Exercises for quantum computing

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1 Week 1

Problem 1. Given the standard (Pauli) single qubit gates,

$$\mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

show that

$$\mathbf{XYX} = -\mathbf{Y}.$$

Problem 2. Considering the additional single qubit gates,

$$\mathbf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

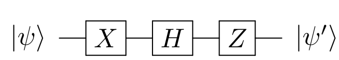
show that

$$\mathbf{HXH} = +\mathbf{Z}, \quad \mathbf{HYH} = -\mathbf{Y}, \quad \mathbf{HZH} = +\mathbf{X}.$$

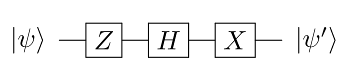
Problem 3. If $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ (a single qubit), what is the outcome of $\mathbf{XYZ}|\psi\rangle$? What about $\mathbf{ZYX}|\psi\rangle$?

Problem 4. Write down the circuit representation for the single qubit operation $|\psi'\rangle = \mathbf{XYZ}|\psi\rangle$.

Problem 5. Starting from $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, what is the final $|\psi'\rangle$ expected from this single qubit circuit?

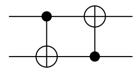


What about this single qubit circuit?



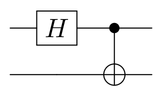
Problem 6. Write the matrix representation of the circuits in Problem 5.

Problem 7. Write the matrix representation of the two qubit circuit with CNOT gates,



Be sure to define first the order of your two qubit basis states $|00\rangle, |01\rangle$, etc.

Problem 8. Write the matrix representation of the two qubit circuits,



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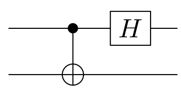
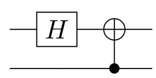


Problem 9. Write the matrix representation of the two qubit circuits,



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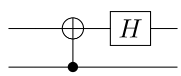


Problem 10. Find the eigenvectors (and eigenvalues, while you're at it) of the matrix

$$\mathbf{A} = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}.$$

For each of the eigenvectors, find the angles θ, ϕ on the Bloch sphere which describe them.

Problem 11. Consider the unitary matrix

$$\mathbf{B} = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}.$$

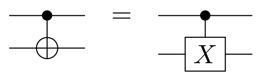
which can be written in the form

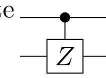
$$\exp\left(-i\frac{\theta}{2}\mathbf{Y}\right)$$

with the Pauli matrix $\mathbf{Y}$ from Problem 1. Find θ, which is not one of the Bloch sphere angles.

2 Weeks 2-4

Note: What we call the *controlled not* gate can also be written as the *controlled* X gate, because $X = \textit{not}$ on qubits:

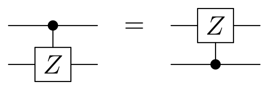


We can also introduce the *controlled* Z gate 

which has the 2-qubitmatrix form

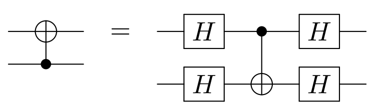
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Problem 12. Show that

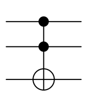


Problem 13. Construct a CNOT (controlled-X) gate from one controlled-Z gate and two Hadamard gates.

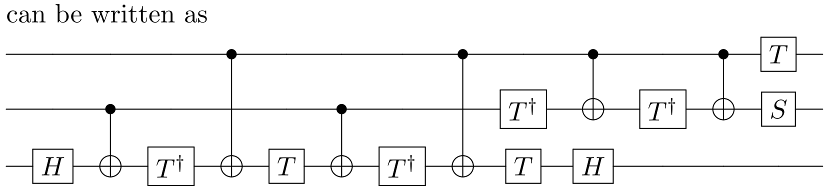
Problem 14. Show that



Problem 15. Show that the Toffoli gate



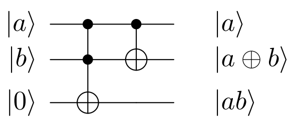
can be written as



Don't forget the definitions of the *phase gate* and $\pi/8$-gate, respectively:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & \exp\left(i\frac{\pi}{4}\right) \end{pmatrix}$$

Problem 16. The Toffoli gate (see problem 15) can be used to add two bits a and b:



Remember that $a \oplus b$ is addition mod 2, so that $0 \oplus 1 = 1$ but $1 \oplus 1 = 0$. In addition ab is the "carry", as $1 \times 1 = 1$.

Confirm the above. Then design a circuit to add *three* bits.

Problem 17. In class we designed a circuit to carry out the two-qubit selective phase transformation,

$$\mathbf{\Phi}_2 = \begin{pmatrix} e^{i\theta_0} & 0 & 0 & 0 \\ 0 & e^{i\theta_1} & 0 & 0 \\ 0 & 0 & e^{i\theta_2} & 0 \\ 0 & 0 & 0 & e^{i\theta_3} \end{pmatrix}$$

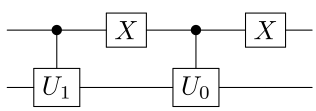
which was written as the product of two controlled single-qubit gates, that is,

$$\mathbf{\Phi}_2 = \mathbf{A}_1\mathbf{A}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\theta_2} & 0 \\ 0 & 0 & 0 & e^{i\theta_3} \end{pmatrix} \begin{pmatrix} e^{i\theta_0} & 0 & 0 & 0 \\ 0 & e^{i\theta_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

by defining

$$\mathbf{U}_0 = \begin{pmatrix} e^{i\theta_0} & 0 \\ 0 & e^{i\theta_1} \end{pmatrix}, \quad \mathbf{U}_1 = \begin{pmatrix} e^{i\theta_2} & 0 \\ 0 & e^{i\theta_3} \end{pmatrix},$$

and our circuit is



(we also worked out the nontrivial circuits for the controlled-U gates; you do not have to do that here).

Now do the same for three qubits, that is, find the circuit for

$$\mathbf{\Phi}_3 = \begin{pmatrix} e^{i\theta_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{i\theta_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i\theta_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\theta_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{i\theta_4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{i\theta_5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{i\theta_6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{i\theta_7} \end{pmatrix}$$

Hint: write this as a product of four matrices $\mathbf{A}_3\mathbf{A}_2\mathbf{A}_1\mathbf{A}_0$, and find the circuits for each of those matrices.