varphi(c') = \varphi(c)$ and $|c'| = |c|$.

For example, consider $2a + 4x$, where $t = 4$. We can represent this element in the following two forms:

$$(1,0,0,0,1) = (0,1,0,1,0).$$

We can turn the first factorization into the second by sliding one copy of a up one space and sliding one copy of $a+4x$ down one space. Note that this works only with arithmetic sequences because distances between generators are the same. That is, when we replace a with $a+x$, we gain x, and when we replace $a+4x$ with $a+3x$ we lose x. If we remove certain blocks of generators, however, we lose the ability to do this if the blocks are close to the edges, as in the following example.

Example 4.2. Let $T = \{5, 8, 11, 14, 17\}$ and $U = \{5, 8, 11, 17\}$ (i.e., $a = 5, x = 3, t = 4$, removed $a+3x$). It is obvious that we have $\mathcal{L}_{\langle U \rangle}(y) \subseteq \mathcal{L}_{\langle T \rangle}(y)$ for all $y \in \langle U \rangle$. In fact, the length sets are identical for all elements of $\langle U \rangle$ except those which are congruent to $14 \pmod{17}$. If we do have $y \equiv 14 \pmod{17}$, however, $\mathcal{L}_{\langle T \rangle}(y) \setminus \mathcal{L}_{\langle U \rangle}(y) = \{\min \mathcal{L}_{\langle T \rangle}(y)\}$. For example, if $y = 14 + 2 * 17$, then we have

$$\mathcal{L}_{\langle T \rangle}(y) = \{3, 6, 9\}$$
$$\mathcal{L}_{\langle U \rangle}(y) = \{6, 9\}.$$

Thus the delta set remains exactly the same.

In terms of sliding, we're looking at the following type of factorization:

$$(0, \ldots, 0, 1, c). \tag{12}$$

Since we've removed $a+(t-1)x$, we need to slide our single copy of $a+(t-1)x$ out of that slot (marked in red) if we want to find a factorization in T of equal length. With a factorization of any other form, we can do this without any problems. However, if the factorization is of the form in (12), we can't slide $a+(t-1)x$

anywhere. If we try sliding it down, we need to slide something else up. But the only other atoms present are already in the $a + tx$ slot, so they can't slide up. If we try to slide $a + (t - 1)x$ up, however, we can only move it up one space. This is problematic because the only option available is to slide a copy of $a + tx$ down one space into the $a + (t - 1)x$ slot. Therefore, the factorization in (12) is the only factorization of that length.

We don't run into this problem with other types of factorizations. If we have nonzero coefficients in any other slot, we can rearrange things via sliding. Also, if we have anything other that 1 in the $a + (t - 1)x$ slot, we can split it up and slide some copies up and others down (see Lemma 4.7). So this particular type of factorization is the only problematic one when we've removed $a + (t - 1)x$.

The issues discussed in Example 4.2 can be extended to deal with the removal of more generators. For example, take 77 in $T = \{7, 16, 25, 34, 43, 52\}$ and $U = \{7, 16, 43, 52\}$. It has a factorization of $(11, 0, 0, 0, 0, 0)$, so it is present in $\langle U \rangle$. However, in $\langle T \rangle$ we also have the factorizations

$$(0, 0, 0, 1, 1, 0) = (0, 0, 1, 0, 0, 1).$$

This factorization can't be formed by sliding in $\langle U \rangle$ because it's close to the edge (the righthand parenthesis) and we've removed a block of generators from that area:

$$(0, 0, 0, 1, 1, 0) = (0, 0, 1, 0, 0, 1). \tag{$*$}$$

We can't slide far enough to get out of the block of missing generators because we run into the edge.

Suppose that we're removing 4 generators (not a or $a + tx$). Then the only problematic type of factorization can be written as follows:

$$(0, 0, 0, 1, 1, 0, 0, 0, 0, \ldots, 0) \tag{13}$$
$$(0, 0, 1, 0, 0, 1, 0, 0, 0, \ldots, 0) \tag{14}$$
$$(0, 1, 0, 0, 0, 0, 1, 0, 0, \ldots, 0) \tag{15}$$
$$(1, 0, 0, 0, 0, 0, 0, 1, 0, \ldots, 0). \tag{16}$$

(Note that if we reverse each of these vectors, we see that we run into the same issue on the other side of the semigroup.) Therefore, if we assume that the length set will change when we remove four generators, we have certain restrictions on what those generators can be. We need to remove $a + 3x$ or $a + 4x$ if we want to eliminate factorization (13). Similarly, we need to remove $a + 2x$ or $a + 5x$, $a + x$ or $a + 6x$, and $a + 7x$ (since we're not removing a). Note that some of these can be reduced; for example, we already know that removing $a + x$ by itself alters length sets. We can also do this from the back end of the semigroup. For instance, we could remove $a + (t - 2)x$, $a + (t - 4)x$, $a + (t - 6)x$, and $a + (t - 7)x$ to change length sets.

The definitions in the following section will help us describe the issues we've run into in the preceding examples more explicitly.

Definition 4.3. A *happy set* is a set $G \subset \mathbb{Z}^+$ such that for $i \in [1, \max G - 1]$, G contains i or $\max(G) - i$.

Definition 4.4. A *golden set* is a happy set G for which $\max(G)$ is odd. We call G *minimal* if no proper subset of G is golden. The *index* of a golden set is a positive integer $i(G) = N$ such that we have $2N - 1 = \max(G)$. Note that we have $N \leq \#G$, and equality is attained if G is minimal.

Definition 4.5. A *reflected golden set* with respect to a positive integer t is a set H for which there is a golden set G such that $H = \{t - g : g \in G\}$.

Lemma 4.6. *A set G is golden if and only if there is a positive integer N for which the following hold:*

(i) $2N - 1 = \max G$, *and*

(ii) for $i \in [1, N - 1]$, *G contains i or* $2N - 1 - i$.

When we say that we have removed a golden set or a reflected golden set from T to form U, we mean that some subset of $T \setminus U$ is golden or reflected golden. Essentially, golden sets are sets of generators where the sliding technique breaks down, as in $(*)$. Of course, we have to remember that you can remove other generators in addition to these. This is simply a minimal process for changing length sets.

The following table includes all minimal golden sets with index at most 6.

Index	Elements of G
1	1
2	$2, 3$
3	$2, 4, 5$
	$3, 4, 5$
4	$2, 4, 6, 7$
	$3, 5, 6, 7$
	$4, 5, 6, 7$
5	$2, 4, 6, 8, 9$
	$2, 5, 6, 8, 9$
	$3, 4, 7, 8, 9$
	$3, 5, 7, 8, 9$
	$4, 6, 7, 8, 9$
	$5, 6, 7, 8, 9$

Index	Elements of G
6	$2, 4, 6, 8, 10, 11$
	$2, 5, 7, 8, 10, 11$
	$2, 6, 7, 8, 10, 11$
	$3, 4, 6, 9, 10, 11$
	$3, 5, 7, 9, 10, 11$
	$3, 6, 7, 9, 10, 11$
	$4, 5, 8, 9, 10, 11$
	$4, 6, 8, 9, 10, 11$
	$5, 7, 8, 9, 10, 11$
	$6, 7, 8, 9, 10, 11$

4.2 Length Sets

I've shown explicitly that when we remove no more than three generators, the length sets remain exactly the same unless we remove a golden or reflected golden set (of course, we exclude elements which are no longer in the semigroup; for example, if $T \setminus U = \{a + 2x\}$, then $\mathcal{L}_{\langle U\rangle}(a + 2x) = \emptyset \neq \mathcal{L}_{\langle T\rangle}(a + 2x)$). These proofs give an idea of the basic technique used in the proof of the main theorem of this section. In Theorem 4.11, I demonstrate the same thing for the removal of any number of generators.

The next two lemmas assume that we have $\#(T \setminus U) = 1$. They explicitly characterize the elements of $\langle U\rangle$ whose length sets in $\langle T\rangle$ and $\langle U\rangle$ are not identical. In doing so, they show that if such elements exist then we have $T \setminus U \subset \{a + x, a + (t - 1)x\}$. That is, the only way to change length sets by removing one generator is to remove the golden set of index 1 or its reflection.

Lemma 4.7. *Suppose that we have $T \setminus U = \{a + nx\}$ for $1 \leq n \leq t - 1$. Given $y \in \langle U\rangle$ and a factorization c of y in $\langle T\rangle$, if we have $c_n \neq 1$, then there is a factorization of y in $\langle U\rangle$ with the same length.*

Proof. Let r be the length of the factorization. Clearly $n = 0$, $n = t$, and $c_n = 0$ are trivial, so assume $0 < n < t$ and $c_n > 1$. Then we have $c_n \in \langle 2, 3\rangle$, so we can write $c_n = 2u + 3v$. If $n \neq 1$, let

$$c' = c + v \cdot e_{n-2} + u \cdot e_{n-1} - (2u + 3v) \cdot e_n + (u + 2v) \cdot e_{n+1}.$$

If $n = 1$, then set

$$c' = c + (u + 2v) \cdot e_0 - (2u + 3v) \cdot e_1 + u \cdot e_2 + v \cdot e_3.$$

Then c' is a factorization of y in $\langle U\rangle$ with $|c'| = r$. □

Lemma 4.8. *Let T and U be as above (and assume once again that we have $1 \leq n \leq t - 1$). Suppose that we have $y \in \langle U\rangle$ and that we can write*

$$y = \sum_{i=0}^{t} c_i(a + ix)$$

with $c_n = 1$. Then there is a factorization of y in $\langle U\rangle$ with the same length unless one of the following conditions holds:

(i) $n = 1$ and $y = c_0(a) + (a + x)$, or

(ii) $n = t - 1$ and $y = (a + nx) + c_t(a + tx)$.

Proof. Let $r = |c|$. Since we have $a + nx \notin \langle U \rangle$, we can find $k \in \{0, \ldots, t\}$ with $k \neq n$ and $c_k > 0$. If $k \neq n - 1$ and $k \neq t$, then set

$$c' = c - e_k + e_{k+1} + e_{n-1} - e_n.$$

If $k = n - 1$ or $k = t$, then set

$$c' = c + e_{k-1} - e_k - e_n + e_{n+1}.$$

Then c' is a factorization of y in $\langle U \rangle$ whose length is r. Note that this method does not work for $(n, k) \in \{(1, 0), (t - 1, t)\}$, but those are precisely the cases excluded in the statement of the lemma. □

Now we will demonstrate that if we remove 2 or 3 generators but no golden or reflected golden sets, then no length sets change (except those which become empty, as stated in the first paragraph of this subsection).

Lemma 4.9. *Suppose that we have $T \setminus U = \{a + mx, a + nx\}$ with $0 < m < n < t$. Assume that the satisfy none of the following conditions:*

(i) $m = 1$,

(ii) $n = t - 1$,

(iii) $\{m, n\} = \{2, 3\}$, *or*

(iv) $\{m, n\} = \{t - 2, t - 3\}$.

Then the length sets will not change for any element of U.

Proof. Pick $y \in \langle U \rangle$ and $r \in \mathcal{L}_{\langle T \rangle}(y)$. Let c be a factorization of y with length r. It is sufficient to show that there is a factorization of y in $\langle U \rangle$ of length r. Let $T' = U \cup \{a + nx\}$. Then we have already shown in Lemmas 4.7 and 4.8 show that there is a factorization in $\langle T' \rangle$ of length r. This allows us to assume that we have $c_m = 0$. Clearly the case in which we have $c_n = 0$ is trivial, so assume that we have $c_n > 0$.

First suppose that we have $c_n > 1$; then write $c_n = 2u + 3v$. If we have $n - m > 1$, let

$$c' = c + (u + 2v) \cdot e_{n-1} - (2u + 3v) \cdot e_n + u \cdot e_{n+1} + v \cdot e_{n+2}.$$

If we have $n - m = 1$, then set

$$c' = c + (u + v) \cdot e_{n-2} - (2u + 3v) \cdot e_n + 2v \cdot e_{n+1} + u \cdot e_{n+2}.$$

Then we have a factorization of length r.

Now suppose that we have $c_n = 1$. Since we are working with a minimal generating set, we have $a + nx \notin \langle U \rangle$. Therefore, we can find some $k \in \{0, \ldots, t\} \setminus \{m, n\}$ with $c_k > 0$. We have a few different cases, but we will show that regardless of what k is, we can rearrange factors in a such a way that we no longer use $a + nk$.

Suppose that we have $n \neq m + 1$. If we have either $k = m + 1$ or $k = n + 1$, then set

$$c' = c + e_{k-2} - e_k - e_n + e_{n+2}$$

(note that we have $k > m \geq 2$ and $n \leq t - 2$). Assume now that we have $k \neq m + 1$ and $k \neq n + 1$. If we have $k > 0$, then set

$$c' = c + e_{k-1} - e_k - e_n + e_{n+1}.$$

If we have $k = 0$, then set

$$c' = c - e_0 + e_1 + e_{n-1} - e_n.$$

Therefore, if we have $n \neq m + 1$, then we have found a factorization of y in U of length r.

Suppose now that we have $n = m+1$. Then, by the assumptions of the lemma, we have $3 \leq m < n \leq t-3$. Since we know $n \leq t - 3$, if we have $k > n$ with $k \neq n + 1$ then we can set

$$c' = c - e_n + e_{n+1} + e_{k+1} - e_k.$$

If we have $k = n + 1$, then set

$$c' = c + e_{n-2} - e_n - e_k + e_{k+2}$$

(recall that we have $k + 2 = n + 3 \leq t$). If we have $0 < k < m$, set

$$c' = c + e_{k-1} - e_k - e_n + e_{n+1}.$$

Finally, we have the case in which we have $n = m + 1$ and $k = 0$. Then set

$$c' = c - e_0 + e_2 + e_{n-2} - e_n$$

(note that we know $m > 2$).

Now we have shown that wherever k, m, and n fall, we can slide factors around to get a factorization of the same length unless m and n are too close to the edge of the semigroup. The cases where they are too close are precisely those produced by removing golden and reflected golden sets. □

Lemma 4.10. *Suppose that we have $T \setminus U = \{a + mx, a + nx, a + qx\}$, with $0 < m < n < q < t$ and $t \geq 6$. Assume that none of the following conditions hold:*

(i) $m = 1$ *or* $q = t - 1$,

(ii) $\{2, 3\} \subset \{m, n, q\}$ *or* $\{t - 2, t - 3\} \subset \{m, n, q\}$,

(iii) $\{m, n, q\} = \{2, 4, 5\}$ *or*$\{t - 2, t - 4, t - 5\}$, *or*

(iv) $\{m, n, q\} = \{3, 4, 5\}$ *or*$\{t - 3, t - 4, t - 5\}$.

Then for any $y \in \langle U\rangle$, we have $\mathcal{L}_{\langle U\rangle}(y) = \mathcal{L}_{\langle T\rangle}(y)$.

Proof. Let $T' = U \cup \{a + qx\}$. Then T' must satisfy the conditions in Lemma 4.9, so all of its length sets are the same as in $\langle T\rangle$. Therefore, if we pick $y \in \langle U\rangle$ with

$$y = \sum_{i=0}^{t} c_i(a + ix),$$

we can assume without loss of generality that we have $c_m = c_n = 0$. Suppose that we have $c_q > 1$, and write $c_q = 2u+3v$. Then we can use exactly the same sliding technique as in Lemma 4.9 unless we have $q = m+2$. In this case, by condition (ii) in the statement of the lemma, we have $q \leq t - 3$. This allows us to set

$$c' = c + (u + v) \cdot e_{q-3} - (2u + 3v) \cdot e_q + v \cdot e_{q+1} + v \cdot e_{q+2} + u \cdot e_{q+3}.$$

Thus if we have $c_q \neq 1$, then we are done.

Suppose that we have $c_q = 1$. Then we can always slide factors around to produce a factorization of y in $\langle U\rangle$ of the appropriate length. The following table assigns a value α based on k, m, n, and q:

$k = 0$	$k = m + 1$	$k = n + 1$	$k = q + 1$	$n = m + 1$	$q = n + 1$	$q = n + 2$	$m = 2$	α
F	F	F	F					$e_{k-1} + e_{n+1}$
T					F			$e_{k+1} + e_{n-1}$
T					T		F	$e_{k+2} + e_{n-2}$
T					T		T	$e_{k+3} + e_{n-3}$
	T							$e_{k-2} + e_{n+2}$
		T		F				$e_{k-2} + e_{n+2}$
		T		T		F		$e_{k+1} + e_{n-1}$
		T		T		T		$e_{k-3} + e_{n+3}$
			T		F			$e_{k+1} + e_{n-1}$
			T	F	T			$e_{k-3} + e_{n+3}$
			T	T	T			$e_{k-4} + e_{n+4}$

Now set $c' = c - e_n - e_k + \alpha$; then c' is a factorization of y in $\langle U\rangle$ of the correct length, completing the proof. □

Finally, we have the general case.

Theorem 4.11. *Let $T = \{a, a + x, \ldots, a + tx\}$, and let $U \subset T$ with $a, a + tx \in U$. Set $M = \#(T \setminus U)$. If there exists some $y \in \langle U\rangle$ with $\mathcal{L}_{\langle U\rangle}(y) \neq \mathcal{L}_{\langle S\rangle}(y)$, then some subset of $T \setminus U$ is either a golden set with index at most M or the reflection of such a set with respect to t.*

Proof. We will induct on M. We've already finished the base case (i.e., if we remove one generator, the length sets will not change unless we remove $a + x$ or $a + (t - 1)x$). Suppose that the proposition holds whenever we remove at most $K - 1$ generators. Now suppose that we have $\#(T \setminus U) = K$, and set $j_0 = \min T \setminus U$. Let $T' = U \cup \{a + j_0x\}$. By the inductive hypothesis, if there is any $y \in \langle T'\rangle$ with $\mathcal{L}_{\langle\rangle}\langle T'\rangle(y) \neq \mathcal{L}_{\langle T\rangle}(y)$, then we can find a golden or reflected set G with $G \subset (T \setminus T') \subset (T \setminus U)$. Suppose that we have $\mathcal{L}_{\langle\rangle}\langle T'\rangle(y) = \mathcal{L}_{\langle T\rangle}(y)$ for all $y \in \langle T'\rangle$.

Consider any $y \in \langle U\rangle$ and a factorization $u = (u_0, \ldots, u_t)$ of y with length $r \in \mathcal{L}_{\langle\rangle}\langle T'\rangle(y)$ (where if $a + ix \notin T'$ then we require $u_i = 0$; allowing factorizations in $\langle T'\rangle$ and $\langle U\rangle$ to have $t + 1$ slots makes notation significantly less complicated). Assume that we have removed no golden sets and no reflections of golden sets. We aim to show that we must have $r \in \mathcal{L}_{\langle U\rangle}(y)$. Then we will have $\mathcal{L}_{\langle U\rangle}(y) = \mathcal{L}_{\langle\rangle}\langle T'\rangle(y) = \mathcal{L}_{\langle T\rangle}(y) = \{x\}$ for all $y \in \langle U\rangle$.

In order to do this, we need to address a number of different cases. Let $c = u_{j_0}$. First, we will reduce (via sliding) to the case in which we have $c \in \{0, 1\}$. Clearly if $c = 0$ then we know $r \in \mathcal{L}_{\langle U\rangle}(y)$, so we are done. If we have $c = 1$, then we can find some $k \neq j_0$ such that $u_k \neq 0$. If we have $k < j_0$, then we can show without too much difficulty that have $r \in \mathcal{L}_{\langle U\rangle}(y)$. If we have $k > j_0$, then we will need to consider the value of u_k. We will argue that if $u_k = 1$, then we can restrict k to a particular range, or else we would have $y \notin U$. Once we have this restrictions, we can show $r \in \mathcal{L}_{\langle U\rangle}(y)$. Finally, if we have $u_k > 1$, we will show that we can slide factors around until we have a factorization in $\langle U\rangle$, demonstrating that we must have $r \in \mathcal{L}_{\langle U\rangle}(y)$.

First, write $c = 2v + w$, with $v \in \mathbb{N}$ and $w \in \{0, 1\}$. Assume $2j_0 - 1 \leq t$. Since we haven't removed a golden set with index j_0, we can find $i \in \{1, \ldots, j_0 - 1\}$ with $a + (j_0 + i)x \in U$. Then, if we let $u' = u + ve_{j_0-i} - 2ve_{j_0} + ve_{j_0+i}$, u' is a factorization of y in $\langle U\rangle$ of length r. If we have $2j_0 - 1 > t$, then set $u' = u + ve_{j_0+1-t} - 2ve_{j_0} + ve_{t-1}$ (note that $a + (t - 1)x$ must be in U because $\{1\}$ is golden). Thus we are left with only w copies of $a + j_0x$. If $w = 0$, then we are done because we have shown $r \in \mathcal{L}_{\langle U\rangle}(y)$.

If $w = 1$, we have to consider which other atoms are in use (since we have $y \in \langle U\rangle$, we cannot have $y = a + j_0x$). Suppose first that we can find $k \in \{0, \ldots, j_0 - 2\}$ with $u_k > 0$. Then $u - e_k + e_{k+1} + e_{j_0-1} - e_{j_0}$ is a factorization in $\langle U\rangle$ of length r. If we have $u_{j_0-1} > 0$ and $u_j = 0$ for $j < j_0 - 1$, then find $i \in \{1, \ldots, j_0 - 1\}$ with $a + (j_0 + i)x \in U$ (as in the paragraph above) and set $u' = u + e_{j_0-1-i} - e_{j_0-1} - e_{j_0} + e_{j_0+i}$. Finally, we are left with the case in we have $c = 1$ and $u_j = 0$ for $j < j_0$.

If this is the case, then we must be able to find $k \in \{j_0 + 1, \ldots, t\}$ with $u_k > 0$. First suppose that we have $c_k = 1$. If we have $y = (a + j_0x) + (a + kx)$ for $k < t - 2j_0$ and $r \notin \mathcal{L}_{\langle U\rangle}(y)$, then we claim that we have $y \notin \langle U\rangle$, yielding a contradiction. Assume that we do have $y = (a + j_0x) + (a + kx)$ with $k < t - 2j_0$. Then we can write $2a + (j_0 + k)x = na + \beta x$, which implies $(n - 2)a = (j_0 + k - \beta)x$. If we have $n = 1$, then we must have $x = 1$ since we know $\gcd(a, x) = 1$. But then we have $a = \beta - (j_0 + k) < t$ (since $n = 1$, $a + \beta x$ is an atom), which is a contradiction because we wouldn't have a minimal generating set. If $n = 2$ then we have $2 \in \mathcal{L}_{\langle U\rangle}(y)$, and we're done. For $n > 2$, we must have $a \mid (j_0 + k - \beta)$. However, we know $j_0 + k - \beta < t - j_0 - \beta < t < a$, which is a contradiction. Therefore, we have finished the case in which we have $k < t - 2j_0$.

Now assume that we have $k \geq t - 2j_0$. If we can find $i \in \{1, \ldots, j_0\}$ with $a + (i + k)x \in U$, then $u + e_{j_0-i} - e_{j_0} - e_k + e_{k+i}$ is a factorization in $\langle U\rangle$ of length r. If we cannot find such an i, then we must have removed the reflection of a golden set with index $\lfloor(t - k)/2\rfloor$, which is a contradiction. Therefore, we have shown $r \in \mathcal{L}_{\langle U\rangle}(y)$ for $c_k = 1$.

Now suppose that we have $u_k > 1$. We claim that if we have not removed any golden sets or their reflections, then we can find $d \in \mathbb{Z}^+$ and $l \in \{0, 1\}$ such that $u + e_l - e_{j_0} + e_{k-d} - 2e_k + e_{k+d+j_0-l}$ is a factorization of y in $\langle U\rangle$.

Set $N_0 = k + \lceil j_0/2\rceil$ and assume that we have $2N_0 - 1 \leq t$. Then we can find $i \in \{0, \ldots, N_0 - 1\}$ such that U contains $a + ix$ and $a + (2N_0 - 1 - i)x$. Then for $l \in \{0, 1\}$ (depending on the parity of j), $u + e_l - e_{j_0} + e_i - 2e_k + e_{2N_0-1-i}$ is a factorization of y in $\langle U\rangle$ of length r.

Now assume that we have $2N_0 - 1 > t$. Then we claim that we can repeat the process from the preceding paragraph because we haven't removed a reflection of a golden set. Set $N_1 = t - k + 1 - \lceil j_0/2 \rceil$; it follows that we have $t + 1 - 2N_1 > 0$. Then we can find $i \in \{0, \ldots, N_1 - 1\}$ such that U contains $a + (t-i)x$ and $a + (t + 1 - 2N_1 + i)x$. Then for $l \in \{0, 1\}$, $u + e_l - e_{j_0} + e_{t-i} - 2e_k + e_{t+1-2N_1+i}$ is a factorization of y in $\langle U \rangle$ of length r. □

Theorem 4.11 helps us characterize $\Delta(\langle U \rangle)$ in two situations:

(i) if $\#(T \setminus U)$ is relatively small, or

(ii) if we remove generators of the form $a + ix$ with i close to $t/2$.

For example, if we remove generators from the middle of T (meaning close to $t/2$), we can remove approximately one third of all generators without changing any length sets, as stated explicitly in the following corollary. However, this theorem is significantly less useful when we remove large numbers of generators and when we remove generators from the edges of the semigroup (i.e., near a or $a + tx$).

Corollary 4.12. *Suppose that for some $M \in \mathbb{Z}^+$ we have*

$$U = \langle a, a + x, \ldots, a + ix, a + (i + M + 1)x, \ldots, a + tx \rangle.$$

If we have $M < \min\{i + 1, t - (i + M + 1)\}$, then we do not change the length set of any element.

4.3 Golden Sets and Delta Sets

Now we will move on to discuss when delta sets change. If $\Delta(\langle U \rangle) \neq \{x\}$, then there must be some $y \in \langle U \rangle$ with $\mathcal{L}_{\langle U \rangle}(y) \neq \mathcal{L}_{\langle T \rangle}(y)$. Therefore, we have removed a golden set or its reflection. However, this is not a complete characterization because removing a golden set is not sufficient to guarantee that we have $\Delta(\langle U \rangle) \neq \{x\}$.

We will begin with an easy lemma which will be uesful when length sets do change. When we remove golden sets, we do lose elements from length sets, but we only lose extremal elements (i.e., the minimum or maximum). If this is true, then the delta set cannot change.

Lemma 4.13. *Let $S_1 = \{n_1, \ldots, n_k\}$ be a minimal generating set for $\langle S_1 \rangle$ with $\Delta(\langle S_1 \rangle) = \{d\}$. If we have $S_2 \subset S_1$ and*

$$\mathcal{L}_{\langle S_1 \rangle}(y) \setminus \mathcal{L}_{\langle S_2 \rangle}(y) \subset \{\min(\mathcal{L}_{\langle S_1 \rangle}(y)), \max(\mathcal{L}_{\langle S_1 \rangle}(y))\}$$

for some $y \in \langle S_2 \rangle$, then we have $\Delta_{\langle S_2 \rangle}(y) \subset \{x\}$.

Proof. It is clear that we have $\mathcal{L}_{\langle S_2 \rangle}(y) \subset \mathcal{L}_{\langle S_1 \rangle}(y)$. If $\Delta_{\langle S_1 \rangle}(y)$ is nonempty, then we can write

$$\mathcal{L}_{\langle S_1 \rangle}(y) = \{r_0 + ix : 0 \leq i \leq k\}$$

for some $k \in \mathbb{N}$ and $r_0 = \min(\mathcal{L}_{\langle S_1 \rangle}(y))$. We have assumed that the only elements which might not be in $\mathcal{L}_{\langle S_2 \rangle}(y)$ are r_0 and $r_0 + kx$. Therefore, if $\Delta_{\langle S_2}(y) \neq \emptyset$, then we have

$$\mathcal{L}_{\langle S_2 \rangle}(y) = \{r_0 + ix : \alpha \leq i \leq \beta\}$$

for some $(\alpha, \beta) \in \{(0, k), (0, k-1), (1, k), (1, k-1)\}$, and we are done. □

Theorem 4.14 assumes $t = 4$ and $T \setminus U = \{a + 3x\}$. We show explicitly how we can rearrange factors to make up for the loss of $a + 3x$. It is inefficient, but it is more concrete than the more general proofs.

Theorem 4.14. *Suppose that we have $U = \{a, a + x, a + 2x, a + 4x\}$. Then $\Delta(\langle U \rangle) = \{x\}$.*

Proof. Set $T = U \cup \{a + 3x\}$. We know $\Delta(\langle T \rangle) = \{x\}$ (Theorem 4.1). Pick $y \in \langle T \rangle$ and $r \in \mathcal{L}_{\langle T \rangle}(y)$. We aim to show

$$\mathcal{L}_{\langle T \rangle}(y) \setminus \mathcal{L}_{\langle U \rangle}(y) \subset \{\min(\mathcal{L}_{\langle T \rangle}(y))\}.$$

Let c be a factorization of y in $\langle T \rangle$ with $|c| = r$. We will show that unless we have $c = (0, 0, 0, 1, c_4)$, we can find a factorization d with $|d| = r$ and $d_3 = 0$. Obviously the case in which we have $c_3 = 0$ is trivial.

Consider the case in which we have $c_3 > 1$. Then we have $c_3 \in \langle 2, 3\rangle$, so we can write $c_3 = 2u + 3v$ for non-negative integers u and v. Then set

$$d = c + v \cdot e_1 + u \cdot e_2 - (2u + 3v) \cdot e_3 + (u + 2v) \cdot e_4,$$

and we are done. Thus if $c_3 \neq 1$ then we can easily construct a factorization in $\langle U\rangle$ of length r.

Now suppose that we have $c_3 = 1$. If we have $c_0 > 0$, then the factorization

$$d = c - e_0 + e_1 + e_2 - e_3$$

has length r. Suppose that we have $c_0 = 0$. If we have $c_1 > 0$, then we can write

$$d = c + e_0 - e_1 - e_3 + e_4,$$

and we are done. Now assume that we have $c_0 = c_1 = 0$. If we have $c_2 > 0$, then

$$d = c + e_1 - e_2 - e_3 + e_4$$

is a factorization with length r. Thus we have shown that we must have $r \in \mathcal{L}_{\langle U\rangle}(y)$ except when we have $c = (0, 0, 0, 1, c_4)$. If we have $y \not\equiv (a + 3x) \pmod{a + 4x}$, then we are done. Suppose that we have $c = (0, 0, 0, 1, c_4)$; then it suffices to show that we have $1 + c_4 = \min \mathcal{L}_{\langle T\rangle}(y)$.

Assume for the sake of contradiction that we can find some $q \in \mathcal{L}_{\langle T\rangle}(y)$ with $q < 1 + c_4$. Then find a factorization z with length q. But then we have $q \leq c_4$, which implies

$$y \leq q(a + 4x) \leq c_4(a + 4x) < y,$$

yielding a contradiction. Therefore, we must have $1 + c_4 = \min \mathcal{L}_{\langle T\rangle}(y)$.

Now, we have

$$\mathcal{L}_{\langle T\rangle}(y) \setminus \mathcal{L}_{\langle U\rangle}(y) \subset \{\min(\mathcal{L}_{\langle T\rangle}(y))\}$$

for all $y \in \langle T\rangle$, which implies $\Delta_{\langle U\rangle}(y) = \Delta_{\langle T\rangle}(y) \subset \{x\}$. It follows that we have $\Delta(\langle U\rangle) = \{x\}$. □

Now we will address the case in which we remove exactly one generator.

Theorem 4.15. *Let $T = \{a, a + x, \ldots, a + tx\}$ with $\gcd(a, x) = 1$ and $t \geq 3$. If $U = T \setminus \{a + nx\}$ for $n \in [0, t]$, then $\Delta(\langle U\rangle) = \{x\}$.*

Proof. If $n = 0$, let $b = a + x$. Then we have $U = \{b, b + x, \ldots, b + (t - 1)x\}$, which is an arithmetic progression. Clearly U is also an arithmetic progression if we have $n = t$. In either case, we must have $\Delta(\langle U\rangle) = \{x\}$ by Theorem 4.1.

Assume that we have $1 \leq n \leq t - 1$. Choose any $y \in \langle T\rangle$ and $r \in \mathcal{L}_{\langle T\rangle}(y)$. We aim to show that if we have $y \in U$ and $r \notin \mathcal{L}_{\langle U\rangle}(y)$, then we must have $r \in \{\min(\mathcal{L}_{\langle T\rangle}(y)), \max(\mathcal{L}_{\langle T\rangle}(y))\}$.

Let $c = (c_0, \ldots, c_t)$ be a factorization of y in $\langle T\rangle$ with $|c| = r$. Suppose that we have $c_n \neq 1$. Then we can apply Lemma 4.7 to demonstrate that we have $r \in \mathcal{L}_{\langle U\rangle}(y)$. Suppose $c_n = 1$ and neither of the following conditions hold:

(i) $n = 1$ and $y = c_0(a) + (a + x)$, or

(ii) $n = t - 1$ and $y = (a + nx) + c_t(a + tx)$.

Then we can apply Lemma 4.8, and we have again shown $r \in \mathcal{L}_{\langle T\rangle}(y)$. Therefore, are doen unless one of the two conditions listed above is true.

First consider condition (i). We want to show $c_0 + 1 = \max \mathcal{L}_{\langle T\rangle}(y)$. Suppose that we can find a factorization q with $|q| > c_0 + 1$. Then we can write $y = |q|a + \lambda x$ for $\lambda \geq 0$. Then we have

$$|q|a + \lambda x = (c_0 + 1)a + x,$$

which implies

$$0 \leq (|q| - (c_0 + 1))a = (1 - \lambda)x.$$

However, we know $\gcd(a, x) = 1$, so we have $a|(1 - \lambda)$, which implies $a \leq 1$. This is impossible, however, as we are working with a minimal generating set. Therefore, we must have $c_0 + 1 = \max \mathcal{L}_{\langle T \rangle}(y)$.

Now suppose that condition (ii) holds. As with condition (i), if suffices to show that we have $c_t + 1 = \min \mathcal{L}_{\langle S \rangle}(y)$. Suppose that we can find a factorization q of y with $|q| < c_t + 1$; then we have $|q| \leq c_t$. Then we have

$$y \leq |q|(a + tx) \leq c_t(a + tx) < y,$$

which is clearly false. Therefore, we have $c_t + 1 = \min \mathcal{L}_{\langle S \rangle}(y)$. Thus we have demonstrated that if we have $r \notin \mathcal{L}_{\langle U \rangle}(y)$, then we must have $r \in \{\min(\mathcal{L}_{\langle T \rangle}(y)), \max(\mathcal{L}_{\langle T \rangle}(y))\}$.

Now we can apply Lemma 4.13, so we have $\Delta_{\langle U \rangle}(y) \subset \{x\}$ for all $y \in \langle U \rangle$, which implies $\Delta(\langle U \rangle) = \{x\}$. □

Next we will move on to the case in which we remove two generators. Let $T \setminus U = \{a + mx, a + nx\}$ with $1 \leq m < n \leq t - 1$. We have shown in Theorem 4.11 that if we do not have

$$(m, n) \in \{(1, n), (m, t - 1), (2, 3), (t - 3, t - 2)\},$$

then the length sets must stay the same, so we have $\Delta(\langle U \rangle) = \{x\}$. Now, we will consider the various ways to remove golden sets in a series of lemmas.

Lemma 4.16. *Let $T = \{a, a + x, \dots, a + tx\}$ (with $t \geq 5$) and $U \subset T$ with $T \setminus U = \{a + x, a + nx\}$ for $2 \leq n \leq t - 2$. Then $\Delta(\langle U \rangle) = \{x\}$.*

Proof. Let $T' = U \cup \{a + nx\}$; then we know $\Delta(\langle T' \rangle) = \{x\}$ by Theorem 4.15. Pick any y in U and $r \in \mathcal{L}_{\langle T \rangle}(y)$. Choose a factorization c in T. We will first demonstrate that if we have $r \notin \mathcal{L}_{\langle T \rangle}(y)$, then we must have $c_n = 1$ and $c_i = 0$ for $i \notin \{0, n\}$.

Clearly if we have $c_n = 0$, then we have $r \in \mathcal{L}_{\langle U \rangle}(y)$, and we are done. Now suppose that we have $c_n > 1$. Write $c_n = 2u + 3v$. If $n \neq 2$, then set

$$d = c + (u + 2v) \cdot e_{n-1} - (2u + 3v) \cdot e_n + u \cdot e_{n+1} + v \cdot e_{n+2}.$$

If $n = 2$, then set

$$d = c + (u + v) \cdot e_0 - (2u + 3v) \cdot e_2 + 2v \cdot e_3 + u \cdot e_4.$$

Then d is a factorization of y in U with $|d| = r$. Thus we are finished except when we have $c_n = 1$.

If we do have $y \in U$ and $c_n = 1$, then we can pick $k \in \{0, \dots, t\} \setminus \{1, n\}$ with $c_k > 0$. Suppose that we can find such a k in $[3, t] \setminus \{n, n + 1\}$; then set

$$d = c + e_{k-1} - e_k - e_n + e_{n+1}.$$

Now consider the case in which we have $c_k = 0$ for $k \notin \{0, 2, n, n + 1\}$. If we have either $c_2 > 0$ and $n \neq 3$ or $c_{n+1} > 0$ and $n \neq 2$, set

$$d = c - e_k + e_{k+1} + e_{n-1} - e_n.$$

If we have either $c_2 > 0$ and $n = 3$ or $c_{n+1} > 0$ and $n = 2$, then set

$$d = c + e_0 - e_2 - e_3 + e_5.$$

Thus we have found a factorization d of y in $\langle U \rangle$ with $|d| = r$, except in the case where $c_n = 1$ and $c_k = 0$ for $k \notin \{0, n\}$.

In this last case, we will demonstrate as in the proof of Theorem 4.15 that we must have $c_0 + 1 = \max \mathcal{L}_{\langle T' \rangle}(y)$. By the same reasoning used in that proof, we will have shown $\Delta(\langle U \rangle) = \Delta(\langle T' \rangle) = \{x\}$.

Assume that there is some $r' \in \mathcal{L}_{\langle T' \rangle}(y)$ such that $r' > r = c_0 + 1$. Then we can write

$$c_0 a + (a + nx) = y = r'a + \beta x$$

for some $\beta \in \mathbb{N}$, which implies

$$(r' - c_0 - 1)a = (n - \beta)x.$$

We know $\gcd(x, a) = 1$, so we have $a \mid n - \beta$. But then we have $a \leq n - \beta \leq n$, which contradicts the minimality of the generating set. Therefore, we have $c_0 + 1 = \max \mathcal{L}_{\langle T' \rangle}(y)$, and the proof follows as in Theorem 4.15. □

Lemma 4.17. *Let $T = \{a, a+x, \ldots, a+tx\}$ (with $t \geq 5$) and $U \subset T$ with $T \setminus U = \{a+mx, a+(t-1)x\}$ for $2 \leq m \leq t-2$. Then $\Delta(\langle U \rangle) = \{x\}$.*

Proof. Let $T' = U \cup \{a+mx\}$; then we know $\Delta(\langle T' \rangle) = \{x\}$ by Theorem 4.15. Pick any y in U and $r \in \mathcal{L}_{\langle T \rangle}(y)$. Choose a factorization c in T. We will first demonstrate that if we have $r \notin \mathcal{L}_{\langle T \rangle}(y)$, then we must have $c_m = 1$ and $c_i = 0$ for $i \notin \{m, t\}$.

Clearly if we have $c_m = 0$, then we have $r \in \mathcal{L}_{\langle U \rangle}(y)$, and we are done. Now suppose that we have $c_m > 1$. Write $c_m = 2u + 3v$. If $m \neq t-2$, then set

$$d = c + v \cdot e_{m-2} + u \cdot e_{m-1} - (2u+3v) \cdot e_m + (u+2v) \cdot e_{m+1}.$$

If $m = t-2$, then set

$$d = c + u \cdot e_{m-2} + 2v \cdot e_{m-1} - (2u+3v) \cdot e_m + (u+v) \cdot e_{m+2}.$$

Then d is a factorization of y in $\langle U \rangle$ with $|d| = r$.

Assume that we have $c_m = 1$. Since we picked $y \in U$, we can find $k \in [0,t] \setminus \{m, t-1\}$ with $c_k > 0$. If we have $k \in [0, t-3] \setminus \{m-1, m\}$, set

$$d = c - e_k + e_{k+1} + e_{m-1} - e_m,$$

and we are done. Suppose that we have either $c_{t-2} > 0$ and $m \neq t-3$ or $c_{m-1} > 0$ and $m \neq t-2$. Then set

$$d = c + e_{k-1} - e_k - e_m + e_{m+1}.$$

If we have either $c_{t-2} > 0$ and $m = t-3$ or $c_{m-1} > 0$ and $m = t-2$, then set

$$d = c + e_{t-5} - e_{t-3} - e_{t-2} + e_t.$$

Thus we have found a factorization d of y in $\langle U \rangle$ with $|d| = r$, except in the case where $c_m = 1$ and $c_k = 0$ for $k \notin \{m, t\}$.

Suppose that we do have $y = (a+mx) + c_t(a+tx)$. If we have $m \neq t-3$, then set

$$d = c - e_m + e_{m+2} + e_{t-2} - e_t,$$

and we have $r \in \mathcal{L}_{\langle U \rangle}(y)$. Now suppose that we have $m = t-3$. As in the proof of Lemma 4.16, it suffices to show that we have $c_t + 1 = \min \mathcal{L}_{\langle T' \rangle}(y)$. Assume that we can find some $r' \in \mathcal{L}_{\langle T' \rangle}(y)$ with $r' < c_t + 1$. Let z be a factorization of y with $|z| = r'$. If $z_t = r'$, then we have

$$y = r'(a+tx) = (a+mx) + c_t(a+tx),$$

which implies

$$(r' - c_t)(a+tx) = a + mx > 0.$$

But then we must have $0 < r' - c_t < 1$, which is is impossible since we are dealing only with integers. Therefore, we must have $z_t \leq r' - 1$. But this means that we have

$$y \leq (r' - z_t)(a+mx) + z_t(a+tx) \leq (a+mx) + (r'-1)(z+tx) < (a+mx) + c_t(a+tx) = y,$$

which is a contradiction (note that here we use the fact that $a+mx = a+(t-2)x$ is the largest generator in $\langle T' \rangle$ other than $a+tx$). We have now demonstrated that we must have $c_t + 1 = \min \mathcal{L}_{\langle T' \rangle}(y)$, which implies $\Delta(\langle U \rangle) = \{x\}$. $\square$

Lemma 4.18. *Let $T = \{a, a+x, \ldots, a+tx\}$ (with $t \geq 5$) and $U \subset T$ with $T \setminus U = \{a+x, a+(t-1)x\}$. Then $\Delta(\langle U \rangle) = \{x\}$.*

Proof. Set $T' = U \cup \{a+x\}$; then we have $\Delta(\langle T' \rangle) = \{x\}$ by Theorem 4.15. Pick $y \in \langle T' \rangle$ and $r \in \mathcal{L}_{\langle T' \rangle}(y)$. We aim to show that if we have $y \in \langle U \rangle$ and $r \notin \mathcal{L}_{\langle U \rangle}(y)$, then we must have $r \in \{\min \mathcal{L}_{\langle T' \rangle}(y), \max \mathcal{L}_{\langle T' \rangle}(y)\}$. Let c be a factorization of y in $\langle T' \rangle$ with $|c| = r$. If $c_1 = 0$, then we are done. Suppose that we have $c_1 > 1$. Then write $c_1 = 2u + 3v$ and set

$$d = c + (u + 2v) \cdot e_0 - (2u + 3v) \cdot e_1 + u \cdot e_2 + v \cdot e_3$$

(note that we have $t - 1 \geq 4$). Now d is a factorization of y in $\langle U \rangle$ with $|d| = r$.

Assume that we have $c_1 = 1$. If we have $y \in \langle U \rangle$, then we can find some $k \in \{0\} \cup [2, t]$ with $c_k > 0$. If we can find $k \in [2, t-3] \cup \{t-1\}$ with $c_k > 0$, then set

$$d = c + e_0 - e_1 - e_k + e_{k+1},$$

and we are done. If we have $c_{t-2} > 0$, then set

$$d = c - e_1 + e_2 + e_{t-3} - e_{t-2},$$

and we have produced a factorization of y in $\langle U \rangle$ with length r. If we have $c_t > 0$, then set

$$d = c - e_1 + e_3 + e_{t-2} - e_t$$

(which we can do since $t - 1 \geq 4$).

Now we are left with the case in which we have $c = (c_0, 1, 0, \ldots, 0)$. In this case, we claim that we have $c_0 + 1 = \max \mathcal{L}_{\langle T' \rangle}(y)$. Suppose that we can find $r' \in \mathcal{L}_{\langle T' \rangle}(y)$ with $r' > c_0 + 1$. Then we can find $\lambda \in \mathbb{N}$ with $y = r'a + \lambda x$, so we have

$$(r' - c_0 - 1)a = (1 - \lambda)x.$$

But we know $\gcd(a, x) = 1$ and $a > 1 \geq 1 - \lambda$, so this is impossible. Therefore, if we have $y \in \langle U \rangle$ and $r \notin \mathcal{L}_{\langle U \rangle}(y)$, then we must have $r = \max \mathcal{L}_{\langle T' \rangle}(y)$. It follows that $\Delta(\langle U \rangle) = \{x\}$. □

Lemma 4.19. *Let $T = \{a, a+x, \ldots, a+tx\}$ with $t \geq 5$ and $U \subset T$ with $T \setminus U = \{a+2x, a+3x\}$. Then we have $\Delta(\langle U \rangle) = \{x\}$.*

Proof. We will show that if we have $y \in \langle U \rangle$ with $\mathcal{L}_{\langle T \rangle}(y) \neq \mathcal{L}_{\langle U \rangle}(y)$, then we have $\mathcal{L}_{\langle T \rangle}(y) \setminus \mathcal{L}_{\langle U \rangle}(y) \subset \{\min \mathcal{L}_{\langle T \rangle}(y)\}$. This implies $\Delta_{\langle U \rangle}(y) \subset \Delta_{\langle T \rangle}(y)$ for all $y \in \langle U \rangle$ and hence $\Delta(\langle U \rangle) = \{x\}$.

Pick $y \in \langle T \rangle$ and $r \in \mathcal{L}_{\langle T \rangle}(y)$. Let c be a factorization of y in $\langle T \rangle$ with $|c| = r$. We will show via sliding that we must have $r \in \mathcal{L}_{\langle U \rangle}(y)$ unless we have

(i) $c \in \{(0, 1, 1, 0, \ldots, 0), (1, 0, 0, 1, 0, \ldots, 0)\}$, or

(ii) $t = 5$ and $c \in \{(0, 0, 0, 1, 1, 0), (0, 0, 1, 0, 0, 1)\}$.

Suppose that we have $c_2 > 1$, and write $c_2 = 2u_2 + 3v_2$. Then set

$$d = c + u_2 \cdot e_0 + 2v_2 \cdot e_1 - (2u_2 + 3v_2) \cdot e_2 + (u_2 + v_2) \cdot e_4,$$

so d is a factorization of y with $d_2 = 0$ and $|d| = r$.

Suppose that we have $c_2 = 1$. If we have $y \in U$, then we can pick $k \in \{0, 1, 3, 4, \ldots, t\}$ with $c_k > 0$. If we have $k \in \{0, 3, \ldots, t-1\}$, set

$$d = c + e_1 - e_2 - e_k + e_{k+1}.$$

If we have $k = 1$ or $k = t$, set

$$d = c + e_{k-1} - e_k - e_2 + e_3.$$

Now, regardless of k, d is a factorization of y such that $|d| = r$ and $d_2 = 0$.

If $d_3 = 0$, then we are done because we have $r \in \mathcal{L}_{\langle U \rangle}(y)$. Suppose that we have $d_3 > 1$; then write $d_3 = 2u_3 + 3u_3$. Set

$$f = d + (u_3 + v_3) \cdot e_1 - (2u_3 + 3v_3) \cdot e_3 + 2v_3 \cdot e_4 + u_3 \cdot e_5,$$

so f is a factorization of y in $\langle U \rangle$ with $|f| = r$.

Finally, we are left with the case in which we have $d_2 = 0$ and $d_3 = 1$. If we have $y \in U$, then we can pick $k \in \{0, 1\} \cup [4, t]$ with $d_k > 0$. Suppose that we can find such a k with $k \neq 0$ and $k \neq 4$. Then set

$$f = d - e_3 + e_4 + e_{k-1} - e_k.$$

Suppose that we have $t \neq 5$ and $k = 4$; then set

$$f = d + e_1 - e_3 - e_4 + e_6.$$

If we have $t = 5$ and $d_4 > 1$, set

$$f = d + e_1 - e_3 - e_4 + 2e_5.$$

Suppose that we have $t = 5$ and $d_4 = 1$. If we have $d_0 > 0$, set

$$f = d - e_0 + 2e_1 - e_3 - e_4 + e_5.$$

If we have $t = 5$, $d_4 = 1$, and $d_0 = 0$, then condition (ii) must hold (stated explicitly in the beginning of the proof). We will address this case momentarily.

Now suppose that we have $d_k = 0$ for $k \notin \{0, 3\}$. If we have $d_0 = 0$, then $y \notin \langle U \rangle$. If we have $d_0 \geq 2$, set

$$f = d - 2e_0 + 3e_1 - e_3,$$

so we have $r \in \mathcal{L}_{\langle U \rangle}(y)$. Now the only other case is when we have $d = (1, 0, 0, 1, 0, \ldots, 0)$, which is just condition (i).

Now we must address the cases in which conditions (i) and (ii) hold. Note that in each of the conditions, we consider two factorizations whose length and image under φ are identical. In condition (i), for example, if we have $c = (0, 1, 1, 0, \ldots, 0)$, then we will have $d = (1, 0, 0, 1, 0, \ldots, 0)$.

In either condition, we have $|d| = 2$ and $y = 2a + \mu x$ for some $\mu \in \{3, 7\}$. We wish to show that we must have $2 = \min \mathcal{L}_{\langle T \rangle}(y)$. Suppose that we have $1 \in \mathcal{L}_{\langle T \rangle}(y)$. Then we can write $a + \lambda x = 2a + \mu x$ for some $\lambda \in \mathbb{N}$, which implies $a = (\lambda - \mu)x$. We know $\gcd(a, x) = 1$, so we must have $x = 1$. But then we have $a = \lambda - \mu < \lambda \leq t$, which contradicts the minimality of the generating set. Therefore, we must have $r = \min \mathcal{L}_{\langle T \rangle}(y)$.

Thus we have demonstrated that given any $y \in \langle U \rangle$, we must have $\mathcal{L}_{\langle T \rangle}(y) \setminus \mathcal{L}_{\langle U \rangle}(y) \subset \{\min \mathcal{L}_{\langle T \rangle}(y)\}$. It follows that $\Delta(\langle U \rangle) = \Delta(\langle T \rangle) = \{x\}$. $\square$

Lemma 4.20. *Let $T = \{a, a+x, \ldots, a+tx\}$ with $t \geq 5$ and $U \subset T$ with $T \setminus U = \{a+(t-2)x, a+(t-3)x\}$. Then we have $\Delta(\langle U \rangle) = \{x\}$.*

Proof. First, note that if we have $t = 5$ then $\{t-2, t-3\} = \{2, 3\}$, so we can apply Lemma 4.19. Now assume that we have $t \geq 6$. We will take the same approach used in the proof of Lemma 4.19.

Pick $y \in \langle T \rangle$ and $r \in \mathcal{L}_{\langle T \rangle}(y)$. Let c be a factorization of y in $\langle T \rangle$ with $|c| = r$. If $c_{t-3} = 0$, set $d = c$. If $c_{t-3} > 1$, write $c_{t-3} = 2u_{t-3} + 3v_{t-3}$ and set

$$d = c + u_{t-3} \cdot e_{t-5} + 2v_{t-3} \cdot e_{t-4} - (2u_{t-3} + 3v_{t-3}) \cdot e_{t-3} + (u_{t-3} + v_{t-3}) \cdot e_{t-1}.$$

If we have $y \in \langle U \rangle$ and $c_{t-3} = 1$, pick $k \in \{0, \ldots, t-4, t-2, t-1, t\}$ with $c_k > 0$. If we have $k \in \{0, \ldots, t-5, t-2, t-1\}$, set

$$d = c - e_k + e_{k+1} + e_{t-4} - e_{t-3}.$$

If we have $k \in \{t-4, t\}$, set

$$d = c + e_{k-1} - e_k - e_{t-3} + e_{t-2}.$$

Now d is a factorization of y with $c_{t-3} = 0$ and $|d| = r$. If $d_{t-2} = 0$, set $f = d$. If $d_{t-2} > 1$, write $d_{t-2} = 2u_{t-2} + 3v_{t-2}$ and set

$$f = d + (u_{t-2} + v_{t-2}) \cdot e_{t-4} - (2u_{t-2} + 3v_{t-2}) \cdot e_{t-2} + 2v_{t-2} \cdot e_{t-1} + u_{t-2} \cdot e_t.$$

Thus f is a factorization of y in $\langle Urangle$ with length r.

Now assume that we have $d_{t-2} = 1$. If we have $y \in \langle U \rangle$, then we can pick $k \in \{0, \ldots, t-4, t-1, t\}$ with $d_k > 0$. If we can find such a k in $\{1, \ldots, t-4, t\}$, then set

$$f = d + e_{k-1} - e_k - e_{t-2} + e_{t-1}.$$

If we have $d_0 > 0$, set

$$f = d - e_0 + e_2 + e_{t-4} - e_{t-2}$$

(note that we have assumed $t \geq 6$, so $2 < t - 3$). Now we are left with the case in which we have $d_{t-2} = 1$ and $d_k = 0$ for $k \notin \{t-2, t-1\}$.

If we have $y \in \langle U \rangle$, then we know $d_{t-1} > 0$. If $d_{t-1} > 1$, set

$$f = d + e_{t-4} - e_{t-2} - 2e_{t-1} + e_t,$$

and we have produced a factorization of y in $\langle U \rangle$ with length r. Therefore, we have only to deal with the factorization $d = (0, \ldots, 0, 1, 1, 0)$.

In this case, we have $y = 2a + (2t-3)x$ and $r = 2$. We claim that we must have $2 = \min \mathcal{L}_{\langle T \rangle}(y)$. Suppose that we have $1 \in \mathcal{L}_{\langle T \rangle}(y)$. Then we can find some $\lambda \leq t$ with $y = a + \lambda x$. But it follows that

$$a = (\lambda - 2t + 3)x.$$

As in the proof of Lemma 4.19, we must have $x = 1$ and hence $a \leq \lambda - 2t + 3 \leq \lambda - 9 < t$, yielding a contradiction. Therefore, if we have $y \in \langle U \rangle$ and $r \notin \mathcal{L}_{\langle U \rangle}(y)$, then we have $r = \min \mathcal{L}_{\langle T \rangle}(y)$. It follows that $\Delta(\langle U \rangle) = \{x\}$. □

Theorem 4.21. *Let $T = \{a, a+x, \ldots, a+tx\}$ with $t \geq 5$ and $U \subset T$ with $\#(T \setminus U) = 2$. Then $\Delta(\langle U \rangle) = \{x\}$.*

Proof. If we have $\{a, a+tx\} \cap (T \setminus U) \neq \emptyset$, then we can apply Theorem 4.15. Otherwise, write $\{a + mx, a + nx\} = T \setminus U$ and consult the following table to find the correct lemma:

$m = 1$	$n = t - 1$	Lemma 4.18
$m = 1$	$n < t - 1$	Lemma 4.16
$m > 1$	$n = t - 1$	Lemma 4.17
$m = 2$	$n = 3$	Lemma 4.19
$m = t - 2$	$n = t - 3$	Lemma 4.20
otherwise		Lemma 4.9

□

We can now totally characterize delta sets for $t \leq 4$. Let $S_t = \langle a, a + x, \ldots, a + tx \rangle$.

For S_2, removing a or $a + 2x$ leaves us with an arithmetic progression of step size x, so the delta set is just $\{x\}$. If we remove $a + x$, then we have an arithmetic progression of step size $2x$, so the delta set is $\{2x\}$.

For S_3, removing a single generator cannot change the delta set. Now suppose that we remove two generators. Removing a or $+3x$ reduces to the case described directly above. If we retain both a and $a + 3x$, then we must have removed the other two, so we are left with an arithmetic progression of step size $3x$ and a delta set of $\{3x\}$.

For S_4, removing one generator cannot change the delta set. Now suppose that we remove two generators. If we remove either a or $a + 4x$, see above paragraph. Suppose that we keep both a and $a + 4x$; then we also keep one from the middle. If we keep $a + 2x$, then we are left with a delta set of $\{2x\}$ because it is an arithmetic progression. If we keep $a + x$ or $a + 3x$, then we can apply the results discussed in Section 5. In these two cases, the delta set can only change for small a. We have not determined whether the value of x affects the delta set in this situation.

4.4 The Goldmember Conjecture

The ultimate goal is to characterize exactly which golden sets will result in a delta set of more than one element. We have not managed to do this. However, it appears that we need to remove a large number of generators in order to change the delta set. We would like to someday show something like the following conjecture (although it probably needs tweaking).

Conjecture 4.22 (The Goldmember Conjecture)**.** *If T and U are define as above, the $\Delta(\langle U\rangle) \neq \{x\}$ only if some subset of $T \setminus U$ is a golden set of index $\lfloor t/2 \rfloor$ or the reflection of such a set.*

The following conjecture is a result of Conjecture 4.22, but it may be simpler to prove.

Conjecture 4.23. *If we have $\#(T \setminus U) < \lfloor t/2 \rfloor$, then we have $\Delta(\langle U\rangle) = \{x\}$.*

Here are some notes on a possible method of proof for Conjecture 4.23. They are based on the proof in [2] that the delta set for a semigroup generated by an arithmetic progression contains exactly one element. However, there are parts of that proof which will certainly no longer work when we begin to remove generators (namely, the bounds he establishes on ℓ and L).

Induct on t. As in Scott's paper [2], let U' be U with $a + tx$ removed and U'' be U with a removed.

We claim that for $m \in U$, we have the following:

(i) $\mathcal{L}_{\langle U\rangle}(m) = \mathcal{L}_{\langle U'\rangle}(m)$ if $m \in U' \setminus U''$,

(ii) $\mathcal{L}_{\langle U\rangle}(m) = \mathcal{L}_{\langle U''\rangle}(m)$ if $m \in U'' \setminus U'$, and

(iii) $\mathcal{L}_{\langle U\rangle}(m) = \mathcal{L}_{\langle U'\rangle}(m) \cup \mathcal{L}_{\langle U''\rangle}(m)$ if $m \in U' \cap U''$.

This can be shown via sliding, as in [2], Proposition 3.2, Corollary 3.3. The only thing that changes is where we slide copies of a and $a + tx$. In the paper, they simply slide up or down one space. We can't necessarily do this, but we can slide if we have $M < t/2$ because we can find $i \in \{1, \ldots, t/2\}$ such that both $a + ix$ and $a + (t - i)x$ are still in U. The rest of Scott's proof still holds.

We know that U' and U'' are arithmetic sequences up through the $(t-1)^{\text{th}}$ term with M terms removed. But we know $\lfloor (t-1)/2 \rfloor = \lfloor t/2 \rfloor > M$, so by the inductive hypothesis we have $\Delta(U') = \Delta(U'') = \{x\}$. Then we are done unless we have $m \in U' \cap U''$. As in Scott's paper, it suffices to show that if we have $\mathcal{L}_{\langle U'\rangle}(m) \cap \mathcal{L}_{\langle U''\rangle}(m) = \emptyset$, then we have $\min \mathcal{L}_{\langle U'\rangle}(m) = \max \mathcal{L}_{\langle U''\rangle}(m) + x$.

Set $w = \max \mathcal{L}_{\langle U''\rangle}(m)$. Suppose that we have $w + x \notin \mathcal{L}_{\langle U'\rangle}(m)$. But we know $w + x \in \mathcal{L}_{\langle S\rangle}(m)$, so we can write $y = b_1(a + jx) + b_2(a + (j+1)x)$ for $j \in \{0, \ldots, t-1\}$, with $b_1 + b_2 = w + x$.

Now we have a factorization with length in the middle of the delta set. We can show that we must have $j \leq M - 1$ or $j \geq t - M$, or else we can slide to find a factorization of equal length in U. My goal is to show that if j is that close to the edge of the semigroup, the factorization length must be close to the edge of the delta set, yielding a contradiction.

4.5 Removing Blocks of Generators

It seems that the most reliable way to change delta sets is to remove a block of generators from the beginning of the semigroup— that is, to have $\{a + x, \ldots, a + kx\} \subset T \setminus U$ for some k. In this section, we examine what happens when we remove this type of block. As it turns out, as long as a is large enough, removing such a block has no effect on the delta set.

In the proof, we use induction. The basic idea is that we can write any factorization either without $a + qx$ or with only a, $a + qx$, and $a + (q+1)x$. In the latter case, if the coefficients for $a + qx$ and $a + (q+1)x$ are large enough, we can rewrite them in terms of a and a pair of generators $a + mx$ and $a + nx$ in U, with $\gcd(m, n) = 1$ (thus the bound arises from the semigroups of the form $\langle m, n\rangle$ with $a + mx, a + nx \in U$ and $m, n \geq q + 1$). If the coefficients for $a + qx$ and $a + (q+1)x$ are too small to be rewritten in such a manner, then they must also be too small to be rewritten using more copies of a, so this must be the longest factorization of this element.

What is particularly interesting about the presence of Frobenius numbers in this theorem is that we also found that for $a \geq (t-1)t = F(\langle t, t+1\rangle) + 1$, we have $\Delta(\langle a, a + qx, a + tx\rangle) = \{x\}$. We are hoping to be able to find a similar pattern to character semigroups between $\langle a, a + qx, \ldots, a + tx\rangle$ and $\langle a, a + qx, a + tx\rangle$.

Theorem 4.24. *Let $T = \{a, a+x, \ldots, a+tx\}$ and $U_q = \{a, a+(q+1)x, \ldots, a+tx\}$ with $t \geq q+2$. Set*

$$\mathscr{F}_q = \mathbb{N} \setminus \bigcup_{\substack{m,n \in [q+1,t] \\ \gcd(m,n)=1}} \langle m, n \rangle.$$

If $\max(\mathscr{F}_q) < a$, then $\Delta(\langle U_q \rangle) = \{x\}$.

Proof. Let $P(q)$ be the proposition that we have $\Delta(\langle U_q \rangle) = \{x\}$. Note that we have $\#(T \setminus U_1) = 1$, so Theorem 4.15 tells us that $P(1)$ is true. Now suppose that $P(q-1)$ is true for some $q \in \mathbb{Z}^+$. We know

$$\mathscr{F}_{q-1} = \mathbb{N} \setminus \left(\mathscr{F}_q \bigcup_{n \in [q+1,t]} \langle q, n \rangle \right),$$

so it follows that we have $\mathscr{F}_{q-1} \subset \mathscr{F}_q$ and hence $\max(\mathscr{F}_{q-1}) \leq \max(\mathscr{F}_q) < a$. Therefore, the inductive hypothesis states that we have $\Delta(\langle U_{q-1} \rangle) = \{x\}$.

Pick any $y \in \langle U_{q-1} \rangle$ and $r \in \mathcal{L}_{\langle U_{q-1} \rangle}(y)$. Let z be a factorization of y with $|z| = r$. If $z_q = 0$, then we are done. Suppose $z_q > 0$. For $q+2 \leq i \leq t$, set $\alpha_i = \min(z_q, z_i)$. Then set

$$z^{(0)} = z - \alpha_t \cdot e_q + \alpha_t \cdot e_{q+1} + \alpha_t \cdot e_{t-1} - \alpha_t \cdot e_t,$$

and for $1 \leq n \leq t-q-2$, set

$$z^{(n)} = z^{(n-1)} - \alpha_{t-n} \cdot e_q + \alpha_{t-n} \cdot e_{q+1} + \alpha_{t-n} \cdot e_{t-n-1} - \alpha_{t-n} \cdot e_{t-n}.$$

Let $u = z^{(t-q-2)}$. If we have $u_q = 0$, then $r \in \mathcal{L}_{\langle U_q \rangle}(y)$. Suppose that we have $u_q > 0$; then we have $u_i = 0$ for $i \in [q+2, t]$. We will now argue that if this is the case, then we have either $r \in \mathcal{L}_{\langle U_q \rangle}(y)$ or $r = \max \mathcal{L}_{\langle U_{q-1} \rangle}(y)$.

Suppose that we have $r \neq \max \mathcal{L}_{\langle U_{q-1} \rangle}(y)$. Then we can find $r' > r$ with $y = r'a + \lambda x$ for some $\lambda \in \mathbb{N}$. It follows that we have

$$(r' - r)a = (qu_q + (q+1)u_{q+1} - \lambda)x.$$

Since we know $\gcd(a, x) = 1$, a must divide $qu_q + (q+1)u_{q+1} - \lambda$. Therefore, we must have $\max(\mathscr{F}_q) < a \leq qu_q + (q+1)u_{q+1}$, so we can find $m, n \in [q+1, t]$ with $qu_q + (q+1)u_{q+1} \in \langle m, n \rangle$. Write $qu_q + (q+1)u_{q+1} = \mu_1 m + \mu_2 n$. We claim that we must have $u_q + u_{q+1} \geq \mu_1 + \mu_2$. Otherwise, we would have

$$\mu_1 m + \mu_2 n \geq (\mu_1 + \mu_2)(q+1) > (u_q + u_{q+1})(q+1) \geq qu_q + (q+1)u_{q+1},$$

which is clearly false.

Now set $\nu = u_q + u_{q+1} - \mu_1 - \mu_2 \geq 0$. Then we have

$$(u_0+\nu)a+\mu_1(a+mx)+\mu_2(a+nx) = (u_0+u_q+u_{q+1})a+(\mu_1 m+\mu_2 n)x = u_0 a+u_q(a+qx)+u_{q+1}(a+(q+1)x) = y.$$

Furthermore, we have $u_0 + \nu + \mu_1 + \mu_2 = u_0 + u_q + u_{q+1} = r$, so we have shown that we must have $r \in \mathcal{L}_{\langle U_q \rangle}(y)$. Therefore, we have demonstrated that we must have $\Delta_{\langle U_q \rangle}(y) \subset \Delta_{\langle U_{q-1} \rangle}$ for all $y \in \langle U_q \rangle$, so we have $\Delta(\langle U_q \rangle) = \{x\}$. The proof follows by induction. ☐

Corollary 4.25. *Suppose that we have $T = \{a, a+x, \ldots, a+tx\}$ and $U_q \subset T$ with*

$$T \setminus U_q = \{a+x, \ldots, a+qx\}$$

for $q \leq t-2$. If we have either $t = q+2$ and $a \geq q(q+1)$ or $t \geq q+3$ and $a \geq (q-1)(q+1)$, then we have $\Delta(\langle U_q \rangle) = \{x\}$.

Proposition 4.26. *Let $S = \langle a, a+(t-1)x, a+tx \rangle$ with $t \geq 4$. Set $k = t-2$, $a = k(k+1) - 1 = t^2 - 3t + 1$, and $x = 1$. Then for $y = 2t^3 - 9t^2 + 11t - 3$, we have $2x \in \Delta_S(y)$.*

Proof. Consider the following vectors in $\mathbb{N}^{t+1}$:

$$\begin{aligned}\alpha &= (2t-3, 0, \ldots, 0)\\ \beta &= (t-2, 0, \ldots, 0, 1, t-3, 0)\\ \gamma &= (0, \ldots, 0, t-2, t-3).\end{aligned}$$

Then we have

$$\begin{aligned}\varphi(\alpha) &= (2t-3)(t^2-3t+1)\\ &= 2t^3 - 6t^2 + 2t - 3t^2 + 9t - 3\\ &= y.\end{aligned}$$

We also have

$$\begin{aligned}\varphi(\beta) &= (t-2)(t^2-3t+1) + (1)((t^2-3t+1)+(t-2)x) + (t-3)((t^2-3t+1)+(t-1)x)\\ &= t^3 - 3t^2 + t - 2t^2 + 6t - 2 + t^2 - 3t + 1 + t - 2 + t^3 - 3t^2 + t + t^2 - t - 3t^2 + 9t - 3 - 3t + 3\\ &= y.\end{aligned}$$

Finally, we have

$$\begin{aligned}\varphi(\gamma) &= (t-2)(t^2-3t+1+(t-1)x) + (t-3)(t^2-3t+1+tx)\\ &= t^3 - 2t^2 - 2t^2 + 4t + t^3 - 2t^2 + t - 3t^2 + 6t - 3\\ &= y.\end{aligned}$$

Thus α, β, and γ are all factorizations of y. We also have

$$\begin{aligned}|\alpha| &= 2t-3,\\ |\beta| &= 2t-4, \text{ and}\\ |\gamma| &= 2t-5,\end{aligned}$$

which implies $|\alpha - \beta| = |\beta - \gamma| = x$. Thus there are no factorizations of y between these. Clearly both α and γ are factorizations in S. We aim to show that there is no factorization of length $|\beta|$ in S, which would mean that we have $2x \in \Delta_S(y)$.

Suppose that we do have some factorization z of y with $z_1 = \ldots = z_{t-2} = 0$ and $|z| = 2t-4$. Then we can write

$$y = (2t-3)a = (z_0 + z_{t-1} + z_t)a + z_{t-1}(t-1) + z_t t = (2t-4)a + z_{t-1}(t-1) + z_t t,$$

which implies $z_{t-1}(t-1) + z_t t = a = t^2 - 3t + 1$. If $z_t = 0$, then we have

$$z_{t-1} = t - 2 - \frac{1}{t-1},$$

which is a contradiction because z_{t-1} is an integer. Therefore, we must have $z_t > 0$. Then we have

$$z_{t-1}(t-1) = t^2 - (3 + z_t)t + 1 = (t^2 - 2t + 1) - (1 + z_t)t,$$

so it follows that

$$z_{t-1} = t - 1 - \frac{(1+z_t)t}{t-1}.$$

We know that z_{t-1} must be a non-negative integer and that the rational number on the right side of the above equation is nonzero, so we can write

$$(1 + z_t)t = c(t-1)$$

for some $c \in \mathbb{Z}^+$. Therefore, we must have

$$z_t = c - 1 - \frac{c}{t},$$

and since z_t is also an integer we can write $c = dt$ for some $d \in \mathbb{Z}^+$. Then we have $z_t = d(t-1) - 1$ for $d > 0$. If we have $d \geq 3$, then we have

$$2t - 4 \geq z_t \geq 3(t-1) - 1 = 3t - 4 > 2t - 4,$$

which is false. Therefore, we have $d \in \{1, 2\}$ and hence $z_t \in \{t-2, 2t-3\}$. If $z_t = 2t-3$, then we have

$$(2t-3)a = y \geq z_t(a + tx) = (2t-3)(a+tx) > (2t-3)a,$$

which is impossible. Therefore, we must have $z_t = t - 2$. Then we have $z_0 + z_{t-1} = t - 2$. We know

$$y = z_0(t^2 - 3t + 1) + z_{t-1}(t^2 - 2t) + (t-2)(t^2 - 2t + 1) = 2t^3 - 9t^2 + 11t - 3,$$

which implies

$$(z_0 + z_{t-1})t^2 - 2(z_0 + z_1)t - z_0 t + z_0 = t^3 - 5t^2 + 6t - 1.$$

Then, since know $z_0 + z_{t-1} = t - 2$, we have

$$z_0(-t+1) = t^2 + 2t - 1 = (-t+1)(-t-3) + 2,$$

so it follows that

$$z_0 = -t - 3 + \frac{2}{-t+1}.$$

But z_0 is a non-negative integer, so this is false. Therefore, we cannot have $2t - 4 \in \mathcal{L}_{\langle\rangle} S(y)$. But since we do have $2t-3, 2t-5 \in \mathcal{L}_{\langle\rangle} S(y)$, we must have $2x \in \Delta(S)$. ☐

Therefore, our bound for the $t = k+2$ case is sharp.

5 Delta Sets of Numerical Semigroups of Embedding Dimension Three

5.1 Bounds on Delta Sets in Embedding Dimension Three

We wish to determine when the delta set of a numerical semigroup in embedding dimension three is a just a single element. For this section assume all numerical semigroups are primitive.

The following proposition is a restatement of [2, Proposition 2.10]:

Proposition 5.1. *Let $S = \langle n_1, n_2, \ldots, n_t\rangle$ where $\{n_1, n_2, \ldots, n_t\}$ is the minimal set of generators. Then*

$$min\Delta(S) = \gcd\{n_i - n_{i-1} | i \in \{2, 3, \ldots, t\}\}.$$

The following theorem is a restatement of [5, Theorem 3.1]:

Theorem 5.2. *Let $S = \langle n_1, n_2, n_3\rangle$. Set*

$$c_1 = \min\{c \in \mathbb{Z}^+ : cn_1 \in \langle n_2, n_3\rangle\}$$
$$c_3 = \min\{c \in \mathbb{Z}^+ : cn_3 \in \langle n_1, n_2\rangle\}.$$

Then we can write $c_1 n_1 = r_{12}n_2 + r_{13}n_3$ and $c_3 n_3 = r_{31}n_1 + r_{32}n_2$, where $r_{12} + r_{13}$ is maximal and $r_{31} + r_{32}$ is minimal. Set $K_1 = c_1 - (r_{12} + r_{13})$ and $K_3 = (r_{31} + r_{32}) - c_3$. Then $\max(\Delta(S)) = \max\{K_1, K_3\}$.

Lemma 5.3. *Let S be a numerical semigroup minimally generated by $S = \langle a, a+qx, a+tx\rangle$ where $\gcd(q,t) = 1$ and K_1, K_3 be as defined in Theorem 5.2 then $K_1, K_3 \equiv 0 \pmod{x}$.*

Proof. We first examine K_1. For any multiple of a in terms of $a+qx$ and $a+tx$ we have

$$\begin{aligned}\gamma_1 a &= \delta_{12}(a+qx)+\delta_{13}(a+tx)\\ (\gamma_1-(\delta_{12}+\delta_{13}))a &= (q\delta_{12}+t\delta_{13})x\end{aligned}$$

Note that since S is primitive $\gcd(a,x)=1$ thus $\gamma_1-(\delta_{12}+\delta_{13}) \equiv 0 \pmod{x}$. Specifically, $K_1 = c_1-(r_{12}+r_{13}) \equiv 0 \pmod{x}$.

Next we examine K_3. For any multiple of $a+tx$ in terms of a and $a+qx$ we have

$$\begin{aligned}\gamma_3(a+tx) &= \delta_{31}(a)+\delta_{32}(a+qx)\\ (\delta_{31}+\delta_{32}-\gamma_3)a &= (t\gamma_3-q\delta_{32})x\end{aligned}$$

Again since $\gcd(a,x)=1$, $(\delta_{31}+\delta_{32})-\gamma_3 \equiv 0 \pmod{x}$. In particular, $K_3=(r_{31}+r_{32})-c_3 \equiv 0 \pmod{x}$. ☐

Lemma 5.4. *Let S be a numerical semigroup minimally generated by $S=\langle a, a+qx, a+tx\rangle$ where* $\gcd(q,t)=1$ *and K_1, K_3 be as defined in Theorem 5.2 then $K_1, K_3 \geq x$.*

Proof. Note that $c_1(a)=r_{12}(a+qx)+r_{13}(a+tx) > r_{12}(a)+r_{13}(a)$ so $K_1=c_1-(r_{12}+r_{13})>0$ and since $K_1 \equiv 0 \pmod{x}$ by Lemma 5.3, $K_1 \geq x$. Similarly, note that $c_3(a+tx)=r_{31}(a)+r_{32}(a+qx) < r_{31}(a+tx)+r_{32}(a+tx)$ so $K_3=(r_{31}+r_{32})-c_3>0$ and since $K_3 \equiv 0 \pmod{x}$ by Lemma 5.3, $K_3 \geq x$. ☐

Lemma 5.5. *Let Sbe a numerical semigroup minimally generated by $S=\langle a, a+qx, a+tx\rangle$ where* $\gcd(q,t)=1$ *and K_1, K_3 be as defined in Theorem 5.2, then $\Delta(S)=\{x\}$ if and only if $K_1=K_3=x$.*

Proof. Let K_1, K_3, c_i, r_{ij} be as defined in Theorem 5.2. From Proposition 5.1 we have that $\min(\Delta(S)) = \gcd(qx,tx)=x$. We will show $\max(\Delta(S))=x$ if and only if $K_1=K_3=x$.

Clearly if $K_1=K_3=x$ then $\max(\Delta(S))=\max\{x,x\}=x$. Now let $\max(\Delta(S))=x$. We have $\max(\Delta(S))=\max\{K_1,K_3\}=x$ which implies $K_1,K_3 \leq x$, but since $K_1,K_3 \geq x$, by Lemma 5.4 we have $K_1=K_3=x$. ☐

We will now present several cases when the delta set of a numerical semigroup in embedding dimension three is just a single element.

5.2 The Smallest and Largest Generators Are Not Coprime

In this section we deal with numerical semigroups minimally generated by $S=\langle a, a+qx, a+tx\rangle$ and show that if $a \equiv 0 \pmod{t}$ then $\Delta(S)=\{x\}$.

Proposition 5.6. *Let S be a numerical semigroup minimally generated by $S=\langle a, a+qx, a+tx\rangle$ where* $\gcd(q,t)=1$*. If $a \equiv 0 \pmod{t}$ then $K_1=x$.*

Proof. Let S be as above and $a \equiv 0 \pmod{t}$.

For any multiple of a in terms of $a+qx$ and $a+tx$ we have

$$\begin{aligned}\gamma_1 a &= \delta_{12}(a+qx)+\delta_{13}(a+tx)\\ &= (\delta_{12}+\delta_{13})a+(q\delta_{12}+t\delta_{13})x. \end{aligned} \tag{17}$$

The left hand side is equivalent to $0 \pmod{a}$ thus, since S is primitive and $\gcd(a,x)=1$, we have $q\delta_{12}+t\delta_{13} \equiv 0 \pmod{a}$. Write $q\delta_{12}+t\delta_{13}=na$ where $n>0$ since either δ_{12} or δ_{13} must be positive. Notice that since $\gcd(q,t)=1$, $\delta_{12} \equiv 0 \pmod{t}$ so

$$\begin{aligned}\delta_{12} &= mt\\ \delta_{13} &= \frac{na}{t}-mq. \end{aligned} \tag{18}$$

where $0 \leq m \leq \frac{na}{qt}$. Substituting into equation 17 and factoring out an a we find

$$\gamma_1 a = \left(mt + \frac{na}{t} - mq\right) a + (na)x$$

$$\gamma_1 = n\left(\frac{a}{t} + x\right) + m(t-q) \tag{19}$$

$$\geq n\left(\frac{a}{t} + x\right). \tag{20}$$

Notice that the right hand side is positive since $t > q > 0$ and $n, a, x > 0$. Thus γ_1 is bounded from below by the case when $n = 1$ and from equation 20 we have

$$\gamma_1 \geq \frac{a}{t} + x. \tag{21}$$

Now notice that γ_1 achieves this lower bound when $\delta_{12} = 0$ and $\delta_{13} = \frac{a}{t}$.

$$\gamma_1(a) = 0(a + qx) + \frac{a}{t}(a + tx)$$

$$\gamma_1(a) = \left(\frac{a}{t} + x\right)(a)$$

Thus by the definition of c_1, $c_1 = \frac{a}{t} + x$. Recall that from the definition of r_{12}, r_{13} in Theorem 5.2 $r_{12} + r_{13} \geq \delta_{12} + \delta_{13}$ for all δ_{12}, δ_{13} satisfying $\delta_{12}(a + qx) + \delta_{13}(a + tx) = \left(\frac{a}{t} + x\right)a$. Since $(0)(a + qx) + \left(\frac{a}{t}\right)(a + tx) = \left(\frac{a}{t} + x\right)a$ we have that $r_{12} + r_{13} \geq \frac{a}{t}$. That is, $K_1 = c_1 - (r_{12} + r_{13}) \leq \left(\frac{a}{t} + x\right) - \frac{a}{t} = x$. By Lemma 5.4 $K_1 \geq x$ thus $K_1 = x$ as desired. □

Proposition 5.7. *Let S be a numerical semigroup minimally generated by $S = \langle a, a + qx, a + tx\rangle$ where* $\gcd(q, t) = 1$. *If $a \equiv 0 \pmod{t}$ then $K_3 = x$.*

Proof. For any multiple of $a + tx$ in terms of a and $a + qx$ we have

$$\gamma_3(a + tx) = \delta_{31}(a) + \delta_{32}(a + qx)$$

$$(\delta_{31} + \delta_{32} - \gamma_3)a = (t\gamma_3 - q\delta_{32})x \tag{22}$$

Note that $\gamma_3(a + tx) = \delta_{31}(a) + \delta_{32}(a + qx) < \delta_{31}(a + tx) + \delta_{32}(a + tx)$. Thus $\delta_{31} + \delta_{32} - \gamma_3 > 0$ and from equation 22 it follows that $t\gamma_3 - q\delta_{32} > 0$. Notice also that $t\gamma_3 - q\delta_{32} \equiv 0 \pmod{a}$. That is

$$t\gamma_3 = na + q\delta_{32} \tag{23}$$

$n > 0$. Observe that γ_3 is bounded by

$$\gamma_3 \geq \frac{a}{t}.$$

We see that γ_3 achieves this lower bound in the following equation

$$\gamma_3(a + tx) = \left(\frac{a}{t} + x\right)(a) + (0)(a + qx)$$

$$\gamma_3 = \frac{a}{t}$$

From the definition of c_3, $c_3 = \frac{a}{t}$. Since $q, r_{32} \geq 0$, we see $tc_3 - qr_{32} = a - qr_{32} \leq a$. From equation 23 we have that $tc_3 - qr_{32} \geq a$. Thus $tc_3 - qr_{32} = a$ which, when substituted into equation 22 shows

$$(r_{31} + r_{32} - c_3)a = ax$$

$$K_3 = r_{31} + r_{32} - c_3 = x.$$

□

Theorem 5.8. *Let S be a numerical semigroup minimally generated by $S = \langle a, a + qx, a + tx\rangle$ where* $\gcd(q, t) = 1$. *If $a \equiv 0 \pmod{t}$ then $\Delta(S) = \{x\}$.*

Proof. Let $a \equiv 0 \pmod{t}$. Then by Proposition 5.6, $K_1 = x$ and by Proposition 5.7, $K_3 = x$. Since $K_1 = K_3 = x$, by Lemma 5.5 $\Delta(S) = \{x\}$. □

5.3 A Specialized Bound on the Smallest Generator

In this section we deal with numerical semigroups minimally generated by $S = \langle a, a+x, a+tx\rangle$ and show that if $a \geq t(t-4)+2$ then $\Delta(S) = \{x\}$.

Proposition 5.9. *Let S be a numerical semigroup minimally generated by $S = \langle a, a+x, a+tx\rangle$. Let $a \not\equiv 0 \pmod{t}$ and write $a = nt+m$ where $m \in \{1, 2, \ldots, t-1\}$. If $n+m+x \geq t-1$ then $K_1 = x$.*

Proof. Let S be as above, $a = nt+m$, where $m \in \{1, 2, \ldots, t-1\}$, and $n+m+x \geq t-1$.

For any multiple of a in terms of $a+x$ and $i+tx$ we have

$$\begin{aligned}\gamma_1 a &= \delta_{12}(a+x) + \delta_{13}(a+tx)\\ &= (\delta_{12}+\delta_{13})a + (\delta_{12}+t\delta_{13})x \end{aligned}\tag{24}$$

The left hand side is equivalent to $0 \pmod{a}$ thus, since S is primitive and $\gcd(a, x) = 1$, we have $\delta_{12}+t\delta_{13} \equiv 0 \pmod{a}$. Write $\delta_{12}+t\delta_{13} = \alpha a = \alpha(nk+m)$ where $\alpha > 0$ since either δ_{12} or δ_{13} must be positive. Notice that $\delta_{12} \equiv \alpha m \pmod{k}$ so

$$\begin{aligned}\delta_{12} &= \alpha m + \beta t\\ \delta_{13} &= \alpha n - \beta\end{aligned}\tag{25}$$

where $-\lfloor \frac{\alpha m}{t} \rfloor \leq \beta \leq \alpha n$. Substituting into equation 24 and factoring out an a we find

$$\gamma_1 a = (\alpha m + \beta t + \alpha n - \beta)a + (\alpha a)x$$

$$\gamma_1 = \alpha(n+m+x) + \beta(t-1)\tag{26}$$

$$\geq \alpha(n+m+x) - \left\lfloor \frac{\alpha m}{t} \right\rfloor (t-1)\tag{27}$$

Notice that when $\alpha = 1$, $\gamma_1 \geq n+m+x$. We wish to show that $n+m+x$ is indeed a lower bound. That is, from equation 27, we wish to show that for all $\alpha > 0$, $\alpha(n+m+x) - \lfloor \frac{\alpha m}{t} \rfloor (t-1) \geq n+m+x$. Note that from our assumptions, $0 < t-1 \leq n+m+x$ and since $m < t$, $0 \leq \lfloor \frac{\alpha m}{t} \rfloor < \alpha$. Since both of these values are non-negative, we see

$$\begin{aligned}\left\lfloor \frac{\alpha m}{t} \right\rfloor (t-1) &\leq (\alpha-1)(n+m+x)\\ n+m+x &\leq \alpha(n+m+x) - \left\lfloor \frac{\alpha m}{t} \right\rfloor (t-1)\end{aligned}$$

Hence γ_1 is bounded from below by

$$\gamma_1 \geq n+m+x\tag{28}$$

Now notice that γ_1 achieves this lower bound when $\delta_{12} = m$ and $\delta_{13} = n$.

$$\begin{aligned}\gamma_1(a) &= (m)(a+x) + (n)(a+tx)\\ \gamma_1(a) &= (n+m+x)(a)\end{aligned}$$

Thus by the definition of c_1, $c_1 = n+m+x$.

Now recall that from the definition of r_{12}, r_{13} in Theorem 5.2 $r_{12}+r_{13} \geq \delta_{12}+\delta_{13}$ for all δ_{12}, δ_{13} satisfying $\delta_{12}(a+x) + \delta_{13}(a+tx) = (n+m+x)a$. Since $(m)(a+x) + (n)(a+tx) = (n+m+x)a$ we have that $r_{12}+r_{13} \geq n+m$. That is, $K_1 = c_1 - (r_{12}+r_{13}) \leq (n+m+x) - (n+m) = x$. By Lemma 5.4 $K_1 \geq x$ thus $K_1 = x$ as desired. □

Proposition 5.10. *Let S be a numerical semigroup minimally generated by $S = \langle a, a+x, a+tx\rangle$. Let $a \not\equiv 0 \pmod{t}$ and write $a = nt+m$ where $m \in \{1, 2, \ldots, t-1\}$. If $n+x+m \geq t-1$ then $K_3 = x$.*

Proof. For any multiple of $a + tx$ in terms of a and $a + x$ we have

$$\gamma_3(a + tx) = \delta_{31}(a) + \delta_{32}(a + x)$$
$$(\delta_{31} + \delta_{32} - \gamma_3)a = (t\gamma_3 - \delta_{32})x \tag{29}$$

Note that $\gamma_3(a + tx) = \delta_{31}(a) + \delta_{32}(a + x) < \delta_{31}(a + tx) + \delta_{32}(a + tx)$. Thus $\delta_{31} + \delta_{32} - \gamma_3 > 0$ and from equation 29 it follows that $t\gamma_3 - \delta_{32} > 0$. Notice also that $t\gamma_3 - \delta_{32} \equiv 0 \pmod{a}$. That is

$$t\gamma_3 = \alpha a + \delta_{32} \tag{30}$$

$\alpha > 0$. Write $a = nt + m$ where $m \in 1, 2, \ldots, t$. Then γ_3 is bounded by

$$\gamma_3 \geq \left\lceil \frac{a}{t} \right\rceil = n + 1.$$

We see that γ_3 achieves this lower bound in the following equation

$$\gamma_3(a + tx) = (n + x + 1 - (t - m))(a) + (t - m)(a + x)$$
$$\gamma_3 = n + 1$$

From the definition of c_3, $c_3 = n + 1$ and by equation 30

$$tc_3 = t(n + 1) = a + r_{32} = nt + m + r_{32}$$

which shows $r_{32} = t - m$. Substituting these values into equation 29 we find

$$(r_{31} + r_{32} - c_3)a = ax$$
$$K_3 = r_{31} + r_{32} - c_3 = x$$

□

Proposition 5.11. *Let S be a numerical semigroup minimally generated by $S = \langle a, a+x, a+tx \rangle$. Let $a \not\equiv 0 \pmod{t}$ and write $a = nt + m$ where $m \in \{1, 2, \ldots, t - 1\}$. Then $K_3 = \lambda x$ where*

$$\lambda = \min\{z \in \mathbb{Z}^+ : z(n + m + x) - \left\lceil \frac{zm}{t} \right\rceil (t - 1) \geq 0\}.$$

Proof. For any multiple of $a + tx$ in terms of a and $a + x$ we have

$$\gamma_3(a + tx) = \delta_{31}(a) + \delta_{32}(a + x)$$
$$(\delta_{31} + \delta_{32} - \gamma_3)a = (t\gamma_3 - \delta_{32})x \tag{31}$$

Note that $\gamma_3(a + tx) = \delta_{31}(a) + \delta_{32}(a + x) < \delta_{31}(a + tx) + \delta_{32}(a + tx)$. Thus $\delta_{31} + \delta_{32} - \gamma_3 > 0$ and from equation 31 it follows that $t\gamma_3 - \delta_{32} > 0$. Notice also that $t\gamma_3 - \delta_{32} \equiv 0 \pmod{i}$. That is

$$t\gamma_3 = \alpha a + \delta_{32} \tag{32}$$

$\alpha > 0$. For a fixed factorization determined by $\gamma_3, \delta_{31}, \delta_{32}$ (note that this fixes α as well), γ_3 is bounded by

$$\begin{aligned}\gamma_3 &\geq \left\lceil \frac{\alpha a}{t} \right\rceil \\ &\geq \left\lceil \frac{\alpha(nt + m)}{t} \right\rceil \\ &\geq \alpha n + \left\lceil \frac{\alpha m}{t} \right\rceil\end{aligned}$$

We see that γ_3 achieves this lower bound in the following equation

$$\begin{aligned}
\gamma_3(a+tx) &= \left(\alpha(n+m+x) - \left\lceil \frac{\alpha m}{t} \right\rceil (t-1)\right)(a) + \left(\left\lceil \frac{\alpha m}{t} \right\rceil t - \alpha m\right)(a+x) \\
&= \alpha na + \alpha xa + \left\lceil \frac{\alpha m}{t} \right\rceil a + \left\lceil \frac{\alpha m}{t} \right\rceil tx - \alpha mx \\
&= \alpha(na + x(nt+m) - mx) + \left\lceil \frac{\alpha m}{t} \right\rceil (a+tx) \\
&= \left(\alpha n + \left\lceil \frac{\alpha m}{t} \right\rceil\right)(a+tx) \\
\gamma_3 &= \alpha n + \left\lceil \frac{\alpha m}{t} \right\rceil \qquad (33)
\end{aligned}$$

Now we wish to find a lower bound for all γ_3. That is, we wish to find a lower bound for α. For equation 33 to be a valid factorization α must satisfy the following inequalities

$$\begin{aligned}
\alpha(n+m+x) - \left\lceil \frac{\alpha m}{t} \right\rceil (t-1) &\geq 0 \\
\left\lceil \frac{\alpha m}{t} \right\rceil t - \alpha m &\geq 0
\end{aligned}$$

The second inequality holds for all α since $\left\lceil \frac{\alpha m}{t} \right\rceil \geq \frac{\alpha m}{t}$. Note from the first inequality that λ as defined in the statement of the proposition is precisely the lowerbound for α that we are looking for. That is, for all γ_3

$$\gamma_3 \geq \lambda n + \left\lceil \frac{\lambda m}{t} \right\rceil$$

From equation 33 γ_3 achieves this lower bound, so from the definition of c_3, $c_3 = \lambda n + \left\lceil \frac{\lambda m}{t} \right\rceil$. Substituting λ into equation 32 gives

$$tc_3 - r_{32} = \lambda a$$

Substituting this into equation 31 we find

$$\begin{aligned}
(r_{31} + r_{32} - c_3)a &= \lambda a x \\
K_3 = r_{31} + r_{32} - c_3 &= \lambda x
\end{aligned}$$

□

Corollary 5.12. *Let S be a numerical semigroup minimally generated by $S = \langle a, a+x, a+tx \rangle$. Let $a \not\equiv 0 \pmod{t}$ and write $a = nt + m$ where $m \in \{1, 2, \ldots, t-1\}$. $K_3 = x$ if and only if $n + x + m \geq t - 1$.*

Proof. Let $K_3 = x$ then by Proposition 5.11 $1 = \min\{z \in \mathbb{Z}^+ : z(n+m+x) - \left\lceil \frac{zm}{t} \right\rceil (t-1) \geq 0\}$, which shows $n + x + m \geq t - 1$. Going the other way, let $n + x + m \geq t - 1$. Then again $1 = \min\{z \in \mathbb{Z}^+ : z(n+m+x) - \left\lceil \frac{zm}{t} \right\rceil (t-1) \geq 0\}$ and by Proposition 5.11, $K_3 = x$ □

Note that Corollary 5.12 implies Propositon 5.10.

Theorem 5.13. *Let S be a numerical semigroup minimally generated by $S = \langle a, a+x, a+tx \rangle$. Let $a \not\equiv 0 \pmod{t}$ and write $a = nt + m$ where $m \in \{1, 2, \ldots, t-1\}$. $\Delta(S) = \{x\}$ if and only if $n + x + m \geq t - 1$.*

Proof. Let $n+x+m \geq t-1$. Then by Proposition 5.9, $K_1 = x$ and by Corollary 5.12, $K_3 = x$. By Lemma 5.5 since $K_1 = K_3 = x$, we have that $\Delta(S) = \{x\}$. Now let $\Delta(S) = \{x\}$. By Lemma 5.5 we have $K_3 = x$ and by Corollary 5.12 $n + x + m \geq t - 1$. □

Corollary 5.14. *Let S be a numerical semigroup minimally generated by $S = \langle a, a+x, a+tx \rangle$. If $a \geq t(t-4) + 2$ then $\Delta(S) = \{x\}$*

Proof. If $a \equiv 0 \pmod{t}$ then by Theorem 5.8 $\Delta(S) = \{x\}$. Otherwise, write $a = nt+m, m \in \{1, 2, \ldots, t-1\}$ and let $a \geq t(t-4) + 2$. Since $m, x \geq 1$, when $n \geq t - 3$ we have $n + m + x \geq t - 3 + m + x \geq t - 1$. Additionally when $n = t - 4$ and $m \geq 2$ we also have $n + m + x \geq t - 4 + 2 + x \geq t - 1$. That is, when $a \geq t(t-4) + 2$, we have $n + m + x \geq t - 1$ and by Theorem 5.13 above, $\Delta(S) = \{x\}$. □

5.4 Generalized Bound on the Smallest Generator

In this section we show that if S is a numerical semigroup minimally generated by $S = \langle a, a+qx, a+tx\rangle$ with $\gcd(q,t) = 1$ and $a \geq t(t-1)$ then $\Delta(S) = \{x\}$.

Proposition 5.15. *Let S be a numerical semigroup minimally generated by $S = \langle a, a+x, a+tx\rangle$. Let $a \not\equiv 0 \pmod t$ and write $a = nt+m$ where $m \in \{1, 2, \ldots, t-1\}$. Set $\zeta = \min\{z \in \mathbb{N} \cup \{0\} : q \mid zt+m\}$. If $\zeta \leq n$ and $q(n+x+\zeta+1) \geq (\zeta+1)t - m$ then $K_1 = x$.*

Proof. Let S, ζ be as described above and let $\zeta \leq n$ and $q(n+x+\zeta+1) \geq (\zeta+1)t-m$.

For any multiple of a in terms of $a+qx$ and $a+tx$ we have

$$\begin{aligned}\gamma_1 a &= \delta_{12}(a+qx) + \delta_{13}(a+tx)\\ &= (\delta_{12}+\delta_{13})a + (q\delta_{12}+t\delta_{13})x \end{aligned} \tag{34}$$

The left hand side is equivalent to $0 \pmod a$ thus, since S is primitive and $\gcd(a,x) = 1$, we have $q\delta_{12}+t\delta_{13} \equiv 0 \pmod a$. Write $q\delta_{12}+t\delta_{13} = \alpha a = \alpha(nk+m)$ where $\alpha > 0$ since either δ_{12} or δ_{13} must be positive. Notice that $q\delta_{12} \equiv \alpha m \pmod t$ so

$$\begin{aligned}\delta_{12} &= \frac{\alpha m + \beta t}{q}\\ \delta_{13} &= \alpha n - \beta\end{aligned} \tag{35}$$

where $-\lfloor \frac{\alpha m}{t} \rfloor \leq \beta \leq \alpha n$ and $q \mid \alpha m + \beta t$. Substituting into equation 34 and factoring out an a we find

$$\begin{aligned}\gamma_1 i &= \left(\frac{\alpha m+\beta t}{q} + \alpha n - \beta\right) a + (\alpha a)x\\ \gamma_1 &= \alpha\left(n+x+\frac{m}{q}\right) + \beta\left(\frac{t-q}{q}\right)\end{aligned} \tag{36}$$

We will show that γ_1 is bounded below by the case when $\alpha = 1$. Note that when $\alpha = 1$, we have $\beta \geq \zeta \geq 0$ by definition of ζ. Thus we wish to show that for all γ_1, $\gamma_1 \geq n+x+\frac{m}{q}+\zeta\frac{t-q}{q}$. Since for all values of α we have $\beta \geq -\lfloor \frac{\alpha m}{t} \rfloor$ it suffices to show that for all $\alpha > 1$, the following inequality holds

$$n+x+\frac{m}{q}+\zeta\frac{t-q}{q} \leq \alpha\left(n+x+\frac{m}{q}\right) - \left\lfloor \frac{\alpha m}{t}\right\rfloor \left(\frac{t-q}{q}\right) \tag{37}$$

Recall that $m < t$ so $0 \leq \left\lfloor \frac{\alpha m}{t} \right\rfloor < \alpha$ and further, from our assumptions, $0 < (\zeta+1)(t-q) \leq q(n+x)+m$. Thus we see, by cases, that for all $\alpha > 1$ inequality 37 holds.

(i) *Case 1:* $\lfloor \frac{\alpha m}{t} \rfloor = 0$.

$$\begin{aligned}\left(\zeta + \left\lfloor \frac{\alpha m}{t}\right\rfloor\right)(t-q) &= \zeta(t-q)\\ &\leq (\zeta+1)(t-q)\\ &\leq qn+qx+m\\ &\leq (\alpha-1)(qn+qx+m)\\ n+x+\frac{m}{q}+\zeta\frac{t-q}{q} &\leq \alpha\left(n+x+\frac{m}{q}\right) - \left\lfloor\frac{\alpha m}{t}\right\rfloor \frac{t-q}{q}\end{aligned}$$

(ii) *Case 2:* $\lfloor \frac{\alpha m}{t} \rfloor > 0$.

$$\left(\zeta + \left\lfloor \frac{\alpha m}{t}\right\rfloor\right)(t-q) \leq \left\lfloor \frac{\alpha m}{t}\right\rfloor (\zeta+1)(t-q) \leq (\alpha-1)(qn+qx+m)$$

$$n+x+\frac{m}{q}+\zeta\frac{t-q}{q} \leq \alpha\left(n+x+\frac{m}{q}\right) - \left\lfloor\frac{\alpha m}{t}\right\rfloor \frac{t-q}{q}$$

Hence γ_1 is bounded from below by

$$\gamma_1 \geq n + x + \frac{m}{q} + \zeta\frac{t-q}{q} \tag{38}$$

Now notice that γ_1 achieves this lower bound when $\delta_{12} = \frac{\zeta t+m}{q}$ and $\delta_{13} = n - \zeta$.

$$\gamma_1(a) = \frac{\zeta t + m}{q}(a + qx) + (n-\zeta)(q+tx)$$
$$\gamma_1(a) = \left(n + x + \frac{m}{q} + \zeta\frac{t-q}{q}\right)(a)$$

Thus by the definition of c_1, $c_1 = n + x + \frac{m}{q} + \zeta\frac{t-q}{q}$. Next we wish to show $r_{12} + r_{13} = n + \frac{m}{q} + \zeta\frac{t-q}{q}$. Recall that we choose $r_{12} + r_{13}$ to be maximal, thus by the factorization above we know $r_{12} + r_{13} \geq n + \frac{m}{q} + \zeta\frac{t-q}{q}$. That is, $K_1 \leq c_1 - \left(n + \frac{m}{q} + \zeta\frac{t-q}{q}\right) = x$ from Lemma 5.3 we know that $K_1 = c_1 - (r_{12} + r_{13}) \geq x$. Thus $K_1 = x$, as desired. ☐

Corollary 5.16. *Let S be a numerical semigroup minimally generated by $S = \langle a, a+qx, a+tx\rangle$ with* $\gcd(q,t) = 1$. *If $a \geq t(t-2)$ then $K_1 = x$.*

Proof. If $a \equiv 0 \pmod t$ then by Proposition 5.6 $K_1 = x$. Now we will show that if $a \not\equiv 0 \pmod t$ and $a \geq t(t-2)$ then $K_1 = x$.

Write $a = nt + m$ where $m \in \{1, 2, \ldots, t-1\}$ and let $\zeta = \min\{z \in \mathbb{N} \cup \{0\} : q \mid zt + m\}$. We wish to show that if $a \geq t(t-2)$ then $\zeta \leq n$ and $q(n + x + \zeta + 1) \geq (\zeta + 1)t - m$.

We start by showing that if $a \geq t(t-2)$ then $\zeta \leq n$. Note that since $\gcd(q,t) = 1$ there exists some $z \in \{0, 1, \ldots, q-1\}$ such that $zt \equiv -m \pmod q$. That is, there exists some $z \in \{0, 1, \ldots, q-1\}$ such that $q \mid zt + m$. Now by the definition of ζ, we see $\zeta \leq q - 1$. Now since $t > q$, and $a \geq t(t-2)$ (which implies $n \geq t-2$) we have $\zeta \leq q - 1 \leq t - 2 \leq n$.

Next we show that if $a \geq t(t-2)$ then $q(n+x) + m \geq (\zeta+1)(t-q)$. Let $a \geq t(t-2)$. Recall $n \geq t-2$, $x, m, q \geq 1$, and $q - 1 \geq \zeta$.

$$\begin{aligned} q(n+x) + m &\geq q((t-2)+1) + m \\ &\geq q(t-1) \\ &\geq q(t-q) \\ &\geq (\zeta+1)(t-q) \end{aligned}$$

Thus when $a \geq t(t-2)$ all of the conditions in Proposition 5.15 hold and hence $K_1 = x$. ☐

Proposition 5.17. *Let S be a numerical semigroup minimally generated by $S = \langle a, a+qx, a+tx\rangle$ with* $\gcd(q,t) = 1$. *Let $a \not\equiv 0 \pmod t$ and write $a = nt + m$ where $m \in \{1, 2, \ldots, t-1\}$. Set $\beta = \min\{z \in \mathbb{Z}^+ : q \mid zt - m\}$ if $\beta \leq n$ then $K_3 = x$ if and only if $q(x + n + \beta) \geq \beta t - m$.*

Proof. First let $q(x + n + \beta) \geq t - m$. For any multiple of $a + tx$ in terms of a and $a + qx$ we have

$$\begin{aligned} \gamma_3(a+tx) &= \delta_{31}(a) + \delta_{32}(a+qx) \\ (\delta_{31} + \delta_{32} - \gamma_3)a &= (t\gamma_3 - q\delta_{32})x \end{aligned} \tag{39}$$

Now since S is primitive implies $\gcd(a,x) = 1$ we have $t\gamma_3 - q\delta_{32} \equiv 0 \pmod a$.

$$\begin{aligned} t\gamma_3 &= \alpha a + q\delta_{32} \\ \gamma_3 &= \alpha n + \frac{\alpha m + q\delta_{32}}{t} \end{aligned}$$

We will show that $\gamma_3 \geq n + \beta$. Let $\alpha = 1$, then

$$\gamma_3 = n + \frac{m + q\delta_{32}}{t}.$$

Since γ_3 is an integer, we have $q\delta_{32} \equiv -m \pmod{t}$. Recall that we define $\beta = \min\{z \in \mathbb{Z}^+ : q \mid zt - m\}$ and by this definition, $\gamma_3 \geq n + \beta$. Now let α be arbitrary. For every value of $\alpha > 1$ we will see that $n + \beta < \alpha n + \frac{\alpha m + q\delta_{32}}{t}$. Recall $\beta \leq n$ and since $\frac{\alpha m + q\delta_{32}}{t} > \frac{\alpha m}{t} > 0$, then for all $\alpha > 1$, $n + \beta \leq 2n < \alpha n + \frac{\alpha m + q\delta_{32}}{t}$. Now we see that $\gamma_3 \geq n + \beta$.

Note that by our assumption $n + x + \beta - \frac{\beta t - m}{q} \geq 0$. Then we see that this lower bound is achieved in the following equation

$$\begin{aligned}\gamma_3(a + tx) &= \left(n + x + \beta - \frac{\beta t - m}{q}\right)(a) + \frac{\beta t - m}{q}(a + qx)\\ &= (n + \beta)(a + tx)\end{aligned}$$

That is, $c_3 = n + \beta$. Thus $c_3 = n + \frac{m + qr_{32}}{t}$ and from this we see $tc_3 - qr_{32} = nt + m = i$, and substituting this into equation 39 we see that $K_3 = r_{31} + r_{32} - c_3 = x$.

Now let $K_3 = x$. That is,

$$r_{31} + r_{32} - c_3 = x \tag{40}$$

For any multiple of $a + tx$ in terms of a and $a + qx$ we have

$$\begin{aligned}\gamma_3(a + tx) &= \delta_{31}(a) + \delta_{32}(a + qx)\\ (\delta_{31} + \delta_{32} - \gamma_3)a &= (t\gamma_3 - q\delta_{32})x\end{aligned}$$

Specifically, $xa = K_3 a = (r_{31} + r_{32} - c_3)a = (tc_3 - qr_{32})x$, so

$$tc_3 = a + qr_{32}. \tag{41}$$

From equation 40 and equation 41 we rearrange to find

$$\begin{aligned}r_{31} &= x + c_3 - r_{32}\\ &= x + c_3 - \frac{tc_3 - a}{q}\end{aligned} \tag{42}$$

Since $r_{31} \geq 0$ we see

$$\begin{aligned}x + c_3 - \frac{tc_3 - a}{q} &\geq 0\\ q(x + c_3) &\geq tc_3 - a\\ qx + a &\geq c_3(t - q)\end{aligned} \tag{43}$$

Now for any γ_3, δ_{32} from a valid factorization, which give $t\gamma_3 = a + q\delta_{32}$ we have that

$$\gamma_3 = \frac{nt + m + q\delta_{32}}{t} = n + \frac{m + q\delta_{32}}{t}.$$

Now let $\beta = \min\{z \in \mathbb{Z}^+ : q \mid zt - m\}$. Then since $\gamma_3, r_{32} \in \mathbb{Z}$, $\gamma_3 \geq n + \beta$. In particular, $c_3 \geq n + \beta$. From equation 43 we see

$$\begin{aligned}qx + a &\geq (n + \beta)(t - q)\\ qx + qn + q\beta + m &\geq \beta t\\ q(x + n + \beta) &\geq \beta t - m\end{aligned}$$

This completes the proof. $\square$

Corollary 5.18. *Let S be a numerical semigroup minimally generated by $S = \langle a, a + qx, a + tx\rangle$ with $\gcd(q, t) = 1$. If $a \geq t(t - 1)$ then $K_3 = x$.*

Proof. If $a \equiv 0 \pmod{t}$ then by Proposition 5.7 $K_1 = x$. Now we will show that if $a \not\equiv 0 \pmod{t}$ and $a \geq t(t-1)$ then $K_3 = x$.

Let $a \not\equiv 0 \pmod{t}$, write $a = nt + m$ where $m \in \{1, 2, \ldots, t-1\}$, and set $\beta = \min\{z \in \mathbb{Z}^+ : q \mid zt - m\}$. We wish to show that when $a \geq t(t-1)$ (which implies $n \geq t-1$) then $\beta \leq n$ and $q(x+n) + m \geq \beta(t-q)$.

We show first that $\beta \leq n$. Since $\gcd(q,t) = 1$, for some $z \in \{1, 2, \ldots, q\}$ we will have $zt \equiv m \pmod{q}$ and for this z, $q \mid zt - m$. Then by the definition of β and by our assumption, $\beta \leq q \leq t-1 \leq n$.

Now we show that $q(x+n) + m \geq \beta(t-q)$. Recall that $n \geq t-1$, $x, m, q \geq 1$ and $q \geq \beta$

$$\begin{aligned} q(x+n) + m &\geq q(1 + (t-1)) + m \\ &\geq q(t) \\ &\geq q(t-q) \\ &\geq \beta(t-q) \end{aligned}$$

Thus when $q \geq t(t-1)$ all of the conditions in Proposition 5.17 hold, hence $K_3 = x$. □

Theorem 5.19. *Let S be a numerical semigroup minimally generated by $S = \langle a, a+qx, a+tx \rangle$ with* $\gcd(q,t) = 1$. *If $a \geq t(t-1)$ then $\Delta(S) = \{x\}$.*

Proof. Let $a \geq t(t-1)$. Then by Corollary 5.16, $K_1 = x$ and by Corollary 5.18, $K_3 = x$. By Lemma 5.5 since $K_1 = K_3 = x$, we have that $\Delta(S) = \{x\}$. □

5.5 Technical Results

In this section we present some results on the specific values of $c_1, c-3, r_{12}, r_{13}, r_{31}, r_{32}$ when $K_1 = K_3 = x$.

Proposition 5.20. *Let S be a numerical semigroup minimally generated by $S = \langle a, a+qx, a+tx \rangle$ with* $\gcd(q,t) = 1$. *Let $ia = ntk + m, m \in \{1, 2, \ldots, kt-1\}$ and set*

$$\zeta = \min\{z \in \mathbb{N} \cup \{0\} : q \mid zt + m)\}, \textit{and } \zeta \leq n.$$

$K_1 = x$ if and only if

$$c_1 = x + n - \zeta + \frac{\zeta t + m}{q}, r_{12} = \frac{\zeta t + m}{q}, \textit{and } r_{13} = n - \zeta.$$

Proof. Let $c_1 = x + n - \zeta + \frac{\zeta t + m}{q}, r_{12} = \frac{\zeta t + m}{q}$, and $r_{13} = n - \zeta$. Then clearly

$$K_1 = c_1 - (r_{12} + r_{13}) = x.$$

Now let $K_1 = x$. For any multiple of a in terms of $a + qx$, $a + tx$, we have

$$\begin{aligned} \gamma_1(a) &= \delta_{12}(a + qx) + \delta_{13}(a + tx) \\ (\gamma_1 - (\delta_{12} + \delta_{13}))(a) &= (q\delta_{12} + t\delta_{13})(x) \end{aligned}$$

Thus for any factorization such that $\gamma_1 - (\delta_{12} + \delta_{13}) = x$ we have $q\delta_{12} + t\delta_{13} = a = nt + m$. Since $\gcd(q,t) = 1$, all δ_{12}, δ_{13} will be of the following form

$$\delta_{12} = \frac{\alpha t + m}{q} \tag{44}$$

$$\delta_{13} = n - \alpha \tag{45}$$

Where $n \geq \alpha \in \mathbb{N} \cup \{0\}$ and $q \mid \alpha t + m$. Then for all factorizations such that $\gamma_1 - (\delta_{12} + \delta_{13}) = x$ we have

$$\begin{aligned} \gamma_1 &= x + \frac{\alpha t + m}{q} + n - \alpha \\ \gamma_1 &= x + n + \frac{m}{q} + \alpha\left(\frac{t-q}{q}\right) \end{aligned}$$

Since $t > q$, γ_1 is bounded from below by the case when α is as small as possible. Note that $\alpha \geq \zeta$ by the definition of ζ. That is,

$$\gamma_1 \geq x + n - \zeta + \frac{\zeta t + m}{q}$$

We see that γ_1 achieves this lower bound in the following equation

$$\begin{aligned}\gamma_1(i) &= \frac{\zeta t + m}{q}(a + qx) + (n - \zeta)(a + tx)\\ &= \left(x + n - \zeta + \frac{\zeta t + m}{q}\right)(a)\end{aligned}$$

Then by definition of c_1, $c_1 = x + n - \zeta + \frac{\zeta t+m}{q}$. Recall that $K_1 = x$ implies $c_1 = x = r_{12} + r_{13}$. That is, $r_{12} + r_{13} = n - \zeta + \frac{\zeta t+m}{q}$. From equation 44

$$\begin{aligned}n - \zeta + \frac{\zeta t + m}{q} &= n - \alpha + \frac{\alpha t + m}{q}\\ \zeta\left(\frac{t - q}{q}\right) &= \alpha\left(\frac{t - q}{q}\right)\end{aligned}$$

So $\alpha = \zeta$ and from equation 44, $r_{12} = \frac{\zeta t+m}{q}$, and $r_{13} = n - \zeta$. $\square$

Proposition 5.21. *Let S be a numerical semigroup minimally generated by $S = \langle a, a + qx, a + tx\rangle$ with $\gcd(q, t) = 1$ and write $a = nt + m, m \in \{1, 2, \ldots, t - 1\}$. Let $\beta = \min\{z \in \mathbb{Z}^+ : q \mid zt - m\}$. Then $K_3 = x$ if and only if $c_3 = n + \beta, r_{31} = n + x + \beta - \frac{\beta t-m}{q}, r_{32} = \frac{\beta t-m}{q}$.*

Proof. Let $c_3 = n + \beta, r_{31} = n + x + \beta - \frac{\beta t-m}{q}, r_{32} = \frac{\beta t-m}{q}$. Clearly

$$K_3 = r_{31} + r_{32} - c_3 = \left(n + x + \beta - \frac{\beta t - m}{q}\right) + \left(\frac{\beta t - m}{q}\right) - (n + \beta) = x.$$

Now let $K_3 = x$. For any multiple of $a + tx$ in terms of a and $a + qx$ we have

$$\begin{aligned}\gamma_3(a + tx) &= \delta_{31}(a) + \delta_{32}(a + qx)\\ (\delta_{31} + \delta_{32} - \gamma_3)a &= (t\gamma_3 - q\delta_{32})x\end{aligned}$$

Specifically, $xa = K_3 a = (r_{31} + r_{32} - c_3)a = (tc_3 - qr_{32})x$, so

$$tc_3 = a + qr_{32}. \tag{46}$$

For any γ_3, δ_{32} from a valid factorization, which give $t\gamma_3 = a + q\delta_{32}$ we have that

$$\gamma_3 = \frac{nt + m + q\delta_{32}}{t} = n + \frac{m + q\delta_{32}}{t}.$$

Now let $\beta = \min\{z \in \mathbb{Z}^+ : q \mid zt - m\}$. Then since $\gamma_3, r_{32} \in \mathbb{Z}$,

$$\gamma_3 \geq n + \beta. \tag{47}$$

By Proposition 5.17 since $K_3 = x$, $n + x + \beta - \frac{\beta t-m}{q} \geq 0$. Then we see that the lower bound shown in equation 47 is achieved in the following equation

$$\begin{aligned}\gamma_3(a + tx) &= \left(n + x + \beta - \frac{\beta t - m}{q}\right)(a) + \frac{\beta t - m}{q}(a + qx)\\ &= (n + \beta)(a + tx)\end{aligned}$$

That is,

$$c_3 = n + \beta.$$

Now by equation 46

$$t(n+\beta) = a + qr_{32} = nt + m + qr_{32}$$
$$r_{32} = \frac{\beta t - m}{q}$$

Now since $K_3 = r_{31} + r_{32} - c_3 = x$, we have

$$r_{31} = x + n + \beta - \frac{\beta t - m}{q}$$

□

6 Compound Semigroups

Now, we will examine various invariants related to a certain kind of numerical semigroups, which we call *compound* numerical semigroups:

Definition 6.1. Let $a_1, \ldots, a_x, b_1, \ldots, b_x$ be integers such that $a_i < b_i$ for each $i \in [1, x]$. For $j \in [0, x]$, set $n_j = b_1 b_2 \cdots b_j a_{j+1} a_{j+2} \cdots a_x$. Then $S = \langle n_0, n_1, \ldots, n_x \rangle$ is *compound.*

Example 6.2. Let $a_1 = 5$, $a_2 = 3$, $a_3 = 4$, $b_1 = 7$, $b_2 = 11$, and $b_3 = 9$. Then, we set

$$n_0 = 5 \cdot 3 \cdot 4 = 60$$
$$n_1 = 7 \cdot 3 \cdot 4 = 84$$
$$n_2 = 7 \cdot 11 \cdot 4 = 308$$
$$n_3 = 7 \cdot 11 \cdot 9 = 693.$$

So, $S = \langle 60, 84, 308, 693 \rangle$ is compound. Notice that the generators are increasing, even though it is not the case that $a_1 \le a_2 \le a_3$ nor $b_1 \le b_2 \le b_3$. The condition that $a_i < b_i$ for each i is sufficient for having increasing generators.

Lemma 6.3. *If $S = \langle n_0, n_1, \ldots, n_x \rangle$ is a numerical semigroup, then there exists some $k \in \mathbb{Q}^+$ such that kS is compound.*

Proof. Order the generators of S so that $n_0 < n_1 < \cdots < n_x$. For each $i \in [1, x]$, set $a_i = \frac{n_{i-1}}{\gcd(n_{i-1}, n_i)}$ and $b_i = \frac{n_i}{\gcd(n_{i-1}, n_i)}$. Then, $\frac{a_i}{b_i} = \frac{n_{i-1}}{n_i}$, where $\gcd(a_i, b_i) = 1$ and $a_i < b_i$. Set $m_i = b_1 b_2 \cdots b_i a_{i+1} a_{i+2} \cdots a_x$. Then, $\frac{m_{i-1}}{m_i} = \frac{a_i}{b_i} = \frac{n_{i-1}}{n_i}$. We know that

$$m_0 = a_1 a_2 \cdots a_x = \frac{n_0 n_1 \cdots n_{x-1}}{\gcd(n_0, n_1) \gcd(n_1, n_2) \cdots \gcd(n_{x-1}, n_x)}.$$

Similarly, if $k = \frac{n_1 n_2 \cdots n_{x-1}}{\gcd(n_0, n_1) \gcd(n_1, n_2) \cdots \gcd(n_{x-1}, n_x)}$, then for each $i \in [0, x]$, $m_i = k n_i$, so $\langle m_0, m_1, \ldots, m_x \rangle = kS$, where $\langle m_0, m_1, \ldots, m_x \rangle$ is compound. □

In the proof of Lemma 6.3, notice that because $\frac{a_i}{b_i} = \frac{n_{i-1}}{n_i}$ and $\gcd(a_i, b_i) = 1$, k is the smallest $z \in \mathbb{Q}^+$ such that zS is compound. If we take l to be any positive integer multiple of k (so $l = \alpha k$ where $\alpha \in \mathbb{N}$), then lS is also compound. (To see why this is, multiply both a_1 and b_1 by α to obtain the form necessary for a compound semigroup.) If we had started by considering lS, we would have found that multiplying lS by $\frac{1}{\alpha}$ would have given us kS, the smallest multiple of lS that is compound. Therefore, an easy way to check whether or not a semigroup is compound is to determine k and compare it with 1. If $k \le 1$, then the original semigroup is compound. If $k > 1$, then it is not. This is equivalent to saying that a numerical semigroup $S = \langle n_0, n_1, \ldots, n_x \rangle$ is compound if and only if

$$n_1 n_2 \cdots n_{x-1} \le \gcd(n_0, n_1) \gcd(n_1, n_2) \cdots \gcd(n_{x-1}, n_x).$$

Example 6.4. Suppose we are given the numerical semigroup $S = \langle 40, 60, 81 \rangle$ and want to determine whether or not it is compound. We can check to see that

$$60 = \gcd(40, 60)\gcd(60, 81) = 20 \cdot 3.$$

Therefore, S is compound. Furthermore,

$$a_1 = \frac{40}{\gcd(40,60)} = \frac{40}{20} = 2$$

$$a_2 = \frac{60}{\gcd(60,81)} = \frac{60}{3} = 20$$

$$b_1 = \frac{60}{\gcd(40,60)} = \frac{60}{20} = 3$$

$$b_2 = \frac{81}{\gcd(60,81)} = \frac{81}{3} = 27.$$

Example 6.5. What about the numerical semigroup $S = \langle 60, 70, 77, 88 \rangle$? Applying the method in Lemma 6.3, we get that

$$k = \frac{70 \cdot 77}{\gcd(60,70)\gcd(70,77)\gcd(77,88)} = \frac{70 \cdot 77}{10 \cdot 7 \cdot 11} = 7 \neq 1.$$

Therefore, S is not compound, although $7S$ is, with $a_1 = 6$, $a_2 = 10$, $a_3 = 7$, $b_1 = 7$, $b_2 = 11$, and $b_3 = 8$.

Example 6.6. Finally, consider the numerical semigroup $S = \langle 36, 48, 60 \rangle$. We check to see that

$$48 \leq \gcd(36, 48)\gcd(48, 60) = 12 \cdot 12 = 144,$$

so $k = \frac{48}{144} = \frac{1}{3}$. Therefore, S is compound. In fact, $S/3 = \langle 12, 16, 20 \rangle$ is also compound with $a_1 = 3$, $a_2 = 4$, $b_1 = 4$, and $b_2 = 5$. There is no single set of a_is and b_is for S, but one such set can be obtained by multiplying a_1 and b_1 by $\frac{1}{k} = 3$. Then, we can say that S is compound with $a_1 = 9$, $a_2 = 4$, $b_1 = 12$, and $b_2 = 5$.

The invariants that we will explore (delta set, catenary degree, specialized elasticity, etc.) do not change when a numerical semigroup is multiplied by a constant because numerical semigroups that are constant multiples of one another are isomorphic. Every numerical semigroup is isomorphic to a compound semigroup, and these compound semigroups represent isomorphism classes that partition the set of all numerical semigroups. Therefore, if one were to determine these invariants for all compound numerical semigroups, one would know these invariants for all semigroups. However, the results in this paper do not categorize all semigroups because they rely on the condition that a semigroup is primitive. The following lemma gives a necessary and sufficient condition for a compound numerical semigroup to be primitive:

Lemma 6.7. *Let $a_1, \ldots, a_x, b_1, \ldots, b_x$ be integers such that $a_i < b_i$ for each $i \in [1, x]$. For $j \in [0, x]$, set $n_j = b_1 b_2 \cdots b_j a_{j+1} a_{j+2} \cdots a_x$. If $S = \langle n_0, n_1, \ldots n_x \rangle$, then S is primitive if and only if $\gcd(a_i, b_j) = 1$ whenever $i \geq j$.*

Proof. ($\Rightarrow$) First, to show that the primitivity of S implies that $\gcd(a_i, b_j) = 1$ whenever $i \geq j$, we will prove the contrapositive. Suppose the $\gcd(a_i, b_j) = z > 1$ for some $i \geq j$. Then, every generator is a multiple of a_i or b_j. Thus, since $z \mid a_i$ and $z \mid b_j$, z divides every generator. Therefore, S is not primitive.

($\Leftarrow$) Now, suppose that $\gcd(a_i, b_j) = 1$ whenever $i \geq j$. For the sake of contradiction, assume that $\gcd(n_0, \cdots, n_x) = z > 1$, and let p be a prime divisor of z. Since $p \mid n_0$, there is some maximal i where $p \mid a_i$. Since $p \mid a_i$ but $\gcd(p,\ a_{i+1} \cdots a_x) = 1$, we must have $p \mid b_1 \cdots b_i$. Hence there is some $j \in [1, i]$ such that $p \mid b_j$, but then $\gcd(a_i, b_j) \geq p$, a contradiction. Thus, S must be primitive. ∎

Lemma 6.8. *If $S = \langle n_0, n_1, \ldots, n_x \rangle$ is a numerical semigroup, then there is a unique $k \in \mathbb{N}$ such that kS is primitive.*

Proof. First, we show that such a k exists. If S is primitive, then $k = 1$, so assume S is not primitive. Then, $\gcd(n_0, n_1, \ldots, n_x) = z > 1$. Set $k = \frac{1}{z}$, so $\frac{1}{z} \cdot S = \langle \frac{n_0}{z}, \frac{n_1}{z}, \ldots, \frac{n_x}{z} \rangle$. Then, $\gcd(\frac{n_0}{z}, \frac{n_1}{z}, \ldots, \frac{n_x}{z}) = 1$, so $\frac{1}{z} \cdot S$ is primitive.

Now we show that k is unique. Assume that for some other $k' \in \mathbb{N}$, $k'S$ is also primitive. Without loss of generality, suppose $k \leq k'$. If $k \mid k'$, then $\frac{k'}{k} \in \mathbb{N}$. We can write $k'S = \frac{k'}{k}(kS)$, so $k'S$ is an integer multiple of a primitive semigroup. Since $k'S$ is also primitive, this implies that $k' = k$. Now, if $k \nmid k'$, then $\frac{k'}{k} \in \mathbb{Q} \setminus \mathbb{N}$. Again, we can write $k'S = \frac{k'}{k}(kS)$. All generators in kS and $k'S$ are natural numbers, which implies that k divides all the generators of kS, so the generators of S are integers, and kS is not primitive, a contradiction. Therefore, k is unique. ☐

Each numerical semigroup has exactly one constant multiple that is a primitive semigroup and has an infinite number of multiples that are compound semigroups. In this paper, we will look at the semigroups where these two conditions coincide, that is, where a compound numerical semigroup is also primitive.

One special case of compound numerical semigroups that we will examine in detail is when $a_1 = a_2 = \cdots = a$ and $b_1 = b_2 = \cdots = b$. This results in semigroups of the form $\langle a^x, a^{x-1}b, \ldots, ab^{x-1}, b^x \rangle$, which are generated by geometric progressions.

To see that $a^x, a^{x-1}b, \ldots, ab^{x-1}, b^x$ is a geometric progression, set $n = a^x$ and $r = \frac{b}{a}$. Then,

$$\begin{aligned} &a^x, a^{x-1}b, \ldots, ab^{x-1}, b^x \\ &= a^x, a^x\left(\frac{b}{a}\right), \ldots, a^x\left(\frac{b}{a}\right)^{x-1}, a^x\left(\frac{b}{a}\right)^x \\ &= n, nr, \ldots, nr^{x-1}, nr^x. \end{aligned}$$

Whenever we work with these semigroups, which we call *geometric*, we will assume $a < b$ to ensure the generators are in increasing order, and we will assume $\gcd(a, b) = 1$ to ensure that the greatest common divisor of all the generators is 1.

Example 6.9. Let $a = 3$, $b = 5$, and $x = 3$. Then, we have a geometric semigroup in embedding dimension 4: $\langle 3^3, 3^2 \cdot 5, 3 \cdot 5^2, 5^3 \rangle$. Notice that the embedding dimension of the semigroup is $x + 1$. This is always the case for geometric semigroups.

6.1 Delta Sets

Proposition 6.10. *Let $S = \langle n_0, n_1, \ldots, n_x \rangle$ be a primitive numerical semigroup such that for each $i \in [0, x-1]$, $\beta_i = \gcd(n_0, n_1, \ldots, n_i) > 1$ and $\alpha_i = \frac{\operatorname{lcm}(b_i, n_{i+1})}{n_{i+1}} > 1$. If $m \in \mathbb{Z}$ and z, z' are two linear combinations of $\{n_0, n_1, \ldots, n_x\}$ that equal m, then z and z' can be connected by x equations of the following form:*

$$\begin{aligned} \alpha_0 n_1 &= r_{00} n_0 \\ \alpha_1 n_2 &= r_{10} n_0 + r_{11} n_1 \\ &\cdots \\ \alpha_i n_{i+1} &= r_{i0} n_0 + \cdots + r_{ii} n_i \\ &\cdots \\ \alpha_{x-1} n_x &= r_{(x-1)0} n_0 + \cdots + r_{(x-1)(x-1)} n_{x-1} \end{aligned}$$

where each $r_{ab} \in \mathbb{N}_0$. Additionally, if the coefficients of z, z' are all non-negative, then all intermediate factorizations of m have non-negative coefficients.

Proof. First, notice that because $\gcd(n_0, n_1, \ldots, n_x) = 1$, we can write any integer (even one not in S) as a linear combination of the generators of S. For the sake of this proof, we will allow linear combinations with negative coefficients to be factorizations of a number.

We will prove the result by induction on x. For the base case, let $x = 1$, so $S = \langle n_0, n_1 \rangle$ and clearly $\gcd(n_0) = n_0 > 1$. Also, $\gcd(n_0, n_1) = 1$; otherwise, S is not primitive. So $\alpha_0 = \frac{\operatorname{lcm}(n_0, n_1)}{n_1} = n_0$. Let $m \in \mathbb{Z}$, and let $y_0n_0 + y_1n_1 = z_0n_0 + z_1n_1$ be two factorizations of m. We want to show that we can connect these two factorizations of m by making swaps using the equation $n_0n_1 = n_1n_0$ (the first of the above equations when $\alpha_0 = n_0$ and $r_{00} = n_1$). Since n_0 divides the first terms of the factorizations and $\gcd(n_0, n_1) = 1$, we know that $y_1 \equiv z_1 \pmod{n_0}$. Without loss of generality, assume $y_1 \geq z_1$. Then $y_1 - kn_0 = z_1$ for some $k \geq 0$. If $k = 0$, then we are done because $y_1 = z_1$, which implies that $y_0 = z_0$, and our factorizations are the same. So, assume $k \geq 1$. Using the equation $n_0n_1 = n_1n_0$ to make swaps k times, we arrive at the intermediate factorization:

$$y_0n_0 + y_1n_1 = (y_0 + kn_1)n_0 + (y_1 - kn_0)n_1.$$

Then since $y_1 - kn_0 = z_1$, $y_0 + kn_1 = z_0$, so our factorizations are the same. Notice that if $y_0, y_1, z_0, z_1 \geq 0$, then all intermediate factorizations have non-negative coefficients. Thus, the result holds when $x = 1$.

Now, let $x \geq 1$, and suppose that for any numerical semigroup in embedding dimension $x + 1$ of the form above, any x equations of the form above connect factorizations of every integer in the semigroup. In addition assume that if two factorizations have non-negative coefficients, any intermediate factorizations will have non-negative coefficients. Let $S = \langle n_0, n_1, \ldots, n_{x+1}\rangle$ be a numerical semigroup in embedding dimension $x + 2$ such that for each $i \in [0, x]$, $\beta_i = \gcd(n_0, n_1, \ldots, n_i) > 1$ and $\alpha_i = \frac{\operatorname{lcm}(b_i, n_{i+1})}{n_{i+1}} > 1$. Let $m \in \mathbb{Z}$, and let $\sum_{j=0}^{x+1} y_jn_j = \sum_{j=0}^{x+1} z_jn_j$ be two factorizations of m. We will show that we can connect these two factorizations using swaps generated by the equations above. By our inductive hypothesis, β_x divides all the generators except for n_{x+1}. Since S is a primitive numerical semigroup, $\gcd(\beta_x, n_{x+1}) = 1$, so $\alpha_x = \beta_x$. Therefore, α_x divides all generators except for n_{x+1}. Thus, $y_{x+1} \equiv z_{x+1} \pmod{\alpha_x}$. Without loss of generality, assume that $y_{x+1} \geq z_{x+1}$, so $y_{x+1} - k\alpha_x = z_{x+1}$ for some $k \geq 0$.

If $k = 0$, then $y_{x+1} = z_{x+1}$, and we can write $m - y_{x+1}n_{x+1} = \sum_{j=0}^{x} y_jn_j = \sum_{j=0}^{x} z_jn_j$, which can be factored using the generators of $M = \langle n_0, n_1, \ldots, n_x\rangle$. By hypothesis, M is not primitive because $\gcd(M) = \beta_x$. However, $M/\beta_x = \langle \frac{n_0}{\beta_x}, \frac{n_1}{\beta_x}, \ldots, \frac{n_x}{\beta_x}\rangle$ is primitive and has all the properties necessary for our inductive hypothesis. Since M/β_x has embedding dimension $x + 1$, there are x equations that generate its monoid of swaps. They are of the form:

$$\frac{\alpha_i}{\beta_x}n_{i+1} = r_{i0}n_0 + \cdots + r_{ii}n_i$$

where $0 \leq i \leq x-1$ and $r_{ab} \in \mathbb{N}_0$. In particular, when multiplying these equations by β_x, we obtain a subset of the equations that generate the monoid of swaps for S. These equations have the form above, so we are done.

Suppose, however, that $k \geq 1$. Then we can obtain an intermediate factorization for m using the equation $\alpha_xn_{x+1} = r_{x0}n_0 + \cdots + r_{xx}n_x$, where $r_{ab} \in \mathbb{N}_0$:

$$y_0n_0 + \cdots + y_xn_x + y_{x+1}n_{x+1} = (y_0 + kr_{x0})n_0 + \cdots + (y_x + kr_{xx})n_x + (y_{x+1} - k\alpha_x)n_{x+1}.$$

If each $y_j, z_j \geq 0$, then all intermediate factorizations will also have non-negative coefficients. Now, since $y_x - k\alpha_x = z_{x+1}$, we can connect this new factorization to $\sum_{j=0}^{x+1} z_jn_j$ using the first x of our available equations. Therefore, any two linear combinations of $\{n_0, n_1, \ldots, n_{x+1}\}$ can be connected using equations of the form above, and as long as the coefficients on the linear combinations are non-negative, all intermediate factorizations will have non-negative coefficients as well. □

Corollary 6.11. *Let $a_1, \ldots, a_x, b_1, \ldots, b_x$ be integers such that $a_i < b_i$ for each $i \in [1, x]$. For $j \in [0, x]$, set $n_j = b_1b_2\cdots b_ja_{j+1}a_{j+2}\cdots a_x$. Suppose $S = \langle n_0, n_1, \ldots n_x\rangle$ is primitive. If $m \in S$ and z, z' are two linear combinations of $\{n_0, n_1, \ldots, n_x\}$ that equal m, then z and z' can be connected by x equations of the following form:*

$$a_1n_1 = b_1n_0$$

$$a_2n_2 = b_2n_1$$

$$\cdots$$

$$a_in_i = b_in_{i-1}$$

$$\cdots$$

$$a_x n_x = b_x n_{x-1}.$$

Additionally, if the coefficients z, z' are all non-negative, then all intermediate factorizations of m have non-negative coefficients.

Proof. Since S is primitive, by Lemma 6.7 we know that $\gcd(a_i, b_j) = 1$ whenever $i \geq j$.

Now, for each $i \in [0, x-1]$,

$$\begin{aligned}
&\gcd(n_0, n_1, \dots, n_i) \\
=&\gcd(a_1 a_2 \cdots a_x,\ b_1 a_2 \cdots a_x,\ \dots,\ b_1 b_2 \cdots b_i a_{i+1} \cdots a_x) \\
=&(a_{i+1} a_{i+2} \cdots a_x) \cdot \gcd(a_1 a_2 \cdots a_i,\ b_1 a_2 \cdots a_i,\ \dots,\ b_1 b_2 \cdots b_i).
\end{aligned}$$

Notice that $\gcd(a_1 a_2 \cdots a_i, b_1 a_2 \cdots a_i) = a_2 \cdots a_i$. Also, $\gcd(a_2 \cdots a_i, b_1 b_2 a_3 \cdots a_i) = a_3 \cdots a_i$. In the same way, each new term reduces the greatest common divisor of $(a_1 a_2 \cdots a_i,\ b_1 a_2 \cdots a_i,\ \dots,\ b_1 b_2 \cdots b_i)$ until we have $\gcd(a_i, b_1 b_2 \cdots b_i) = 1$. Therefore:

$$\begin{aligned}
&\gcd(n_0, n_1, \dots, n_i) \\
=&(a_{i+1} a_{i+2} \cdots a_x) \cdot \gcd(a_1 a_2 \cdots a_i,\ b_1 a_2 \cdots a_i,\ \dots,\ b_1 b_2 \cdots b_i) \\
=&(a_{i+1} a_{i+2} \cdots a_x) \cdot 1 \\
>&1.
\end{aligned}$$

Also,

$$\begin{aligned}
&\operatorname{lcm}(a_{i+1} a_{i+2} \cdots a_x,\ n_{i+1}) \\
=&\operatorname{lcm}(a_{i+1} a_{i+2} \cdots a_x,\ b_1 b_2 \cdots b_{i+1} a_{i+2} \cdots a_x) \\
=&a_{i+1} b_1 b_2 \cdots b_{i+1} a_{i+2} \cdots a_x \\
=&a_{i+1} n_{i+1}
\end{aligned}$$

so $\frac{\operatorname{lcm}(a_{i+1} a_{i+2} \cdots a_x, n_{i+1})}{n_{i+1}} = a_{i+1}$. Therefore, by Proposition 6.10, the above equations are of the proper form to connect any two factorizations z, z' of an element $m \in S$. Furthermore, if z and z' have only non-negative coefficients, then each intermediate factorization obtained using these equations will have non-negative coefficients. ☐

Corollary 6.12. *Let $S = \langle a^x, a^{x-1}b, \dots, ab^{x-1}, b^x \rangle$, where $a < b$ and $\gcd(a, b) = 1$. If $m \in S$ and z, z' are two linear combinations of $\{a^x, a^{x-1}b, \dots, b^x\}$ that equal m, then z and z' can be connected by x equations of the following form:*

$$ba^{x-(j-1)}b^{j-1} = aa^{x-j}b^j$$

for $j \in [1, x]$. Additionally, if the coefficients z, z' are all non-negative, then all intermediate factorizations of m have non-negative coefficients.

Proof. The proof follows immediately from Corollary 6.11 since S and the connecting equations have the proper form. ☐

Recall the following proposition from [2, Proposition 2.10]:

Proposition 6.13. *Let $S = \langle n_0, n_1, \dots, n_x \rangle$ where $\{n_0, n_1, \dots, n_x\}$ is the minimal set of generators. Then*

$$\min \Delta(S) = \gcd\{n_i - n_{i-1} : 1 \leq i \leq x\}.$$

The following lemma is from [7, Lemma 3]:

Lemma 6.14. *If S is a primitive numerical semigroup, then*

$$\min \Delta(S) = \gcd \Delta(S).$$

Theorem 6.15. *Let $a_1, \ldots, a_x, b_1, \ldots, b_x$ be integers such that $a_i < b_i$ for each $i \in [1, x]$. For $j \in [0, x]$, set $n_j = b_1 b_2 \cdots b_j a_{j+1} a_{j+2} \cdots a_x$. If $S = \langle n_0, n_1, \ldots n_x \rangle$ is primitive and $M = \max\{b_1 - a_1, \ldots, b_x - a_x\}$ and $m = \gcd(b_1 - a_1, \ldots, b_x - a_x)$, then $\Delta(S) \subseteq [m, M] \cap m\mathbb{N}$.*

Proof. By Proposition 6.13, $\min \Delta(S)$ is the greatest common factor of the differences in consecutive generators. The differences between consecutive generators are:

$$c_1 = (b_1 - a_1) a_2 a_3 \cdots a_x$$

$$c_2 = (b_2 - a_2) b_1 a_3 \cdots a_x$$

$$\cdots$$

$$c_x = (b_x - a_x) b_1 b_2 \cdots b_{x-1}.$$

Clearly $\gcd(c_1, c_2, \ldots, c_x) = k \cdot \gcd(b_1 - a_1, \ldots, b_x - a_x) = km \geq m$ for some $k \in \mathbb{N}$. So, $\min \Delta(S) = km \geq m$.

Let $z \in S$. Then, it is possible to connect any two factorizations of z using swaps of the form $a_i n_i = b_i n_{i-1}$ by Corollary 6.11. The maximum distance between the length of factorizations each time one of these swaps is performed is $\max\{b_1 - a_1, \ldots, b_x - a_x\} = M$. Therefore, $\max \Delta(z) \leq M$. Because z is arbitrary, $\max \Delta(S) \leq M$.

By Lemma 6.14, $\min \Delta(S) = km = \gcd \Delta(S)$. Therefore, all elements in $\Delta(S)$ must be multiples of km and hence multiples of m. So, $\Delta(S) \subseteq [m, M] \cap m\mathbb{N}$. ☐

Corollary 6.16. *Let $a_1, \ldots, a_x, b_1, \ldots, b_x$ be integers such that $0 < k = b_i - a_i$ for each $i \in [1, x]$. For $j \in [0, x]$, set $n_j = b_1 b_2 \cdots b_j a_{j+1} a_{j+2} \cdots a_x$. If $S = \langle n_0, n_1, \ldots n_x \rangle$ is primitive, then $\Delta(S) = \{k\}$.*

Proof. This is a special case of Theorem 6.15 where $\max\{b_1 - a_1, \ldots, b_x - a_x\} = k$ and $\gcd(b_1 - a_1, \ldots, b_x - a_x) = k$. Thus, $\Delta(S) \subseteq [k]$. Also, $\Delta(S) \neq \{\}$ because the element $a_x n_x = b_x n_{x-1}$ has a factorization of two different lengths. So, $\Delta(S) = \{k\}$. ☐

Corollary 6.17. *Let $S = \langle a^x, a^{x-1} b, \ldots, a b^{x-1}, b^x \rangle$, where $a < b$ and $\gcd(a, b) = 1$. Then, $\Delta(S) = \{b - a\}$.*

Proof. This is a special case of Corollary 6.16 when $a_1 = a_2 = \cdots = a_x = a$ and $b_1 = b_2 = \cdots = b_x = b$. For each $i \in [0, x]$, $b_i - a_i = b - a$, so $\Delta(S) = \{b - a\}$. ☐

Corollaries 6.16 and 6.17 guarantee that, given any $m \in \mathbb{N}$, there are in infinite number of numerical semigroups in every embedding dimension that have delta set $\{m\}$ because for any relatively prime numbers a and b such that $b - a = m$, then $\Delta(\langle a^x, a^{x-1} b, \ldots, a b^{x-1}, b^x \rangle) = \{m\}$.

Lemma 6.18. *Let $a_1, \ldots, a_x, b_1, \ldots, b_x$ be integers such that $a_i < b_i$ for each $i \in [1, x]$. For $j \in [0, x]$, set $n_j = b_1 b_2 \cdots b_j a_{j+1} a_{j+2} \cdots a_x$. Suppose $S = \langle n_0, n_1, \ldots n_x \rangle$ is primitive and fix some n_i. If $k = \min\{z \in \mathbb{N} : z n_i \in \langle n_0, \ldots, n_{i-1}, n_{i+1}, \cdots, n_x \rangle\}$ and $k n_i = c_0 n_0 \cdots c_{i-1} n_{i-1} + c_{i+1} n_{i+1} + \cdots + c_x n_x$, then either $c_0 = \cdots = c_{i-1} = 0$ or $c_{i+1} = \cdots = c_x = 0$.*

Proof. Set $A = a_{i+1} \cdots a_x$, $B = b_1 b_2 \cdots b_i$. Then, $n_i = AB$. Note that $\gcd(A, B) = 1$ because S is primitive. If $k n_i$ is the smallest multiple of n_i that can be written in terms of the other generators, we may write $k n_i$ as

$$\begin{aligned} k n_i = kAB &= c_0 n_0 + c_1 n_1 + \cdots + c_{i-1} n_{i-1} + c_{i+1} n_{i+1} + \cdots + c_x n_x \\ &= c_0 (a_1 \cdots a_i) A + c_1 (b_1 a_2 \cdots a_i) A + \cdots + c_{i-1} (b_1 \cdots b_{i-1} a_i) A \\ &+ c_{i+1} (b_{i+1} a_{i+2} \cdots a_x) B + \cdots + c_x (b_{i+1} \cdots b_x) B. \end{aligned}$$

Now, take both sides modulo B. The result is 0, hence

$$0 \equiv c_0 (a_1 \cdots a_i) A + c_1 (b_1 a_2 \cdots a_i) A + \cdots + c_{i-1} (b_1 \cdots b_{i-1} a_i) A \pmod{B}$$

but since $\gcd(A, B) = 1$ we may divide by A to get

$$0 \equiv c_0 (a_1 \cdots a_i) + c_1 (b_1 a_2 \cdots a_i) + \cdots + c_{i-1} (b_1 \cdots b_{i-1} a_i) \pmod{B}.$$

Hence there is some nonnegative integer l such that $lB = c_0(a_1 \cdots a_i) + c_1(b_1a_2 \cdots a_i) + \cdots + c_{i-1}(b_1 \cdots b_{i-1}a_i)$. If $l = 0$, then $c_0 = c_1 = \cdots = c_{i-1} = 0$. If instead $l > 0$, then we can multiply both sides by A to get

$$lAB = c_0(a_1 \cdots a_i)A + c_1(b_1a_2 \cdots a_i)A + \cdots + c_{i-1}(b_1 \cdots b_{i-1}a_i)A.$$

If $0 < l < k$, we have a contradiction because then kn_i is not the smallest multiple of n_i that can be written in terms of the other generators. So, $k \leq l$. Thus,

$$kAB \leq lAB = c_0(a_1 \cdots a_i)A + c_1(b_1a_2 \cdots a_i)A + \cdots + c_{i-1}(b_1 \cdots b_{i-1}a_i)A \leq kAB,$$

which implies that

$$kAB = c_0(a_1 \cdots a_i)A + c_1(b_1a_2 \cdots a_i)A + \cdots + c_{i-1}(b_1 \cdots b_{i-1}a_i)A$$

and $c_{i+1} = \cdots = c_x = 0$. □

Proposition 6.19. *Let $a_1, \ldots, a_x, b_1, \ldots, b_x$ be integers such that $a_i < b_i$ for each $i \in [1, x]$. For $j \in [0, x]$, set $n_j = b_1b_2 \cdots b_ja_{j+1}a_{j+2} \cdots a_x$. If $S = \langle n_0, n_1, \ldots n_x \rangle$ is primitive, then for any generator n_i of S, $\min\{a_i, b_{i+1}\} = \min\{z \in \mathbb{N} : zn_i \in \langle n_0, \ldots, n_{i-1}, n_{i+1}, \cdots, n_x \rangle\}$ (where b_1n_0 is the smallest multiple of n_0 because there is no a_0, and a_xn_x is the smallest multiple of n_x because there is no b_{x+1}).*

Proof. Fix some generator n_i, and set $k = \min\{z \in \mathbb{N} : zn_i \in \langle n_0, \ldots, n_{i-1}, n_{i+1}, \cdots, n_x \rangle\}$. By Lemma 6.18, every factorization of kn_i (not involving n_i) will either be in terms of only generators larger than n_i or only generators smaller than n_i.

Suppose first that we have a factorization of kn_i that is only in terms of generators larger than n_i. Then,

$$kn_i = c_{i+1}n_{i+1} + \cdots + c_xn_x$$

for $c_{i+1}, \ldots, c_x \in \mathbb{N}_0$. We can rewrite this as

$$k(b_1 \cdots b_ia_{i+1} \cdots a_x) = c_{i+1}(b_1 \cdots b_{i+1}a_{i+2} \cdots a_x) + \cdots + c_x(b_1 \cdots b_x).$$

Dividing through by $(b_1 \cdots b_i)$, we obtain

$$k(a_{i+1} \cdots a_x) = c_{i+1}(b_{i+1}a_{i+2} \cdots a_x) + \cdots + c_x(b_{i+1} \cdots b_x).$$

Notice that b_{i+1} divides every term on the right side of the equation. Also, because S is primitive, we have $\gcd(b_{i+1}, a_{i+1} \cdots a_x) = 1$. Thus, $b_{i+1} \mid k$, which implies that $k \geq b_{i+1}$. In fact, $k = b_{i+1}$ because $b_{i+1}n_i = a_{i+1}n_{i+1}$.

Suppose now that that we have a factorization of kn_i that is only in terms of generators smaller than n_i. Then,

$$kn_i = c_0n_0 + \cdots + c_{i-1}n_{i-1}$$

for $c_0, \ldots, c_{i-1} \in \mathbb{N}_0$. We can rewrite this as

$$k(b_1 \cdots b_ia_{i+1} \cdots a_x) = c_0(a_1 \cdots a_x) + \cdots + c_{i-1}(b_1 \cdots b_{i-1}a_i \cdots a_x).$$

Dividing through by $(a_{i+1} \cdots a_x)$, we obtain

$$k(b_1 \cdots b_i) = c_0(a_1 \cdots a_i) + \cdots + c_{i-1}(b_1 \cdots b_{i-1}a_i).$$

Notice that a_i divides every term on the right side of the equation. Also, because S is primitive, we have $\gcd(a_i, b_1 \cdots b_i) = 1$. Thus, $a_i \mid k$, which implies that $k \geq a_i$. In fact, $k = a_i$ because $a_in_i = b_in_{i-1}$.

It is clear that if $i = 0$, then any factorization of kn_0 (other than kn_0) will be in terms of larger generators. So, $k = b_1$. If $i = x$, then any factorization of kn_x (other than kn_x) will be written in terms of smaller generators. In this case, $k = a_x$. Otherwise, $1 \leq i \leq x-1$. It is possible to write some multiple of n_i in terms of exclusively larger generators (consider $b_{i+1}n_i = a_{i+1}n_{i+1}$), and it is possible to write some multiple of n_i in terms of exclusively smaller generators (consider $a_in_i = b_in_{i-1}$). Therefore, since we want to minimize k, $k = \min\{a_i, b_{i+1}\}$. □

Corollary 6.20. *Let $S = \langle a^x, a^{x-1}b, \ldots, ab^{x-1}, b^x \rangle$ where $a < b$ and $\gcd(a,b) = 1$. Then, for each generator $a^i b^{x-i}$ except for a^x, $a = \min\{z \in \mathbb{N} : z \cdot a^i b^{x-i} \in \langle \{a^x, a^{x-1}b, a^{x-2}b^2, \ldots, ab^{x-1}, b^x\} \setminus a^i b^{x-i} \rangle\}$. Furthermore, $b = \min\{z \in \mathbb{N} : z \cdot a^x \in \langle a^{x-1}b, a^{x-2}b^2, \ldots, ab^{x-1}, b^x \rangle\}$.*

Proof. This is a special case of Proposition 6.19 when $a_1 = \cdots a_x = a$ and $b_1 = \cdots = b_x = b$. Notice that because $a < b$, for every $i \in [1, x-1]$, the smallest multiple of $a^{x-i}b^i$ that can be written in terms of the other generators is $\min\{a, b\} = a$. □

6.2 Catenary Degree

Theorem 6.21. *Let $a_1, \ldots, a_x, b_1, \ldots, b_x$ be integers such that $a_i < b_i$ for each $i \in [1, x]$. For $j \in [0, x]$, set $n_j = b_1 b_2 \cdots b_j a_{j+1} a_{j+2} \cdots a_x$. If $S = \langle n_0, n_1, \ldots n_x \rangle$ is primitive, then $c(S) = \max\{b_1, b_2, \ldots, b_x\}$.*

Proof. Let $b = \max\{b_1, b_2, \ldots, b_x\}$. First, we show that $c(S) \leq b$. Let S be a numerical semigroup with the form above, and let $m \in S$. By Corollary 6.11, we can connect any two factorizations of m by making swaps of a_i of one generator for b_i of another. Because $a_i < b_i$, the distance between factorizations separated by one such swap is b_i. Therefore, since each $b_i \leq b$, it is possible to make a b-chain connecting the factorizations of m. So, $c(m) \leq b$ and $c(S) \leq b$.

Now to show that $c(S) = b$, we have only to demonstrate that there exists some $m \in S$ such that $c(m) = b$. Fix some i such that $b_i = b$. Consider the element $m = b_i n_{i-1}$. First we will show that any factorization of m involves only generators less than or equal to n_{i-1} or only generators greater than or equal to n_i.

Set $A = a_i a_{i+1} \cdots a_x$, $B = b_1 b_2 \cdots b_i$. Note that $\gcd(A, B) = 1$ because S is primitive. We may write m as

$$\begin{aligned} m = AB &= c_0 n_0 + c_1 n_1 + \cdots + c_{i-1} n_{i-1} + c_i n_i + c_{i+1} n_{i+1} + \cdots + c_x n_x \\ &= c_0 (a_1 \cdots a_{i-1}) A + c_1 (b_1 a_2 \cdots a_{i-1}) A + \cdots + c_{i-1} (b_1 \cdots b_{i-1}) A \\ &\quad + c_i (a_{i+1} \cdots a_x) B + c_{i+1} (b_{i+1} a_{i+2} \cdots a_x) B + \cdots + c_x (b_{i+1} \cdots b_x) B. \end{aligned}$$

Now, take m modulo B. The result is 0, so

$$0 \equiv c_0 (a_1 \cdots a_{i-1}) A + c_1 (b_1 a_2 \cdots a_{i-1}) A + \cdots + c_{i-1} (b_1 \cdots b_{i-1}) A \pmod{B}.$$

Since $\gcd(A, B) = 1$, we can divide by A to get

$$0 \equiv c_0 (a_1 \cdots a_{i-1}) + c_1 (b_1 a_2 \cdots a_{i-1}) + \cdots + c_{i-1} (b_1 \cdots b_{i-1}) \pmod{B}.$$

Thus, there is some non-negative integer k such that

$$kB = c_0 (a_1 \cdots a_{i-1}) + c_1 (b_1 a_2 \cdots a_{i-1}) + \cdots + c_{i-1} (b_1 \cdots b_{i-1})$$

. If $k = 0$, then $c_0 = c_1 = \cdots = c_{i-1} = 0$. If instead $k \geq 1$, then we multiply both sides by A to get

$$kAB = c_0 (a_1 \cdots a_{i-1}) A + c_1 (b_1 a_2 \cdots a_{i-1}) A + \cdots + c_{i-1} (b_1 \cdots b_{i-1}) A \leq AB.$$

So, $k = 1$ and $c_{i+1} = \cdots = c_x = 0$. Therefore, any factorization of m is either written entirely in terms of $\{n_0, n_1, \ldots, n_{i-1}\}$ or entirely in terms of $\{n_i, n_{i+1}, \ldots, n_x\}$.

Now, let z, z' be two factorizations of m such that z involves only generators less than or equal to n_{i-1} and z' involves only generators greater than or equal to n_i. (There always exist such factorizations. For instance, take z to be $b_i n_{i-1}$ and z' to be $a_i n_i$.) Because $\gcd(z, z') = 1$, $d(z, z') = \max\{|z|, |z'|\}$. We know $|z| > |z'|$ because z involves strictly smaller generators. Thus, $d(z, z') = |z|$. Now $|z| \geq b_i$ because $b_i n_{i-1}$ involves only the largest generator less than or equal to n_{i-1}. So, $d(z, z') \geq b_i$. If we disallow swaps of length b_i or greater, z will be disconnected from z'. Thus, the $c(m) \geq b_i = b$. However, we already know that $c(S) \leq b$. Hence, $c(m) = b$ and $c(S) = b$. □

Corollary 6.22. *If $S = \langle a^x, a^{x-1}b, a^{x-2}b^2, \ldots, ab^{x-1}, b^x \rangle$ for $\gcd(a, b) = 1$ and $a < b$, then $c(S) = b$.*

Proof. This is a special case of Theorem 6.21 when $a_1 = a_2 = \cdots = a_x = a$ and $b_1 = b_2 = \cdots = b_x = b$. So, $c(S) = \max\{b_1, b_2, \ldots, b_x\} = b$. □

6.3 Specialized Elasticity

Definition 6.23. We say an element m of a numerical semigroup S is *k-unique* if there is a unique way to factor m in S and this unique factorization has length k.

Proposition 6.24. *Let $a_1, \ldots, a_x, b_1, \ldots, b_x$ be integers such that $a_i < b_i$ for each $i \in [1, x]$. For $j \in [0, x]$, set $n_j = b_1 b_2 \cdots b_j a_{j+1} a_{j+2} \cdots a_x$. If $S = \langle n_0, n_1, \ldots n_x \rangle$ is primitive, then all sums of k atoms are k-unique if $k < \min\{a_1, a_2, \ldots, a_x\}$.*

Proof. We will prove the result by induction on embedding dimension of S.

First, suppose S has embedding dimension 2, so that $S = \langle a, b \rangle$. Fix some k such that $k < a$. Let m be an element of S that has a factorization of length k. Then we can write $m = c_0 a + c_1 b = d_0 a + d_1 b$, for $c_0, c_1, d_0, d_1 \in \mathbb{N}_0$ where $c_0 + c_1 = k$. Since S is primitive, $\gcd(a, b) = 1$ by Lemma 6.7. So, $c_1 \equiv d_1 \pmod{a}$. If $c_1 = d_1$, then our factorizations are the same, and we are done.

So, suppose $c_1 \neq d_1$. By Corollary 6.11, we can connect these two factorizations using swaps of the form $ab = ba$. However, no such swap can be made on the factorization $c_0 a + c_1 b$ because $c_0, c_1 \leq k < a < b$, so subtracting either a or b from either of the coefficients results in a factorization with a negative coefficient. Therefore, we cannot have $c_1 \neq d_1$, so the factorizations are the same.

Now, for the inductive step, assume the result holds for any compound semigroup of embedding dimension x. Let $a_1, \ldots, a_x, b_1, \ldots, b_x$ be integers such that $a_i < b_i$ for each $i \in [1, x]$. For $j \in [0, x]$, set $n_j = b_1 b_2 \cdots b_j a_{j+1} a_{j+2} \cdots a_x$. Let $S = \langle n_0, n_1, \ldots n_x \rangle$ be a primitive compound semigroup in embedding dimension $x + 1$. Let $a_i = \min\{a_1, a_2, \ldots, a_x\}$, and fix some k such that $k < a_i$.

Let m be an element in S that has a factorization of length k. Then, we can write $m = \sum_{j=0}^{x} c_j n_j = \sum_{j=0}^{x} d_j n_j$, where $c_j, d_j \in \mathbb{N}_0$ and $\sum_{j=0}^{x} c_j = k$. All generators except n_x are divisible by a_x. Further, because S is primitive, by Lemma 6.7, $\gcd(a_x, n_x) = 1$. Thus, $c_x \equiv d_x \pmod{a_x}$. If $c_x = d_x$, then $\frac{m - c_x n_x}{a_x} = \sum_{j=0}^{x-1} c_j \left(\frac{n_j}{a_x}\right)$ is a factorization of $\frac{n - c_x n_x}{a_x}$ in $\langle \frac{n_0}{a_x}, \frac{n_1}{a_x}, \ldots, \frac{n_{x-1}}{a_x} \rangle$ that has length $k - c_x \leq k < a_i$. Notice that $a_i = \min\{a_1, a_2, \ldots, a_x\} \leq \min\{a_1, a_2, \ldots, a_{x-1}\}$. So, by the inductive hypothesis, this factorization is unique. Therefore, the factorizations $\frac{m - c_x n_x}{a_x} = \sum_{i=0}^{x-1} c_i \left(\frac{n_i}{a_x}\right) = \sum_{j=0}^{x-1} d_j \left(\frac{n_j}{a_x}\right)$ are identical, so $m = \sum_{i=0}^{x} c_i n_i = \sum_{j=0}^{x} d_j n_j$ are the same factorizations of m in S.

Suppose now that $c_x \neq d_x$. If $c_x > d_x$, then we can subtract some positive multiple of n_x and add a linear combination of the smaller generators to get from $\sum_{j=0}^{x} c_j n_j$ to $\sum_{j=0}^{x} d_j n_j$. That is, $y n_x = \sum_{i=0}^{x-1} y_i n_i$ where $y_i \in \mathbb{Z}$ and $0 < y \leq c_x \leq k$. Since a_x divides each term on the right side of the equation and $\gcd(a_x, n_x) = 1$, $a_x \mid y$. This implies $a_i \leq a_x \leq y \leq k$. However, we assumed $k < a_i$. Therefore, $c_x \not> d_x$.

So, suppose $d_x > c_x$. Then, we can subtract a linear combination of smaller generators and add some positive multiple of n_x to get from $\sum_{j=0}^{x} c_j n_j$ to $\sum_{j=0}^{x} d_j n_j$. That is, $z n_x = \sum_{i=0}^{x-1} z_i n_i$ where $z_i \in \mathbb{Z}$ and $\sum_{i=0}^{x-1} z_i \leq k$. We cannot have $z > k$, because then $z n_x > \sum_{i=0}^{x-1} z_i n_i$, so $z \leq k$. Again, since a_x divides each term on the right side of the equation and $\gcd(a_x, n_x) = 1$, $a_x \mid z$. This implies $a_i \leq a_x \leq z \leq k$. However, we assumed $a_i > k$. Therefore, there are no other factorizations for m.

Thus, all sums of k atoms are k-unique. ☐

The bound in Proposition 6.24 that $k < \min\{a_1, \ldots, a_x\}$ is important because if $k \geq \min\{a_1, a_2, \ldots, a_x\}$, then there is almost always some sum of k atoms will not be k-unique. Consider the following example:

Example 6.25. Let $S = \langle 15, 55, 143 \rangle$, so S is compound with $a_1 = 3$, $a_2 = 5$, $b_1 = 11$, and $b_2 = 13$. Set $k = \min\{a_1, a_2\} = 3$. Then, $165 = 3(55) = 7(15)$ has a factorization of length 3 and of length 7, so S is not 3-unique.

Corollary 6.26. *Let $a_1, \ldots, a_x, b_1, \ldots, b_x$ be integers such that $a_i < b_i$ for each $i \in [1, x]$. For $j \in [0, x]$, set $n_j = b_1 b_2 \cdots b_j a_{j+1} a_{j+2} \cdots a_x$. If $S = \langle n_0, n_1, \ldots n_x \rangle$, then for any k where $k < \min\{a_1, a_2, \cdots, a_x\}$, $\rho_k(S) = k$.*

Corollary 6.27. *Let $S = \langle a^x, a^{x-1} b, a^{x-2} b^2, \ldots, a b^{x-1}, b^x \rangle$ be a numerical semigroup such that $\gcd(a, b) = 1$ and $a < b$. Then all sums of k atoms are k-unique if $k < a$.*

Proof. This is a special case of Proposition 6.24 when $a_1 = a_2 = \cdots = a_x = a$ and $b_1 = b_2 = \cdots = b_x = b$. Thus, all sums of k atoms are k-unique if $k < \min\{a\} = a$. ☐

Corollary 6.28. *Let $S = \langle a^x, a^{x-1}b, a^{x-2}b^2, \ldots, ab^{x-1}, b^x\rangle$ be a numerical semigroup such that* $\gcd(a, b) = 1$ *and $a < b$. For any k where $k < a$, $\rho_k(S) = k$.*

The motivation for the following theorem comes from the result in [8] that

$$\rho(S) = \lim_{k\to\infty} \frac{\rho_k(S)}{k}.$$

It is also shown in [8] that for any numerical semigroup S, $\rho(S)$ is the largest generator divided by the smallest generator. Therefore, if $S = \langle n_0, n_1, \ldots, n_x\rangle$, then

$$\rho(S) = \frac{n_x}{n_0} = \lim_{k\to\infty} \frac{\rho_k(S)}{k},$$

which implies that $\lim_{k\to\infty} \rho_k(S) = \frac{kn_x}{n_0}$. However, specialized elasticity is always an integer and $\frac{n_x}{n_0} \notin \mathbb{Z}$ for any semigroup, so we consider values of k that are multiples of n_0:

Lemma 6.29. *If $S = \langle n_0, n_1, \ldots n_x\rangle$ is a primitive semigroup, then for $k = cn_0$ where $c \in \mathbb{N}$, $\rho_k(S) = cn_x$.*

Proof. Let $c \in \mathbb{N}$, and set $k = cn_0$. Let $m \in S$ such that $m = \sum_{i=0}^{x} c_i n_i$ where $\sum_{i=0}^{x} c_i = k$. We know that $m \le kn_x$. Substituting in $k = cn_0$, we have $m \le (cn_0)n_x = (cn_x)n_0$. Since the longest factorization of m contains only the smallest generator, every factorization of m will have length less than or equal to cn_x. We attain a length of cn_x when $m = kn_x = (cn_x)n_0$, so $\rho_k(S) = cn_x$. □

6.4 Apery Sets

Definition 6.30. Let $a_1, \ldots, a_x, b_1, \ldots, b_x$ be integers such that $a_i < b_i$ for each $i \in [1, x]$. For $j \in [0, x]$, set $n_j = b_1 b_2 \cdots b_j a_{j+1} a_{j+2} \cdots a_x$. Suppose $S = \langle n_0, n_1, \ldots n_x\rangle$ is primitive. A factorization of an element $n \in S$ is *i-basic* if there are integers $c_0, \ldots, c_{i-1}, c_{i+1}, \ldots, c_x$ satisfying:

1. For each $j \in [0, i-1]$, we have $0 \le c_j < b_{j+1}$,
2. For each $j \in [i+1, x]$, we have $0 \le c_j < a_j$, and
3. $n = c_0 n_0 + \cdots + c_{i-1} n_{i-1} + c_{i+1} n_{i+1} + \cdots + c_x n_x$.

Furthermore, if n has an i-basic factorization, we call the element n *i-basic.*

Example 6.31. Let $S = \langle 45, 105, 280, 504\rangle$. Then S is compound with $a_1 = 3$, $a_2 = 3$, $a_3 = 5$, $b_1 = 7$, $b_2 = 8$, and $b_3 = 9$. Any 0-basic element will have a factorization of the form

$$c_1(105) + c_2(280) + c_3(504),$$

where $c_1 < 3$, $c_2 < 3$, and $c_3 < 5$. Similarly, a 1-basic element will have some factorization of the form

$$c_0(45) + c_2(280) + c_3(504),$$

where $c_0 < 7$, $c_2 < 3$, and $c_3 < 5$. Then, any 2-basic element will have a factorization of the form

$$c_0(45) + c_1(105) + c_3(504),$$

where $c_0 < 7$, $c_1 < 8$, and $c_3 < 5$. Finally, a 3-basic element in S will have a factorization of the form

$$c_0(45) + c_1(105) + c_2(280),$$

where $c_0 < 7$, $c_1 < 8$, and $c_2 < 9$.

As we will see, i-basic factorizations are special because the coefficients in these factorizations are too small to allow for certain types of swaps. For this reason, i-basic elements in a compound semigroup can be used to describe the Apéry set and Frobenius number of the semigroup. Before we prove these results, we will need the following lemma:

Lemma 6.32. *Let $a_1, \dots, a_x, b_1, \dots, b_x$ be integers such that $a_i < b_i$ for each $i \in [1, x]$. For $j \in [0, x]$, set $n_j = b_1 b_2 \cdots b_j a_{j+1} a_{j+2} \cdots a_x$. Suppose $S = \langle n_0, n_1, \dots n_x \rangle$ is primitive. If an element $n \in S$ is i-basic, then n has a unique i-basic factorization in S.*

Proof. Let n be i-basic, and suppose n has two i-basic factorizations:

$$\begin{aligned} m &= c_0 n_0 + \cdots + c_{i-1} n_{i-1} + c_{n+1} n_{i+1} + \cdots + c_x n_x \\ &= d_0 n_0 + \cdots + d_{i-1} n_{i-1} + d_{n+1} n_{i+1} + \cdots + d_x n_x. \end{aligned}$$

Then, for each $j \in [0, i-1]$, we have $c_j, d_j < b_{j+1}$, and for each $j \in [i+1, x]$, we have $c_j, d_j < a_j$. All the generators except for n_x are divisible by a_x. Furthermore, $\gcd(a_x, n_x) = 1$, so $c_x \equiv d_x \pmod{a_x}$. Since $0 \le c_x, d_x < a_x$, this implies that $c_x = d_x$. In a similar way, for each $j > i$, it can be argued that $c_j \equiv d_j \pmod{a_j}$ and $c_j = d_j$.

Also, all of the generators except for n_0 are divisible by b_1, and $\gcd(b_1, n_0) = 1$, so $c_0 \equiv d_0 \pmod{b_1}$. Since $0 \le c_0, d_0 \le b_1$, this implies that $c_0 = d_0$. In a similar way, for each $j < i$, it can be argued that $c_j \equiv d_j \pmod{b_{j+1}}$ and $c_j = d_j$. Thus, the two factorizations for m are the same. ☐

Lemma 6.33. *Let $a_1, \dots, a_x, b_1, \dots, b_x$ be integers such that $a_i < b_i$ for each $i \in [1, x]$. For $j \in [0, x]$, set $n_j = b_1 b_2 \cdots b_j a_{j+1} a_{j+2} \cdots a_x$. Suppose $S = \langle n_0, n_1, \dots n_x \rangle$ is primitive. For some $i \in [0, x]$, if n is i-basic in S, then $m = n - n_i \notin S$.*

Proof. Let some integer n be an i-basic element of S, and set $m = n - n_i$. Then, we can write $m = \sum_{j=0}^{x} c_j n_j$, where $c_i = -1$, $c_j < b_{j+1}$ for each $j \in [0, i-1]$ and $c_j < a_j$ for each $j \in [i+1, x]$. To show that $m \notin S$, it suffices to show that m cannot be written as a non-negative linear combination of the generators of S.

Suppose for the sake of contradiction that there is some such factorization of m in S: $m = \sum_{j=0}^{x} d_j n_j$ where each $d_j \ge 0$. All generators except for n_x are multiples of a_x, and $\gcd(a_x, n_x) = 1$ by the primitivity of S, so $c_x \equiv d_x \pmod{a_x}$. By our assumption, $c_x < a_x$, so $c_x \le d_x$. Then, $c_x + k_x a_x = d_x$ for some $k_x \in \mathbb{N}_0$. If $k_x = 0$, then $c_x = d_x$, so the coefficients on n_x in our two factorizations are already the same. So, assume $k_x > 0$. Then, we may perform the swap $a_x n_x = b_x n_{x-1}$ the appropriate number of times to obtain the following equivalent factorizations of m:

$$m = c_0 n_0 + \cdots + (c_{x-1} - k_x b_x) n_{x-1} + (c_x + k_x a_x) n_x = d_0 n_0 + \cdots + d_{x-1} n_{x-1} + d_x n_x.$$

Since $c_x + k_x a_x = d_x$, we can subtract $d_x n_x$ from both equations and divide through by a_x. So, we have two factorizations of the element $\frac{n - d_x n_x}{a_x}$:

$$c_0 \left(\frac{n_0}{a_x}\right) + \cdots + c_{x-2} \left(\frac{n_{x-2}}{a_x}\right) + (c_{x-1} - k_x b_x) \left(\frac{n_{x-1}}{a_x}\right) = d_0 \left(\frac{n_0}{a_x}\right) + \cdots + d_{x-2} \left(\frac{n_{x-2}}{a_x}\right) + d_{x-1} \left(\frac{n_{x-1}}{a_x}\right).$$

Once again, it can be shown that $(c_{x-1} - k_x b_x) \equiv d_{x-1} \pmod{a_{x-1}}$ and that we can form intermediate factorizations that have equal coefficients on $\frac{n_{x-1}}{a_x}$ using the swap $a_{x-1} n_{x-1} = b_{x-1} n_{x-2}$. We can continue this process until we arrive at the equivalent factorizations:

$$\begin{aligned} &c_0 \left(\frac{n_0}{a_{i+1} \cdots a_x}\right) + \cdots + c_{i-1} \left(\frac{n_{i-1}}{a_{i+1} \cdots a_x}\right) + (c_i - k_{i+1} b_{i+1}) \left(\frac{n_i}{a_{i+1} \cdots a_x}\right) \\ &= d_0 \left(\frac{n_0}{a_{i+1} \cdots a_x}\right) + \cdots + d_{i-1} \left(\frac{n_{i-1}}{a_{i+1} \cdots a_x}\right) + d_i \left(\frac{n_i}{a_{i+1} \cdots a_x}\right), \end{aligned}$$

for some $k_{i+1} \in \mathbb{N}_0$. Since all generators in $\{n_0, \dots, n_{i-1}\}$ are multiples of a_i and $\gcd(a_i, \frac{n_i}{a_{i+1} \cdots a_x}) = 1$, $(c_i - k_{i+1} b_{i+1}) \equiv d_i \pmod{a_i}$. We know that $c_i - k_{i+1} b_{i+1} \le c_i = -1 < 0 \le d_i$, so $c_i - k_{i+1} b_{i+1} + k_i a_i = d_i$ for some positive $k_i \in \mathbb{N}$. So, we use the swap $a_i n_i = b_i n_{i-1}$ the appropriate number of times to obtain the equivalent factorizations:

$$c_0 \left(\frac{n_0}{a_{i+1} \cdots a_x}\right) + \cdots + (c_{i-1} - k_i b_i) \left(\frac{n_{i-1}}{a_{i+1} \cdots a_x}\right) + d_i \left(\frac{n_i}{a_{i+1} \cdots a_x}\right)$$

$$= d_0\left(\frac{n_0}{a_{i+1}\cdots a_x}\right)+\cdots+d_{i-1}\left(\frac{n_{i-1}}{a_{i+1}\cdots a_x}\right)+d_i\left(\frac{n_i}{a_{i+1}\cdots a_x}\right).$$

Notice that by our assumption, $c_{i-1} < b_i \le k_i b_i$, so $c_{i-1} - k_i b_i < 0$. Thus, at each remaining step we will have a negative coefficient. If we continue making swaps in this way, we will eventually arrive at the equivalent factorizations:

$$(c_0 - k_1 b_1)\left(\frac{n_0}{a_1\cdots a_x}\right) = d_0\left(\frac{n_0}{a_1\cdots a_x}\right),$$

where $0 < k_1 \in \mathbb{N}$. Therefore, we have that $c_0 - k_1 b_1 = d_0$. However, $c_0 < b_1 \le k_1 b_1$, so $c_0 - k_1 b_1 < 0 \le d_0$, and the two terms cannot be equal. Therefore, there is no non-negative factorization of m in S, and $m \notin S$. □

Corollary 6.34. *Let $S = \langle a^x, a^{x-1}b, \ldots, ab^{x-1}, b^x\rangle$ be a numerical semigroup with $a < b$ and $\gcd(a,b) = 1$. For some $i \in [0,x]$, if n is i-basic in S, then $m = n - a^{x-i}b^i \notin S$.*

Proposition 6.35. *Let $a_1,\ldots,a_x,b_1,\ldots,b_x$ be integers such that $a_i < b_i$ for each $i \in [1,x]$. For $j \in [0,x]$, set $n_j = b_1b_2\cdots b_j a_{j+1}a_{j+2}\cdots a_x$. If $S = \langle n_0, n_1, \ldots n_x\rangle$ is primitive, then for any $i \in [0,x]$:*

$$Ap(S, n_i) = \{n \in S : n \text{ is } i\text{-basic}\}.$$

Proof. Let M denote the set on the right side of the equation, and fix some $i \in [0,x]$. Clearly, $|Ap(S,n_i)| = n_i$. Also, one can see by counting that there are $b_1\cdots b_{i-1}a_i\cdots a_x = n_i$ possible i-basic factorizations. By Lemma 6.32, this implies that there are n_i elements in S that are i-basic. Therefore, because $|M| = |Ap(S,n_i)| < \infty$, to show that the two sets are equal it suffices to show that $M \subseteq Ap(S,n_i)$.
Let $m \in M$. Then, $m \in S$. However, $m - n_i \notin S$ by Proposition 6.33. Therefore, $m \in Ap(S,n_i)$. So, $M \subseteq Ap(S,n_i)$, and the two sets are equal. □

Corollary 6.36. *Let $S = \langle a^x, a^{x-1}b, \ldots, ab^{x-1}, b^x\rangle$ be a numerical semigroup with $a < b$ and $\gcd(a,b) = 1$. For some $i \in [0,x]$,*

$$Ap(S, a^{x-i}b^i) = \{n \in S : n \text{ is } i\text{-basic}\}.$$

Proposition 6.37. *Let $a_1,\ldots,a_x,b_1,\ldots,b_x$ be integers such that $a_i < b_i$ for each $i \in [1,x]$. For $j \in [0,x]$, set $n_j = b_1b_2\cdots b_j a_{j+1}a_{j+2}\cdots a_x$. Suppose $S = \langle n_0, n_1, \ldots n_x\rangle$ is a primitive numerical semigroup. Then, for every $i \in [0,x]$, $F(S) = (b_1-1)n_0 + \cdots + (b_i-1)n_{i-1} - n_i + (a_{i+1}-1)n_{i+1} + \cdots + (a_x-1)n_x$.*

Proof. Fix some $i \in [0,x]$. The Frobenius number of S is equal to $\max Ap\{S,n_i)\} - n_i$. By Proposition 6.35, $Ap(S,n_i)$ is equal to the set of all i-basic elements in S. So, $\max\{Ap(S,n_i)\} = (b_1-1)n_0 + \cdots + (b_i-1)n_{i-1} + (a_{i+1}-1)n_{i+1} + \cdots + (a_x-1)n_x$, and $F(S) = (b_1-1)n_0 + \cdots + (b_i-1)n_{i-1} - n_i + (a_{i+1}-1)n_{i+1} + \cdots + (a_x-1)n_x$. □

Corollary 6.38. *Let $S = \langle a^x, a^{x-1}b, \ldots, ab^{x-1}, b^x\rangle$ be a numerical semigroup with $a < b$ and $\gcd(a,b) = 1$. Then, $F(S) = (b-1)a^x + (b-1)a^{x-1}b + \cdots + (b-1)ab^{x-1} - b^x$.*

Although not every element of a compound semigroup is a member of an Apéry set of some generator, we can assign each element a "normal" factorization with respect to each generator in the following way:

Lemma 6.39. *Let $a_1,\ldots,a_x,b_1,\ldots,b_x$ be integers such that $a_i < b_i$ for each $i \in [1,x]$. For $j \in [0,x]$, set $n_j = b_1b_2\cdots b_j a_{j+1}a_{j+2}\cdots a_x$. Suppose $S = \langle n_0, n_1, \ldots n_x\rangle$ is primitive. Let $i \in [0,x]$, and $n \in S$. Then there is exactly one $\tau \in \mathbb{N}_0$ such that $n - \tau n_i \in S$ and $n - \tau n_i$ is i-basic.*

Proof. First we will show that such a τ exists. There is some element, y, in $Ap(S,n_i)$ such that $y \equiv n \pmod{n_i}$. Since y is the smallest such element in S, $y \le n$. So, $n - \tau n_i = y \in S$ for some $\tau \in \mathbb{N}_0$. By Proposition 6.35, y has an i-basic factorization. Thus, $n - \tau n_i$ is i-basic.
Now suppose that $n - \sigma n_i \in S$ and $n - \sigma n_i$ is i-basic for some $\sigma \in \mathbb{N}_0$ where $\sigma \ne \tau$. Without loss of generality, assume $\sigma < \tau$. Then, $n - \sigma n_i > n - \tau n_i$. By Proposition 6.35, since $n - \sigma n_i$ and $n - \tau n_i$ have i-basic factorizations, $n - \sigma n_i, n - \tau n_i \in Ap(S,n_i)$. Clearly, $n - \sigma n_i \equiv n - \tau n_i \pmod{n_i}$. However, this implies that there are two elements in $Ap(S,n_i)$ that are equivalent $\pmod{n_i}$, which cannot be the case. Therefore, there is only one $\tau \in \mathbb{N}_0$ such that $n - \tau n_i \in S$ and $n - \tau n_i$ is i-basic. □

6.5 Omega Primality

The following proposition is from [9, Proposition 3.2]:

Proposition 6.40. *For a numerical semigroup S,*

$$\omega(n) = \max\{|a| : a \in \mathrm{bul}(n)\}$$

for all $n \in S$.

Proposition 6.41. *Let $S = \langle a^x, a^{x-1}b, \ldots, ab^{x-1}, b^x\rangle$ be a numerical semigroup with $a < b$ and $\gcd(a,b) = 1$. Then $\omega(a^x) = a$.*

Proof. First, we will show that all expressions of the form $aa^{x-i}b^i$, where $1 \le i \le x$, are bullets for a^x. Fix some $i \in [1, x]$. Then, let $m = -a^x + aa^{x-i}b^i$. We want to show that $m \in S$. Since $aa^{x-j}b^j = ba^{x-(j-1)}b^{j-1}$ for each $j \in [1, x]$, we can write

$$\begin{aligned} m &= a^{x-i}(-a^i + ab^i) \\ &= a^{x-i}((b-1)a^i + (b-a)a^{i-1}b + (b-a)a^{i-2}b^2 + \cdots + (b-a)ab^{i-1}) \end{aligned}$$

Therefore, $m \in S$. However, $n = -a^x + (a-1)a^{x-i}b^i \notin S$, by Corollary 6.34 because $(a-1)a^{x-i}b^i$ is x-basic. So, $aa^{x-i}b^i$ is a bullet for a^x.
Additionally, a^x is clearly a bullet for itself because $a^x - a^x = 0 \in S$ and $-a^x \notin S$.
Now, we will demonstrate that these are the only bullets for a^x. Consider that

$$a^x \not\preceq (a-1)a^{x-1}b + (a-1)a^{x-2}b^2 + \cdots + (a-1)ab^{x-1} + (a-1)b^x.$$

This is equivalent to saying that

$$-a^x + (a-1)a^{x-1}b + (a-1)a^{x-2}b^2 + \cdots + (a-1)ab^{x-1} + (a-1)b^x \notin S,$$

which is true by Corollary 6.34. Similarly,

$$a^x \not\preceq c_1a^{x-1}b + \cdots + c_xb^x$$

if each $c_i < a$. Thus, all the bullets for a^x in S (except for a^x itself) are of the form $aa^{x-i}b^i$, where $1 \le i \le x$. According to Proposition 6.40, $\omega(a^x) = \max\{|z| : z \in \mathrm{bul}(a^x)\}$. So, $\omega(a^x) = \max\{1, a\} = a$. ☐

Proposition 6.42. *Let $S = \langle a^x, a^{x-1}b, \ldots, ab^{x-1}, b^x\rangle$ be a numerical semigroup with $a < b$, $\gcd(a,b) = 1$, and $x \ge 1$. Then $\omega(a^{x-1}b) = b$.*

Proof. First, we will show that all expressions of the form $aa^{x-i}b^i$, where $2 \le i \le x$, are bullets for $a^{x-1}b$. Fix some $i \in [2, x]$. Then, let $m = -a^{x-1}b + aa^{x-i}b^i$. We want to show that $m \in S$. Since $aa^{x-j}b^j = ba^{x-(j-1)}b^{j-1}$ for each $j \in [1, x]$, we can write

$$\begin{aligned} m &= a^{x-i}b(-a^{i-1} + ab^{i-1}) \\ &= a^{x-i}b((b-1)a^{i-1} + (b-a)a^{i-2}b + (b-a)a^{i-3}b^2 + \cdots + (b-a)ab^{i-2}) \end{aligned}$$

Therefore, $m \in S$. However, $n = -a^{x-1}b + (a-1)a^{x-i}b^i \notin S$ by Corollary 6.34. So, $aa^{x-i}b^i$ is a bullet for $a^{x-1}b$.

Additionally, $a^{x-1}b$ is clearly a bullet for itself because $a^{x-1}b - a^{x-1}b = 0 \in S$ and $-a^{x-1}b \notin S$.

Also, ba^x is a bullet for $a^{x-1}b$ because $ba^x - a^{x-1}b = (a-1)a^{x-1}b \in S$ and $(b-1)a^x - a^{x-1}b \notin S$ by Corollary 6.34.

Now we will demonstrate that these are the only bullets for $a^{x-1}b$. Consider that

$$a^{x-1}b \not\preceq (b-1)a^x + (a-1)a^{x-2}b^2 + \cdots + (a-1)b^x.$$

This is equivalent to saying that

$$(b-1)a^x - a^{x-1}b + (a-1)a^{x-2}b^2 + \cdots + (a-1)b^x \notin S.$$

which is true by Corollary 6.34. Similarly,

$$a^{x-1}b \not\preceq c_0a^x + c_2a^{x-2}b^2 + \cdots + c_xb^x$$

if $c_0 < b$ and every other $c_i < a$. Thus, $\omega(a^{x-1}b) = \max\{1, a, b\} = b$. □

All data seem to suggest that the following conjectures are true:

Conjecture 6.43. *Let $S = \langle a^2, ab, b^2\rangle$ be a numerical semigroup with $a < b$ and $\gcd(a, b) = 1$. Then $\omega(b^2) = \left\lceil \frac{b}{a} \right\rceil b$.*

Conjecture 6.44. *Let $S = \langle a^x, a^{x-1}b, \ldots, ab^{x-1}, b^x\rangle$ be a numerical semigroup with $a < b$ and $\gcd(a, b) = 1$. Then, $b \mid \omega(a^{x-i}b^i)$ for $i \in [1, x]$.*

The following lemma is powerful because it makes the problem of finding omega primality for any generator of any geometric semigroup much easier. All that needs to be done to find the omega primality of a generator is to find the omega primality of the last generator of a related geometric semigroup. Perhaps this lemma could be helpful for proving Conjecture 6.44 if it could be shown that the last generator of a geometric semigroup always has omega primality divisible by b:

Lemma 6.45. *Let $S = \langle a^x, a^{x-1}b, \ldots, ab^{x-1}, b^x\rangle$ and $M = \langle a^{x+1}, a^xb, \ldots, ab^x, b^{x+1}\rangle$. If $\omega_S(a^{x-i}b^i) = n$, then $\omega_M(a^{x+1-i}b^i) = \max\{a, n\}$.*

Proof. Choose some generator of S, $v = a^{x-i}b^i$, and set $w = av = a^{x+1-i}b^i$, which is a generator of M. Suppose $z = z_1 + z_2 + \cdots + z_k$, where each z_i is a generator of M, is a bullet for w in M. So, $w \preceq z = (z_1 + z_2 + \cdots + z_k)$, and $w \not\preceq (z_1 + z_2 + \cdots + z_k - z_i)$ for each $i \in [0, k]$.

If b^{x+1} is not a factor of z, then $a \mid z_i$ for each $i \in [1, k]$. Dividing through by a, we have that $v = \frac{w}{a} \preceq \frac{z}{a} = \left(\frac{z_1}{a} + \frac{z_2}{a} + \cdots + \frac{z_k}{a}\right)$, where each $\frac{z_i}{a}$ is a generator of S, and $v = \frac{w}{a} \not\preceq \left(\frac{z_1}{a} + \frac{z_2}{a} + \cdots + \frac{z_k}{a} - \frac{z_i}{a}\right)$ for each $i \in [1, k]$. Thus, $\frac{z}{a}$ is a bullet for v in S, and $|z| = \left|\frac{z}{a}\right|$.

Now suppose b^{x+1} is a factor of z. Notice that $w \preceq ab^{x+1}$ because

$$\begin{aligned} ab^{x+1} - w &= bab^x - a^{x+1-i}b^i \\ &= (b-a)ab^x + (b-a)a^2b^{x-1} + \cdots + (b-1)a^{x+1-i}b^i, \end{aligned}$$

which is in M. Also, $w \not\preceq (a-1)b^{x+1}$ because $(a-1)b^{x+1} - a^{x+1-i}b^i \notin M$ by Corollary 6.34. So, ab^{x+1} is a bullet for w in M.

We will conclude by showing that there are no other possibilities for z. If there were, they would be of the form $z = \sum_{j=0}^{x+1} c_ja^{x+1-j}b^j$, where $1 \le c_{x+1} \le a-1$, $c_i = 0$, and every other $c_j \ge 0$. Assume for the sake of contradiction that some z of this form is a bullet for w in M. Then, $w = a^{x+1-i}b^i \preceq \sum_{j=0}^{x+1} c_ja^{x+1-j}b^j$, so $\left(\sum_{j=0}^{x+1} c_ja^{x+1-j}b^j\right) - a^{x+1-i}b^i \in M$. By Proposition 6.10, we can connect $\left(\sum_{j=0}^{x+1} c_ja^{x+1-j}b^j\right) - a^{x+1-i}b^i$ to some non-negative factorization of $z - w$ using swaps of the form $ba^{x-(j-1)}b^{j-1} = aa^{x-j}b^j$ for $j \in [1, x+1]$. Since $a^{x+1-i}b^i$ is the only term with a negative coefficient, we must use a series of swaps to increase this coefficient to a non-negative number. Because b^{x+1} is a larger generator than $a^{x+1-i}b^i$, if this series of swaps does anything to c_{x+1}, it will change it by some multiple of a. To increase c_{x+1} would be absurd, since c_{x+1} is positive to begin with and we are only interested in changing the factorization until we have a non-negative factorization. Thus, any change to c_{x-1} would be a decrease by a multiple of a.

However, $c_{x+1} < a$, so any such change would reduce c_{x+1} to a negative number. Thus, there must be some series of swaps that does not involve b^{x+1} that will connect $\left(\sum_{j=0}^{x+1} c_ja^{x+1-j}b^j\right) - a^{x+1-i}b^i$ to a non-negative factorization of $z - w$. But this implies we can subtract $c_{x+1}b^{x+1}$ and have $\left(\sum_{j=0}^{x} c_ja^{x+1-j}b^j\right) - a^{x+1-i}b^i \in M$, so z is not a bullet.

Therefore, any bullet for w in M either is a constant multiple of a bullet for v in S or is ab^{x+1}. Then, if $\omega_S(v) = n$, by Proposition 6.40, $\omega_M(w) = \max\{a, n\}$. □

Lemma 6.46. *Let $S = \langle n_1, n_2, \ldots, n_k\rangle$ be a primitive numerical semigroup. If z_i is the length of the longest factorization of elements in $Ap(S, n_i)$ for $1 \le i \le k$, then $\omega(n_i) \le z_i + 1$.*

Proof. Fix some $i \in [1,k]$. To find $\omega(n_i)$, we must consider the bullets of n_i. Suppose some $x = x_1 + \cdots + x_m$ is a bullet for n_i. Then, $x_1 + \cdots + x_m - n_i \in S$ but $x_1 + \cdots x_m - x_j - n_i \notin S$ for every $k \in [1,m]$. Since $x_1 + \cdots + x_m - x_j \in S$ but $x_1 + \cdots + x_m - x_j - n_i \notin S$, $x_1 + \cdots x_m - x_j \in Ap(S, n_i)$. Therefore, every bullet, with one element removed, is in $Ap(S, n_i)$. This implies that $\max\{|x| : x \in \text{bul}(n_i)\} - 1$ is no longer than the longest factorization of elements in $Ap(S, n_i)$. Since $\omega(n_i) = \max\{|x| : x \in \text{bul}(n_i)\}$ by Proposition 6.40, $\omega(n_i) - 1$ is no longer than the longest factorization of elements in $Ap(S, n_i)$. Therefore, $\omega(n_i) \leq z_i + 1$. □

For some generators in certain semigroups, this bound is attained, and the omega primality of a generator n_i is exactly one more than the longest factorization in $Ap(S, n_i)$. For instance, in $S = \langle 25, 30, 36\rangle$, $\omega(36) = 12$ and the longest factorization in $Ap(S, 36)$ is 11. Also, in $T = \langle 11, 12, 13, 14\rangle$, $\omega(12) = 6$ and the longest factorization in $Ap(T, 12)$ is 5. In the semigroup $R = \langle 5, 7, 9\rangle$, all generators attain this maximum bound: $\omega(5) = 3$, while the longest factorization in $Ap(R, 5)$ is 2; $\omega(7) = 5$, while the longest factorization in $Ap(R, 7)$ equals 4; and $\omega(9) = 5$, while the longest factorization in $Ap(R, 9)$ is 4.

7 Appendix

7.1 Numerical Semigroups of Embedding Dimension Four with Catenary Degree Three

The following are the generators of all of the numerical semigroups of embedding dimension four which have catenary degree three.

(4, 5, 6, 7), (4, 6, 7, 9), (5, 6, 7, 8), (5, 6, 7, 9), (5, 6, 8, 9), (5, 7, 8, 9), (5, 7, 8, 11), (6, 7, 8, 9), (6, 7, 8, 10), (6, 7, 8, 11),(6, 7, 9, 10), (6, 7, 9, 11), (6, 7, 10, 11), (6, 8, 9, 10), (6, 8, 9, 11), (6, 8, 9, 13), (6, 8, 10, 11), (6, 8, 10, 13), (6, 8, 10, 15), (6, 9, 10, 11), (6, 9, 10, 14), (6, 9, 11, 13), (6, 9, 13, 14), (7, 8, 9, 12), (7, 8, 9, 13), (7, 8, 10, 11), (7, 8, 10, 13), (7, 8, 11, 13), (7,8, 12, 13), (7, 9, 12, 15), (7, 10, 11, 12), (7, 10, 11, 15), (8, 9, 10, 12), (8, 9, 10, 14), (8, 9, 10, 15), (8, 9, 11, 13), (8, 9, 11, 15), (8, 9, 12, 15), (8, 10, 11, 12), (8, 10, 11, 13), (8, 10, 11, 14), (8, 10, 12, 13), (8, 10, 12, 15), (8, 10, 12, 17), (8, 10, 13, 14), (8, 10, 14, 15), (8, 10, 14, 17), (8, 10, 14, 19), (8, 10, 14, 21), (8, 11, 12, 14), (8, 11, 12, 17), (8, 11, 13, 17), (8, 12, 13, 14), (8, 12, 13, 18), (8, 12, 13, 19), (8, 12, 14, 15), (8, 12, 14, 17), (8, 12, 14, 19), (8, 12, 14, 21), (8, 12, 15, 18), (8, 12, 15, 21), (8, 12, 17, 18), (8, 12, 18, 19), (8, 12, 18, 21), (8, 12, 18, 27), (9, 10, 12, 15), (9, 10, 12, 17), (9, 10, 15, 17), (9, 11, 12, 15), (9, 12, 13, 15), (9, 12, 14, 15), (9, 13, 14, 21), (10, 11, 12, 15), (10, 11, 12, 18), (10, 11, 13, 15), (10, 11, 13, 17), (10, 11, 15, 18), (10, 12, 13, 15), (10, 12, 13, 17), (10, 12, 14, 15), (10, 12, 15, 16), (10, 12, 15, 18), (10, 12, 15, 21), (10, 12, 17, 18), (10, 12, 18, 19), (10, 12, 18, 21), (10, 12, 18, 23), (10, 12, 18, 27), (10, 13, 14, 15), (10, 13, 15, 19), (10, 14, 15, 17), (10, 14, 15, 21), (10, 14, 16, 17), (10, 14, 16, 19), (10, 15, 16, 24), (10, 15, 18, 27), (11, 12, 14, 19), (11, 12, 14, 21), (11, 12, 15, 18), (11, 12, 15, 21), (11, 12, 18, 21), (11, 15, 18, 27), (12, 13, 14, 18), (12, 13, 15, 18), (12, 13, 15, 21), (12, 13, 18, 21), (12, 13, 18, 27), (12, 14, 15, 18), (12, 14, 15, 21), (12, 14, 18, 19), (12, 14, 18, 21), (12, 14, 18, 23), (12, 14, 18, 25), (12, 14, 18, 27), (12, 15, 16, 18), (12, 15, 16, 21), (12, 15, 17, 18), (12, 15, 17, 21), (12, 15, 19, 21), (12, 16, 17, 18), (12, 16, 18, 21), (12, 16, 18, 23), (12, 16, 18, 25), (12, 16, 18, 27), (12, 17, 18, 21), (12, 17, 18, 27), (12, 18, 19, 21), (12, 18, 19, 27), (12, 18, 20, 21), (12, 18, 22, 27), (13, 18, 21, 27), (14, 15, 17, 21), (14, 15, 18, 27), (14, 16, 17, 21), (14, 16, 21, 24), (14, 18, 19, 21), (14, 18, 20, 21), (14, 18, 21, 27), (14, 18, 24, 27), (15, 16, 18, 27), (15, 17, 18, 27), (15, 18, 19, 27), (15, 18, 20, 27), (15, 18, 23, 27), (16, 18, 21, 27), (16, 18, 24, 27), (17, 18, 24, 27), (18, 19, 21, 27), (18, 20, 21, 27), (18, 20, 24, 27), (18, 21, 22, 27), (18, 21, 23, 27), (18, 21, 25, 27), (18, 22, 24, 27), (18, 23, 24, 27), (18, 24, 25, 27), (18, 24, 26, 27)

7.2 Special Elasticity Code

The first cell in every sage worksheet should be to load the numerical semigroup package developed by Chris.

```
load('/media/sf_Desktop/NumericalSemigroup.sage')
```

7.2.1 ρ_2

This first definition returns a tuple that has ρ_2 in the first entry and the factorization(s) that give ρ_2. This code can be adjusted for ρ_k but may not be as efficient.

```
def datrhodo(S):
    gens = S.gens
    embdim = len(gens)
    twoelements = []
    for i in gens:
        for j in gens:
            twoelements.append(i+j)
    factorizations = []
    for t in twoelements:
        for f in S.Factorizations(t):
            factorizations.append(f)
    maxlength = 0
    maxfactorizations = []
    for g in factorizations:
        if sum(g) > maxlength:
            maxlength = sum(g)
    for g in factorizations:
        if sum(g) == maxlength:
            maxfactorizations.append(g)
    maxfactelements = Set([])
    for m in maxfactorizations:
        element = 0
        for i in range(0, embdim):
            element += m[i]*gens[i]
        maxfactelements = maxfactelements.union(Set([element]))
    maxfactelementslist = list(maxfactelements)
    maxfactelementslistwfacts = [(m, S.Factorizations(m)) for m in maxfactelementslist]
    maxfacts = S.Factorizations(maxfactelementslist[0])
    return (max([sum(f) for f in maxfacts]), maxfactelementslistwfacts)
```

This definition calls the above one and gives back the semigroup, followed by the value for $\rho_2 \leq 3$ and on the next line(s) it returns the factorization(s). This code can be adjusted to return any value of ρ_2.

```
def dodatrhodo(NumSemigroups):
    for S in NumSemigroups:
        if tuple(S.gens) in Tested.keys():
            print str(S.gens)+",", Tested[tuple(S.gens)][0]
            for tup in Tested[tuple(S.gens)][1]:
                print str(tup[0])+",", tup[1]
            print
            continue
        result = datrhodo(S)
        if result[0]<=3:
            print str(S.gens)+",", result[0]
            for tup in result[1]:
                print str(tup[0])+",", tup[1]
            print
        Tested[tuple(S.gens)] = result

Tested = { }
```

This is one of our example codes to compute many semigroups. Here we are computing all semigroups of dimension 3 such that the multiplicity is 3 or 4 and the other two generators range up to, but not including, 10.

```
for i in [3,4]:
    for j in range(i+1, 10):
        for k in range(j+1, 10):
            if gcd(i, gcd(j, k)) == 1:
                s = NumericalSemigroup([i, j, k])
```

```
            if len(s.gens) == 3:
                NumSemigroups[(i,j,k)] = s
    print str(100.*((i-2)/2))+"%"
dodatrhodo(NumSemigroups)
```

This code was to compute semigroups where the highest common factor between generators is 2.

```
NumSemigroups=[]
i=6
for j in range(i+1, 25):
    for k in range(j+1, 30):
        if gcd(i, gcd(j, k)) == 1 and ((gcd(i,j)==2 and gcd(i,k)==1 and gcd(j,k)==1) or
(gcd(i,j)==1 and gcd(i,k)==2 and gcd(j,k)==1) or (gcd(i,j)==1 and gcd(i,k)==1 and gcd(j,k)==2)):
            s = NumericalSemigroup([i, j, k])
            if len(s.gens) == 3:
                NumSemigroups.append(s)
    print str(100.*((j-i)/(25-i+1)))+"%"
dodatrhodo(NumSemigroups)
```

This code was used for semigroups of the pairwise corpime (symmetric) form. Lemma 3.25 lets us choose one generator to be even.

```
NumSemigroups=[]
for xp in range(2,10):
    for yp in range(1,10):
        for zp in range(yp+1,15):
            if gcd(2*xp, 2*yp+1) == 1 and gcd(2*yp+1, 2*zp+1)==1 and gcd(2*xp, 2*zp+1)==1 :
                s = NumericalSemigroup([2*xp, 2*yp+1, 2*zp+1])
                if len(s.gens) == 3:
                    NumSemigroups.append(s)
    print str(100.*((xp-1)/(8)))+"%"
dodatrhodo3(NumSemigroups)
```

7.2.2 Modified Arithmetic Sequences

These first two definitions return the generators and ρ_k depending on which definition and how you alter them.

```
def rhotwoarith(NumSemigroups):
    for s in NumSemigroups:
        print str(s.gens)+",",s.SpecialElasticity(2)

def rhokarith(NumSemigroups):
    for s in NumSemigroups:
        print str(s.gens)+",",s.SpecialElasticity(3)
```

This code removes one of the middle generators out of a general arithmetic sequence.

```
NumSemigroups=[]
x=4
h=2
d=3
for a in range(5,9):
    for n in range(1,x):
        if gcd(a,d)==1:
            l=[a]+[a*h+i*d for i in [1..x] if i != n]
            s = NumericalSemigroup(l)
            NumSemigroups.append(s)
    print str(100.*((a-4)/4))+"%"
rhotwoarith(NumSemigroups)
```

We used this code to check the factorization that gives back rho_2 and its factorizations and compare it to certain "lower" semigroups.

```
print dodatrhodo([NumericalSemigroup([7,15,16,17,18])])
print dodatrhodo([NumericalSemigroup([7,15,16,18])])
```

This code was used for bulk computations.

```
NumSemigroups=[]
x=5
h=1
d=1
for a in range(6,11):
    for n in range(1,5):
        for m in range(n+1,min(n+3,5)):
            if gcd(a,d)==1:
                l=[a]+[a*h+i*d for i in [1..x] if i != n and i != m]
                s = NumericalSemigroup(l)
                NumSemigroups.append(s)
    print str(100.*((a-5)/5))+"%"
dodatrhodo(NumSemigroups)
```

These two cells were used to compute examples for removing all but the first, second, second to last, and last generator out of an arithmetic sequence $a, a+d, ..., a+xd$.

```
NumSemigroups=[]
x=13
d=3
for a in [x+1,2*x-3,2*x-2,2*x-1,2*x]:
    if gcd(a,d)==1:
        l=[a, a+d, a+(x-1)*d, a+x*d]
        s = NumericalSemigroup(l)
        NumSemigroups.append(s)
    print str(100.*((a-5)/5))+"%"
dodatrhodo(NumSemigroups)
```

```
NumSemigroups=[]
x=13
d=3
for a in [x+2,..,2*x-4]:
    if gcd(a,d)==1:
        l=[a, a+d, a+(x-1)*d, a+x*d]
        s = NumericalSemigroup(l)
        NumSemigroups.append(s)
    print str(100.*((a-5)/5))+"%"
dodatrhodo(NumSemigroups)
```

This code was used to check the possible values in theorem 5.1 which could give $\rho_2(S) < 4$.

```
def rhofrob(amax, bmax):
    for a in range(5, amax+1):
        for b in range(a+1, bmax+1):
            if gcd(a, b) != 1:
                continue
            T = NumericalSemigroup([a, b])
            f = T.frob
            for c in range(b+1, f+1):
                S = NumericalSemigroup([a, b, c])
                if len(S.gens) != 3:
                    continue
                if tuple(S.gens) in Tested.keys():
```

```
            print str(S.gens)+",", Tested[tuple(S.gens)][0]
            for tup in Tested[tuple(S.gens)][1]:
                print str(tup[0])+",", tup[1]
            print
            continue
        result = datrhodo(S)
        print str(S.gens)+",", result[0]
        for tup in result[1]:
            print str(tup[0])+",", tup[1]
        print
        Tested[tuple(S.gens)] = result
```

This code was used to get an idea of whether conjecture 5.10 holds. It calculates $\rho_2(S)$ for semigroups with all generators coprime for embedding dimension 4 up to the frobenius number, and can easily be adjusted to for higher embedding dimensions.

```
def rhoe4(amax, bmax, cmax):
    for a in range(9, amax+1):
        for b in range(a+1, bmax+1):
            for c in range(b+1, cmax+1):
                    if any([gcd(x,y) > 1 for x in [a,b,c] for y in [a,b,c] if x != y]):
                        continue
                    T = NumericalSemigroup([a, b, c])
                    if len(T.gens) != 3:
                        continue
                    result = datrhodo(T)
                    print str(T.gens)+',', result[0]
                    for tup in result[1]:
                        print str(tup[0])+",", tup[1]
                    print
                    f = T.frob
                    for d in range(c+1, f+1):
                        if any([gcd(x,d) > 1 for x in [a,b,c]]):
                            continue
                        S = NumericalSemigroup([a, b, c, d])
                        if len(S.gens) != 4:
                            continue
                        resultwd = datrhodo(S)
                        #if [0,0,0,0,2] not in sum([l[1] for l in resultwd[1]],[]):
                        #print str(T.gens)+',', result[0]
                        #for tup in result[1]:
                            #print str(tup[0])+",", tup[1]
                        #print
                        print str(d)+',', resultwd[0]
                        for tup in resultwd[1]:
                            print str(tup[0])+",", tup[1]
                        print
```

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