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Examining individual and collective level mathematical progress

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Abstract A challenge in mathematics education research is to coordinate different analyses to develop a more comprehensive account of teaching and learning. We contribute to these efforts by expanding the constructs in Cobb and Yackel's (Educational Psychologist 31:175–190, 1996) interpretive framework that allow for coordinating social and individual perspectives. This expansion involves four different constructs: disciplinary practices, classroom mathematical practices, individual participation in mathematical activity, and mathematical conceptions that individuals bring to bear in their mathematical work. We illustrate these four constructs for making sense of students' mathematical progress using data from an undergraduate mathematics course in linear algebra.

Keywords Individual and collective · Emergent perspective · Linear algebra · Practices

Recent work in mathematics education research has sought to integrate different theoretical perspectives to develop a more comprehensive account of teaching and learning (Bikner-Ahsbahs & Prediger, 2014; Cobb, 2007; Hershkowitz, Tabach, Rasmussen, & Dreyfus, 2014; Prediger, Bikner-Ahsbahs, & Arzarello, 2008; Saxe et al., 2009; Sfard, 1998). One of the early efforts at integrating different theoretical perspectives is Cobb and Yackel's (1996) emergent perspective and accompanying interpretive framework. In this paper, we expand the interpretive framework for coordinating social and individual perspectives by offering a set of constructs for how to examine the mathematical progress of both the collective and the individual. Using data from an undergraduate mathematics course in linear algebra, we

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illustrate these constructs by conducting four parallel analyses and make initial steps toward coordinating across the analyses.

The emergent perspective is a version of social constructivism that coordinates the individual cognitive perspective of constructivism (von Glasersfeld, 1995) and the sociocultural perspective based on symbolic interactionism (Blumer, 1969). A primary assumption from this point of view is that mathematical development is a process of active individual construction and a process of mathematical enculturation (Cobb & Yackel, 1996). The interpretive framework, shown in Fig. 1, lays out the central constructs in the emergent perspective. Details on how the constructs of social norms, sociomathematical norms, and associated beliefs are operationalized can be found in Yackel and Cobb (1996). The constructs in the bottom row of the framework are explicated and expanded upon in this paper.

The work of Paul Cobb and associates continues to have a profound effect on the field. The recent book by Yackel, Gravemeijer, and Sfard (2011) traces the evolution and insights of Cobb's work over the past two decades. One of the seminal contributions to the field has been the emergent perspective and the accompanying interpretive framework. Indeed, much of our previous work has theoretically and methodologically benefited from using aspects of the interpretive framework to analyze classroom teaching and learning at the undergraduate level. For example, Yackel and Rasmussen (2002) examined the role of social and sociomathematical norms on undergraduate students' evolving beliefs. Yackel, Rasmussen, and King (2000) documented the emergence and constitution of particular social and sociomathematical norms in a undergraduate differential equations class. Rasmussen (2001) made use of the interpretive framework to analyze the role of the classroom participation structure in individual student learning. Methodologically, Rasmussen and Stephan have developed a three-phase approach for documenting classroom mathematical practices (Rasmussen & Stephan 2008; Stephan & Rasmussen 2002). Wawro (2011) has expanded on this methodology by developing new argumentation structures related to classroom mathematical practices. Rasmussen, Zandieh, and Wawro (2009) examined the role of the teacher in relation to the emergence of classroom mathematical practices.

As this body of work attests, we have found that the interpretative framework offers a useful set of constructs for making sense of individual and collective mathematical activity. The significance of accounting for both individual and collective activity is highlighted by Saxe (2002), who points out that "individual and collective activities are reciprocally related. Individual activities are constitutive of collective practices. At the same time, the joint activity of the collective gives shape and purpose to individuals' goal-directed activities" (p. 276–277). Our work with the interpretative framework has raised our awareness of the opportunity (and need) to ask additional questions regarding individuals and the collective. This report is a first step toward an approach for expanding the ways we can analyze individual and collective mathematical progress. We use the phrase "mathematical progress" as an umbrella term that admits analyses of collective practices and individual conceptions and activity.

On the bottom left hand side of the interpretive framework (Fig. 1), the construct of classroom mathematical practices is a way to conceptualize the collective mathematical progress of the local classroom community. In particular, such an analysis answers the

Social Perspective	**Individual Perspective**
Classroom social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions and activity

Fig. 1 The interpretive framework

question: What are the normative ways of reasoning that emerge in a particular classroom? Such normative ways of reasoning are said to be reflexively related to individual students' mathematical conceptions and activity. In prior work that has used the interpretive framework, individual conceptions and activity has been treated as a single construct that frames the ways that individual students participate in classroom mathematical practices (e.g., Bowers, Cobb, & McClain, 1999; Cobb, 1999; Stephan, Cobb, & Gravemeijer, 2003). Such a framing of the individual is, in our view, compatible with what Sfard (1998) refers to as the "participation metaphor" for learning.

In an effort to be more inclusive of a cognitive framing that would posit particular ways that students think about an idea, we split the bottom right hand cell into two constructs, one for individual participation in mathematical activity and one for mathematical conceptions that individual students bring to bear in their mathematical work. With these two constructs for individual progress, we now can ask the following two questions: How do individual students contribute to mathematical progress that occurs across small group and whole class settings? And what conceptions do individual students bring to bear in their mathematical work?

Our extensive work at the undergraduate level has also highlighted the fact that, in comparison to K-12 students, university mathematics and science majors are more intensely and explicitly participating in the discipline of mathematics. However, the notion of a classroom mathematical practice was never intended to capture the ways in which the emergent, normative ways of reasoning relate to various disciplinary practices (Stephan & Cobb, 2003). In order to more fully account for what often occurs at the undergraduate level, we therefore expand the interpretive framework to explicate how the classroom collective activity reflects and constitutes more general disciplinary practices. Thus, we add an additional cell to the bottom left row of the interpretive framework, resulting in one cell for classroom mathematical practices and one cell for disciplinary practices. With these two constructs, we can now answer two different questions about collective mathematical progress, one related to disciplinary practices (What is the mathematical progress of the classroom community in terms of the disciplinary practices of mathematics?) and one for classroom mathematical practices (What are the normative ways of reasoning that emerge in a particular classroom?).

To summarize, Fig. 2 shows our expansion of the bottom row of the interpretive framework, which now entails four different constructs: disciplinary practices, classroom mathematical practices, individual participation in mathematical activity, and mathematical conceptions. These constructs then do work for us within the interpretive framework, which is a manifestation of the emergent perspective that coordinates individual and social perspectives. In this way, our work is nested within this coordination.

The left hand side of the bottom row comprises two different constructs for examining the mathematical progress of the classroom community, while the right hand side comprises two different constructs for examining the mathematical progress of individual students. The contribution that this expansion makes is in providing researchers with a more comprehensive means of bringing together analyses from different perspectives to document mathematical progress. In particular, the expanded interpretive framework enables a researcher to answer the questions listed in Fig. 3.

Social Perspective		**Individual Perspective**	
Classroom social norms		Beliefs about own role, others' roles, and the general nature of mathematical activity	
Sociomathematical norms		Mathematical beliefs and values	
Disciplinary practices	Classroom mathematical practices	Participation in mathematical activity	Mathematical conceptions

Fig. 2 Expanded interpretive framework

Disciplinary practices	Classroom mathematical practices	Participation in mathematical activity	Mathematical conceptions
What is the mathematical progress of the classroom community in terms of the disciplinary practices of mathematics?	What are the normative ways of reasoning that emerge in a particular classroom?	How do individual students contribute to mathematical progress that occurs across small group and whole class settings?	What conceptions do individual students bring to bear in their mathematical work?

Fig. 3 Four constructs for analyzing mathematical progress and respective research questions

1 Research setting

We illustrate the four constructs and address the respective research questions using data from a semester-long classroom teaching experiment (Cobb, 2000) in linear algebra conducted at a large public university in the USA. The teaching experiment was part of a larger design research project that explored ways of building on students' current ways of reasoning to help them develop more formal and conventional ways of reasoning (Wawro, Rasmussen, Zandieh, & Larson, 2013). We selected data from this teaching experiment based on its strong potential to illustrate all four constructs.

The majority of students in the class had completed at least two semesters of calculus, with some students having completed a third semester of calculus or a discrete mathematics course. Most students were in their second or third year of university and had chosen engineering, mathematics, or computer science as their major course of study. We collected data for analysis by videotaping each class session, collecting student written work, and conducting interviews with students throughout the semester.

In addition to videorecording whole class discussions, three of the eight small groups were videorecorded; we present analysis here of one of the groups (henceforth referred to as the focus group) and its individual members. We chose to analyze data from this group because the members were particularly open to sharing their thinking and willing to challenge others' ideas, which gave us access to their mathematical thinking without having to rely on interview data. The five students in the focus group were as follows: Abraham, a junior statistics major; Aziz, a junior chemical physics major; Giovanni, a senior business major; Justin, a sophomore mathematics major; and Kaemon, a senior computer engineering major.

2 Theoretical and methodological background

In this section, we provide further background on the theoretical constructs and associated methods for making sense of collective and individual mathematical progress. We begin with the two constructs for the collective mathematical progress (starting with classroom mathematical practices), followed by the two constructs for individual mathematical progress (starting with mathematical conceptions). Each of these constructs may be explored with different tools, and hence within each subsection, we specify our choice of methodological tool for each construct. We leave open the possibility that other methodological tools could be used.

Classroom mathematical practices Classroom mathematical practices refer to the normative ways of reasoning that emerge as learners solve problems, explain their thinking, represent their ideas, and so on. By normative we mean that there is empirical evidence that an idea or way of reasoning functions as if it is a mathematical truth in the classroom. This means that particular ideas or ways of reasoning are functioning in classroom discourse *as if* everyone has similar understandings, even though individual differences in understanding may exist. The production of these normative ways of reasoning constitutes the mathematical progress of the classroom community. The empirical evidence needed to document normative ways of reasoning is garnered using the approach developed by Rasmussen and Stephan (2008) and Stephan and Rasmussen (2002). This approach applies Toulmin's argumentation scheme to document the mathematical progress. We next briefly describe this approach.

In his seminal work, Toulmin (1958) created a model to describe the structure and function of argumentation. Figure 4 illustrates that the core of an argument consists of three parts: the data, the claim, and the warrant. In an argument, a speaker or speakers make a claim and present evidence or data to support that claim. Typically, the data consist of facts or procedures that lead to the conclusion that is made. To further improve the strength of the argument, speakers often provide more clarification that connects the data to the claim, which serves as a warrant, or a connector between the two. Finally, the argumentation may also include a backing, which demonstrates why the warrant has authority to support the data-claim pair. Toulmin's model also includes qualifiers and rebuttals.

Methodologically, to document normative ways of reasoning, one begins by using Toulmin's model to code every whole class discussion, resulting in anywhere from a few to more than a dozen coded arguments. The collection of all coded arguments results in an argumentation log for all whole class discussions. The next step involves taking the argumentation log as data itself and looking across all class sessions to see what mathematical ideas become part of the class' normative ways of reasoning. The following two criteria are used to determine when a way of reasoning becomes normative:

Criterion 1 When the backings and/or warrants for a particular claim are initially present but then drop off. For example, criterion 1 is satisfied when the same claim gets debated on more than one class period or more than once during the same class period and in subsequent occurrences the backing or warrants drop off.

Criterion 2 When certain parts of an argument (the warrant, claim, or backing) shift position within subsequent arguments, indicating knowledge consolidation. For example, criterion 2 is satisfied when once-debated conclusions shift function over time and serve as unchallenged data or justification for future conclusions.

The use of this methodology requires classrooms in which genuine argumentation is a norm. That is, students are routinely explaining their reasoning, indicating agreement or

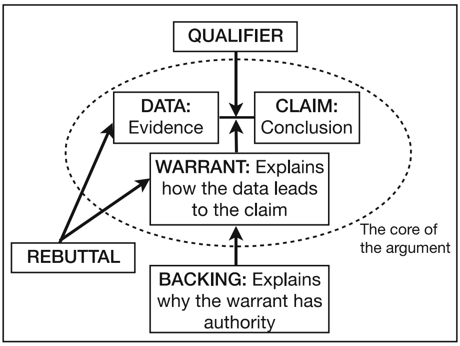


Fig. 4 Toulmin's model of argumentation

disagreement with others' reasoning, and so on. The linear algebra class analyzed in this report is one in which students engaged in genuine argumentation.

Disciplinary practices Disciplinary practices refer to the ways in which mathematicians go about their profession. The following disciplinary practices are among those core to the activity of professional mathematicians: defining, algorithmatizing, symbolizing, and theoremizing (Rasmussen, Zandieh, King, & Teppo, 2005). Other researchers might choose alternative ways to characterize the types of activities in which mathematicians engage. For example, at the K-12 level in the USA, one might instead choose to use the Standards for Mathematical Practice, as described in the Common Core State Standards for Mathematics (Common Core State Standards Initiative, 2010).

Not all classroom mathematical practices are easily or sensibly characterized in terms of a disciplinary practice. This is because classroom mathematical practices capture the emergent and potentially idiosyncratic collective mathematical progress of a local community, whereas disciplinary practices capture how that collective progress reflects and embodies the core practices of the discipline at large. For example, an important algorithm in differential equations is Euler's method, which is a numerical technique for obtaining an approximate solution to an initial value problem. A common instructional approach is simply to tell students what this algorithm is and then to have them practice the method. This kind of approach to teaching Euler's method does not offer students an opportunity to engage in doing mathematics like mathematicians do. In contrast, the inquiry-oriented differential equations class analyzed in Rasmussen et al. (2005) offered students several opportunities to engage in the authentic practice of algorithmatizing. As it played out in the classroom studied, algorithmatizing involved engaging in the goal-directed activity of creating predictions, isolating attributes, forming quantities, creating relationships between quantities, and expressing relationships symbolically.

The term algorithmatizing is similar to the term "theoremizing" in the following way. Each has a noun as the root (algorithm and theorem) made into a verb. The verb form reflects a focus on student activities, namely creating and using algorithms in the former and conjecturing and proving in the latter. When students are engaged in genuine argumentation, it is often the case that conjectures are made and then justifications are created to support or refute the conjectures. The term theoremizing is used to explicitly encompass both conjecturing and steps toward justifying the assertions. Our analysis of theoremizing builds on the work of Rasmussen et al. (2005) and Zandieh and Rasmussen (2010), who analyze the disciplinary practices of algorithmatizing, symbolizing, and defining.

Our use of the term "disciplinary practice" is somewhat similar to how Moschkovich (2007) describes "professional discourse practices," which includes the discourse practices of academic mathematicians. We agree with Moschkovich that such practices are culturally and historically situated. Moreover, while perhaps not all academic mathematicians would characterize their work in terms of defining, algorithmatizing, symbolizing, and theoremizing, we argue that these broad categories do capture much of what professional mathematicians do and represent what Moschkovich (2007) argues are "socially, culturally, and historically produced practices that have become normative" (p. 25). In our analysis of classroom data, however, we employ a grounded approach (Glaser & Strauss, 1967) to characterize the ways in which the students engage in these broader disciplinary practices. That is, we do not impose any set of a priori categories of student activity related to defining, algorithmatizing, symbolizing, or theoremizing, but rather allow the data to shape how we characterize the features of a disciplinary practice that emerge in a particular class.

This analysis is a response to the more general question: What is the mathematical progress of the classroom community in terms of the disciplinary practices of mathematics? We noted above our use of the disciplinary practice construct stays true to the original, emergent nature of the interpretive framework in that defining, symbolizing, algorithmatizing, and theoremizing are seen as being constituted by the students and their instructor, as opposed to being a preexisting way of reasoning and communicating in which students are indoctrinated (Cobb & Yackel, 1996).

Mathematical conceptions As students solve problems, explain their thinking, represent their ideas, and make sense of others' ideas, they necessarily bring forth various conceptions of the ideas being discussed and potentially modify their conceptions. From this point of view, we seek to answer the question: What conceptions do individual students bring to bear in their mathematical work? For example, in the inquiry-oriented differential equations class where students reinvented Euler's method, individual students thought about rate of change in various ways, many of which are exemplified in the literature on ratio and rate (e.g., Harel, Behr, Lesh, & Post, 1994; Thompson, 1994; Zandieh, 2000).

Our analysis of individual student conceptions makes use of constructs from prior work that have characterized different views that students can have of key mathematical ideas. Indeed, there is a rich literature that has characterized various ways that students might think about particular ideas in linear algebra. For example, both Sierpinska (2000) and Hillel (2000) developed overarching frameworks for analyzing student reasoning across the linear algebra curriculum. Other studies analyzed student difficulties with the notions of basis (Hillel, 2000), linear transformation (Dreyfus, Hillel, & Sierpinska, 1999), rank (Dorier, Robert, Robinet, & Rogalski, 2000), linear independence (Bogomolny, 2007; Harel, 1997; Trigueros & Possani, 2013), and span (Stewart & Thomas, 2009). In our work, Selinski, Rasmussen, Zandieh, and Wawro (2014), Larson and Zandieh (2013), and Wawro and Plaxco (2013) developed frameworks analyzing aspects of individual conceptions of linear algebra.

For our individual conceptions analysis, we draw on the work of Wawro and Plaxco (2013), who describe four ways in which students may reason about span and linear (in)dependence (see Table 1). In this paper, we use these different conceptions to capture student reasoning as they explained their thinking, both to their group and to the class.

Participation in mathematical activity This construct for analyzing individual mathematical progress is used to answer the question: How do individual students contribute to the

Table 1 Different student conceptions for span and linear (in)dependence (from Wawro & Plaxco, 2013)

Category	Description
Travel	• Language indicative of purposeful movement • Captures notions of "getting to" or "moving to" locations in the vector space
Geometric	• Language indicative of spatial reasoning or graphical representations without use of travel-oriented language • Included sketches of vectors and/or discussion of objects such as lines and planes
Vector algebraic	• Participants use operations on algebraic representations of vectors to describe concept • Includes linear combination of vectors written as $n\times 1$ matrices or designated by variables (i.e., $2v+3w$)
Matrix algebraic	• Involves explicit attention to the form or properties of a matrix (e.g., size, actual values, pivots) • Participants focus on operations on matrices (e.g., Gaussian elimination)

mathematical progress that occurs across small group and whole class settings? To address this question, our approach draws on recent work by Krummheuer (2007, 2011). Krummheuer characterizes individual learning as participation within a mathematics classroom using the constructs of production design and recipient design. In production design, individual speakers take on various roles, which are dependent on the originality of the content and form of the utterance. The title of *author* is given when a speaker is responsible for both the content and formulation of an utterance. The title of *relayer* is assigned when a speaker is not responsible for the originality of either the content nor the formulation of an utterance (i.e., responsible for neither content nor form). A *ghostee* takes part of the content of a previous utterance and attempts to express a new idea (i.e., is responsible for content but not form), and a *spokesman* is one who attempts to express the content of a previous utterance in his/her own words (i.e., is responsible for form but not content).

Within the recipient design of learning-as-participation, Krummheuer (2011) defines four roles: conversation partner, co-hearer, over-hearer, and eavesdropper. A *conversation partner* is the listener to whom the speaker seems to allocate the subsequent talking turn. Thus, the conversation partner is not only directly addressed but also evidences a high level of engagement. Listeners who are also directly addressed but do not seem to be treated as the next speaker are called *co-hearers*. Whereas the previous two listening roles involved direct participation of the recipient to the utterance, the final two involve indirect participation. Those who seem tolerated by the speaker but do not participate in the conversation are *over-hearers*, and listeners deliberately excluded by the speaker from conversation are *eavesdroppers.*

3 Illustration of the expanded interpretive framework

The data we use to illustrate the four constructs of the expanded interpretive framework come from days 4 to 6 of the linear algebra class. A central challenge we faced in illustrating the four constructs was how best to present the classroom data and the interpretive power of the four different constructs. For the sake of clarity, we decided to first present three episodes from days 4 to 6, beginning with the focus group work from day 4 followed by whole class discussions on days 5 and 6. We refrain from analyzing these episodes with any of the four constructs and provide only the barest of descriptive commentary. We follow the episodes with a sequential interpretation, beginning with the classroom mathematical practice construct, followed by the individual mathematical conception construct, the individual participation in mathematical activity construct, and the disciplinary construct. In the discussion section, we make a start at coordinating across constructs and point to a number of different ways that analyses from these four constructs may be networked.

Episode 1: Small group work on day 4. One of the small group tasks on this day was the example-generation task shown in Fig. 5.

When the focus group discussed the cell in the "3 vectors in $\mathbf{R}^2$" row and "linearly dependent" column, Justin argued that any three vectors in $\mathbf{R}^2$ would form a linearly dependent set (utterances 1–5). He presented an argument based on span and a task from day 1, referred to as the Magic Carpet Ride problem (for details, see Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012) to support the claim that any three vectors in $\mathbf{R}^2$ would form a linearly dependent set (utterances 1–5).

LINEAR INDEPENDENCE AND DEPENDENCE: CREATING EXAMPLES

Fill in the following chart with the requested sets of vectors.

	Linearly dependent set	Linearly independent set
A set of 2 vectors in $\mathbf{R}^2$		
A set of 3 vectors in $\mathbf{R}^2$		
A set of 2 vectors in $\mathbf{R}^3$		
A set of 3 vectors in $\mathbf{R}^3$		
A set of 4 vectors in $\mathbf{R}^3$		

Write at least 2 generalizations that can be made from this table.

Fig. 5 Example generation task

1 Justin: The way I think of it, it could be any three random ones, as long as they're not all on the same line. It'll work, no matter what. For linear dependent, it can be any three you want, they don't have to be multiples, they don't have to be anything.
2 Aziz: For linear dependent, or independent?
3 Justin: For dependent, as long as you have three vectors. Because look [points to a paper], we did it before, where we have two vectors like this, and as long as they're not on the same line, your span is infinite.
4 Aziz: Yeah.
5 Justin: So if we're using three vectors to get back, the third vector can be anything. Because with this infinite span, we can always get to a point where we can get back on the third vector.

In (5), Justin used the phrase "get back," which comes from previous work on the Magic Carpet Ride problem (Wawro et al., 2012) in which students used a set of three vectors in $\mathbf{R}^3$ to try to begin and end a trip at home (i.e., the origin). As the focus group's discussion continued, Aziz conjectured that *no* three vectors in $\mathbf{R}^2$ could form a linearly independent set.

6 Aziz: So is this not applicable, this does not exist? [points to the "3 vectors in $\mathbf{R}^2$" row and "linearly independent" column on the board]
7 Justin: Oh! I don't know. I was talking about on the linear dependent side.
8 Aziz: So this linear independent, it should be no solution, right? There should be no solution for that, right? Yeah? No?
9 Giovanni: No, there would be a solution.
10 Aziz: No, because it's anywhere you go.
11 Justin: What solution would there be?
12 Giovanni: I don't know, now you got me thinking.

In utterance (6), Aziz conjectured that there *cannot* be an example that fits the cell of "linearly independent" and "3 vectors in $\mathbf{R}^2$." Justin's reaction (7) indicates that he had not considered the implications for linear independence, given that any set of three vectors in $\mathbf{R}^2$ would be linearly dependent. Aziz restated his claim (8), Giovanni disagreed, but then Justin immediately challenged Giovanni to create an example that did fit.

Later in the same discussion, Giovanni voiced his uncertainty of the claim discussed in utterances 1–5. Interrupting Aziz, Justin explained a way to justify that problem to everyone in his group.

13 Aziz: I think this is a solution: There's more vectors than there are dimensions, then—
14 Justin: Can I just show this real quick? Alright. So let's go back to our magic carpet problem. So there's an axis, and we have one going off like this, and one going off like that. And with these we have an infinite span.
15 Abraham: You can get anywhere.
16 Justin: So we can go anywhere in this entire plane. Show me any vector that I can't ride back. If it's a horizontal line, I can use these two to get way out here.
17 Giovanni: And then take it back.
18 Justin: If it's a vertical line, same thing, it doesn't. So since I can go anywhere on this plane, as long as I have a vector that originates from the origin, I can get back.
19 Giovanni: Got you.
20 Justin: So if we're in, if we have three vectors in two dimensions, and they're not all parallel, or even if they are all parallel, you can still get back.
21 Giovanni: I see what you're saying.
22 Justin: Does that make sense?
23 Giovanni: Yep.
24 Justin: Alright.

Aziz's utterance in (13) was the first instance that some aspect of the generality (i.e., not particular to three vectors in $\mathbf{R}^2$) surfaced, although no one in the focus group seemed to acknowledge Aziz's statement. As their discussion continued, however, generalities began to emerge.

25 Kaemon: So I'm guessing that if the vectors exceed the dimensions. Because the last one is four vectors, so I'm guessing that one's not going to.
26 Justin: Yeah, I'm guessing there's going to be no solution to that, either. Unless, the zero vector, but then that doesn't really count. At least to me, it doesn't count, I don't think that's fair.
27 Aziz: If more vectors. [writes on the group's whiteboard "If the number of vectors exceeds the number of dimensions, the set is linearly dependent"]
28 Abraham: If the vectors exceed the dimension, you're saying that it can't be linearly independent?

Episode 2: Whole class discussion on day 5 Part of this day's activity included discussing how the various small groups had completed the task from day 4. For example, Justin explained his group's rationale for how they determined there was "no solution" for a set of three linearly independent vectors in $\mathbf{R}^2$. The class discussed Justin's explanation for a few minutes, and many students spoke up in support or raised questions that they then considered together. After a student restated Justin's argument in his own words, the instructor revoiced it one last time and asked, "Does that lead us to anything we can say for another generalization at all?" She paused for a moment, and when no one spoke, she began to restate the class's prior discussion. As she was doing this, however, Abraham and Justin whispered to each other about the generalization that Aziz had written on their board (see 27).

29 Abraham: You should bring up your more vectors.
30 Justin: Than dimensions?
31 Abraham: Than dimensions. I was going to say more.
32 Justin: Go ahead, jump out there, speak, go, put yourself out there.
33 Abraham: I didn't come up with it, though.
34 Justin: You say it, and I'll back you up.

As the instructor completed her revoicing of Justin's argument regarding three vectors in $\mathbf{R}^2$, Justin raised his hand and offered his group's generalization to the whole class:

35 Justin: So using that, we decided that if you ever have more vectors than you have dimensions, it's always going to be linearly dependent.
36 Instructor: If you ever have more vectors than dimensions, it will always be linearly dependent. This course is going to get really confusing if we have a whole lot of pronouns, so let's restate that without some 'it's' and 'you's,' some formalism, say again.
37 Justin: If we have a set of vectors, and the number of vectors exceeds the dimensions that we're working in, then that set of vectors will always be linearly dependent. Is that clear?

Justin's statement in (35) is the first time this generalization, which the focus group had formed on day 4, was mentioned in whole class discussion. By the end of day 5, four generalizations surfaced in whole class discussion, one of which was, "If a set of vectors in $\mathbf{R}^n$ contains more than n vectors, then the set is linearly dependent" (which was labeled #4). The four generalizations were discussed to varying degrees in the whole class, and at the end of class, the instructor requested that students complete a reflection about which generalization they were most and least confident in and why. Both Giovanni and Abraham stated they were least confident in generalization #4. Giovanni wrote, "I think I'm confused with the definitions of linearly dependent and linearly independent," and Abraham wrote, "I've only seen a couple of examples and examples cannot be used to prove anything. Therefore, it seems like it might be right but it has not been completely justified. I'm not completely sold on this generalization."

Episode 3: Whole class discussion on day 6 Upon reviewing student reflections, the research team conjectured that the fourth generalization was the most problematic for students, so the teacher-researcher began class on day 6 discussing this statement.

38 Instructor: … So how about, can I have someone who understands #4 restate #4 in their own words, what do we mean by this generalization?
39 Justin: If you have more vectors than dimensions, you'll always be able to return to your original position.
40 Instructor: Could you say that louder? If you have more vectors than dimensions?
41 Justin: Then you can always return to your original position.
42 Instructor: Does that resonate with anyone else's way of thinking about this problem? Nate, can you say anything about the way you understand what #4 is about?
43 Nate: It's saying if you have, there's no other way of putting it, if there's more vectors than. I don't know another way to say it.
44 Instructor: That's fine. So that's the problem statement, now the question would be, how can we explain why that is true? Jerry, did your table get to talk about a reason why #4 is going to make sense to your table?
45 Jerry: Not really.

46 Instructor: Not really. Saul, how did your table talk about #4?
47 Saul: We were confused on how to prove it, we didn't know where to start, where to go with a proof.
48 Instructor: Okay, why don't we start with an example?

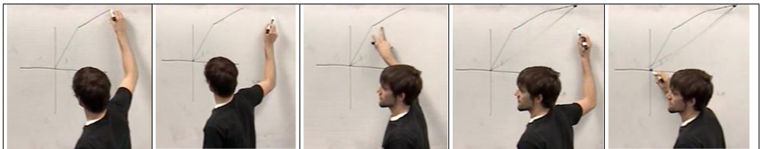
In utterance (39), Justin restated generalization #4 as "if you have more vectors than dimensions, you'll always be able to return to your original position," students commented on this interpretation, and the instructor elicited a justification for generalization #4. With little progress on justifying the generalization, the instructor had the class examine a specific case—that of three vectors in $\mathbf{R}^2$. After working in their small groups on the task, the class discussion involved many students sharing ideas, solution techniques, and justifications. For example, Lawson presented his group's justification shown in Fig. 6.

After Lawson gave his group's justification, Aziz offered the following:

50 Aziz: If two vectors span $\mathbf{R}^2$, if the third one is contained in $\mathbf{R}^2$, then you should be able to reach the origin back. But if the two vectors are multiples of each other, then it already makes the set linearly dependent. Them two making the span makes the third one be able to reach back to the origin.

As the discussion continued, the instructor eventually drew the class's attention back to generalization #4 for any m greater than n, not just when m is 3 and n is 2.

51 Instructor: The original question I was talking about, the generalization #4. So I don't want to beat this thing to death, but let's go back to this. We did a case of three vectors in $\mathbf{R}^2$, but what about four vectors in $\mathbf{R}^3$? I think that was also in the chart. So I think Table 4, you guys were the ones who came up with this generalization, can you guys say a little bit more in general how this makes sense to you, not just in the $\mathbf{R}^2$ case?
52 Justin: So if you start in any $\mathbf{R}^n$, and you just start with one vector and keep adding more. So let's do $\mathbf{R}^3$, just for an example. So we start with one vector. So either, we have two choices: the next vector we add can either be on the same line, which means it's already linearly dependent, so we don't want that, so we're going to put it off somewhere else. Now the span of that is a plane in three dimensions. So now we're going to add another vector in. Our third vector, now it can either be in that span or out of that span. And we want it to be linearly independent, so we're going to put it out of that span. But now that we have that going off of that plane, we just extended our span to all of $\mathbf{R}^3$. So our fourth vector, when we put it in, no matter where we put it, it's going to get us back home.



49 Graphically think about it. You say you have a vector over here, and then plus a vector that's not parallel to it	And we have just any other random vector, say it goes this way.	If you add any two nonparallel vectors	that at some point these vectors are going to come across this other vector, and hit it at a point.	So if that happens, then you can just take the last vector and multiply it by some scalar to get you back to the origin. That's how graphically we got it.

Fig. 6 Lawson's justification for why any three vectors in $\mathbf{R}^2$ are linearly dependent

Because just like in this case, we have to have the last one to get back home, we can get anywhere with those first three that we put in, but we have to have to have that fourth one to come back. And so it works like that in any dimension, because the more you, if you keep adding, eventually you're going to get the span of your dimensions, and then you're going to have that extra one bringing you back. Unless you have two vectors that are lying on the same line, then you won't have the span of all of your dimension, but it's negligible because those two will give you a linearly dependent set. Does that make sense?

After a brief discussion of $\mathbf{R}^{10}$, Aziz continued the explanation:

53 Aziz: I'd like to add on to this, if you're going from the plane to the volume of space, that's with three vectors in three dimensions. With the fourth one, you can cross all those dimensions back to the origin, that's what makes the fourth dimension, the fourth vector, it makes you able to go back to the beginning. Because each vector takes up a different point in space. You're going from the plane to the volume, gives you those three dimensions but you can't get back to origin, unless you have a fourth one that can cross all three.

54 Instructor: I think Aziz and Justin said a lot and it's all accurate, but I think it might have been a lot to take in, so take a minute with the person next to you, and try to restate what you heard from them, try to solidify it while you're thinking of it, then we'll move on.

In utterance (52), Justin built his explanation by choosing $\mathbf{R}^3$ as a generic vector space through which to explain his reasoning. He explained each possibility with the various combinations of vectors, starting with only one and continuing up to four. After he spoke about a fourth vector in $\mathbf{R}^3$, he changed the focus of his explanation and stated, "it would work like that in any dimension, because…eventually you're going to get the span of your dimensions, and then you're going to have that extra one bringing you back." Finally, Justin ended by stating his entire justification did not matter if two of the given vectors were "lying on the same line" because that already made the entire set of vectors linearly dependent.

Classroom mathematical practice analysis In this section, we address the question, what are the normative ways of reasoning that emerge in this classroom? By answering this question, we gain insight into the collective mathematical progress. To do so, we use Toulmin's model to analyze all whole class discussions, keeping track of how ideas function in the discourse and whether either criterion 1 or 2 is met. For illustrative purposes, we highlight two normative ways of reasoning that involved whole class discussions on days 5 and 6. We also bring in analysis of the whole class discussion on other days as needed in order to illustrate the two criteria.

First normative way of reasoning In a whole class discussion on day 5 (see episode 2), the class reviewed the generalizations from the previous day, including the following: For a given set of n vectors in $\mathbf{R}^m$, if $m < n$, the set must be linearly dependent. One of the members of the focus group offered an explanation (37) of what this means, and the Toulmin model for his explanation is shown in Fig. 7. The data offered to support the claim was grounded in the particular case of three vectors in $\mathbf{R}^2$.

As the class was almost over, further discussion of this generalization was taken up on day 6 (see episode 3) when the instructor wrote the generalization on the board and asked for

Data: *Using that* [the discussion that three vectors in $\mathbf{R}^2$ form linearly dependent sets] [Justin, 35]

Claim: If you ever have more vectors than you have dimensions, it's always going to be linearly dependent [Justin, 35]

Fig. 7 Argument 1: more vectors than dimensions

someone to restate the generalization in their own words. In (39), Justin restated the generalization as, "If you have more vectors than dimensions, you'll always be able to return to your original position." The instructor queried if this resonated with how others thought about the generalization and Nate said, "there is no other way of putting it" and the instructor responded, "That's fine. So that's the problem statement" (44). What we want to point out here is that equating linear dependence with being able "to return to your original position" is beyond justification. This contrasts with what occurred on day 3 (transcript omitted due to space constraints) however, in which there was an extensive discussion in which the class used the formal definition of linear dependence to justify why "returning to your original position" or "getting back home" was metaphorically synonymous with linear dependence. In other words, in previous class sessions, the connection between linear dependence and "getting back home" required justification (e.g., data, warrants, or backing), but on day 6, the need for such justification dropped off. This is precisely the empirical evidence needed to satisfy criterion 1, and hence, one normative way of reasoning that emerged in this classroom was the following: A set of vectors being linearly dependent means the same thing as being able to return to your original position.

Second normative way of reasoning Continuing with the whole class discussion on day 6, the instructor then pushed students to explain *why* the generalization is true. In terms of Toulmin's model, the instructor was requesting a warrant for the data-claim in Fig. 7. After querying several students (see utterances 42, 44, 46) for ideas on how to justify the statement (with little to no progress), the instructor transitioned the class to considering specific examples in $\mathbf{R}^2$. Specifically, she stated, "Let's think about why if we had three vectors in $\mathbf{R}^2$, that would mean I should always be able to get back home [i.e., the vectors are linear dependent]."

After discussing in their small groups, Lawson came to the front of the class to explain how his group approached the task. The structure of his explanation is shown in Fig. 8.

In argument 2 (Fig. 8), Lawson explains how he and his group reasoned that three vectors in $\mathbf{R}^2$ would be linearly dependent. In brief, their reasoning hinged on geometrically combining two nonparallel vectors so that they "come across" the third vector. The warrant that explains why this analysis is relevant is that one can then simply take a scalar multiple of the third vector and return to the origin. In (50), Aziz (who was not in Lawson's group)

Data: *Graphically think about it. You say you have a vector over here, and then plus a vector that's not parallel to it [draws two vectors tip-to-tail on the board]. And we have just any other random vector, say it goes this way. If you add any 2 nonparallel vectors that at some point these vectors are going to come across this other vector, and hit it at a point.* [Lawson, 49]

Claim: *If we had 3 vectors in $\boldsymbol{R}^2$, I should always be able to get back home*

Warrant: *So if that happens, then you can just take the last vector and multiply it by some scalar to get you back to the origin. That's how gr aphically we got it.* [Lawson, 49]

Fig. 8 Argument 2: on three nonparallel vectors in $\mathbf{R}^2$

summarized the argument by saying, “If two vectors span $\mathbf{R}^2$, if the third one is contained in $\mathbf{R}^2$, then you should be able to reach the origin back.”

The last relevant argument from day 6 occurred when the instructor asked someone from the focus group to justify their original generalization: If a set of vectors in $\mathbf{R}^n$ contains more than n vectors, then the set is linearly dependent. The Toulmin scheme for Justin and Aziz’s joint explanation is given in Fig. 9.

Justin built his explanation by choosing $\mathbf{R}^3$ as a generic vector space through which to explain his reasoning. His data were dense, going through each possibility with the various combinations of vectors, starting with only one and continuing up to four. After he spoke about a fourth vector in $\mathbf{R}^3$, he provided a warrant that his data supported his claim by stating, “it would work like that in any dimension, because…eventually you’re going to get the span of your dimensions, and then you’re going to have that extra one bringing you back.” Finally, Justin ended his justification with a qualifier that, in essence, his entire justification did not matter in the case that two of the given vectors were “lying on the same line” because that made the entire set of vectors linearly dependent.

Recall that criterion 2 for when a way of reasoning is functioning as-if shared is when elements of a previous argument shift position within subsequent arguments, indicating knowledge consolidation. On multiple instances during the remainder of the semester, the claim in argument 3 (If you have more vectors than dimensions, the set of vectors is linearly dependent) served as data for new claims. For instance, on day 9, Aziz claimed that a specific set of three vectors in $\mathbf{R}^2$ was linearly dependent based on the data that there were more vectors than dimensions. On day 20, the class investigated if transformations from $\mathbf{R}^m$ to $\mathbf{R}^n$ could be one-to-one and onto when $m<n$, $m=n$, and $m>n$. For brevity, we include the Toulmin scheme (Fig. 10) without supporting transcript; full analysis can be found in Wawro (2011).

Here Mitchell claimed that a transformation from $\mathbf{R}^m$ to $\mathbf{R}^n$ could not be one-to-one if m was less than n. His data for this claim, however, was “it’s just not possible.” The instructor

Qualifier: *Unless you have 2 vectors that are lying on the same line, then you won't have the span of all of your dimension, but it's negligible because those 2 will give you a linearly dependent set.* [Justin, 52]

Data: *So if you start in any* $\mathbf{R}^n$*, and you just start with 1 vector and keep adding more. So let's do* $\mathbf{R}^3$*, just for an example. So we start with 1 vector. So either, we have 2 choices: The next vector we add can either be on the same line, which means it's already linearly dependent, so we don't want that, so we're going to put it off somewhere else. Now the span of that is a plane in 3 dimensions. So now we're going to add another vector in. Our 3rd vector, now it can either be in that span or out of that span. And we want it to be linearly independent, so we're going to put it out of that span. But now that we have that going off of that plane, we just extended our span to all of* $\mathbf{R}^3$*. So our 4th vector, when we put it in, no matter where we put it, it's going to get us back home. Because just like in this case, we have to have the last one to get back home, we can get anywhere with those 1st 3 that we put in, but we have to have to have that 4th one to come back.* [Justin, 52]

Claim: If you have more vectors than dimensions, the set of vectors is linearly dependent.

Warrant: *And so it works like that in any dimension, because the more you, if you keep adding, eventually you're going to get the span of your dimensions, and then you're going to have that extra one bringing you back.* [Justin, 52]

Backing: *I'd like to add on to this, if you're going from the plane to the volume of space, that's with 3 vectors in 3 dimensions. With the 4th one, you can cross all those dimensions back to the origin, that's what makes the 4th dimension, the 4th vector, it makes you able to go back to the beginning. Because each vector takes up a different point in space. You're going from the plane to the volume, gives you those 3 dimensions but you can't get back to origin, unless you have a 4th one that can cross all 3.* [Aziz, 53]

Fig. 9 Argument 3: the generalization

Data1: *It's just not possible* [Mitchell]

Data2: having *more vectors than dimensions* means *it has to be linear dependent* [Abraham]

Claim: *You can't be more than 1-1 if m* is less than *n* [Mitchell]

Warrant: *In order for it to be 1-1, it has to be linear independent* [Abraham]

Fig. 10 Argument 4: using the generalization as data for a new claim

asked Abraham for a reason why it wouldn't be possible. Abraham's data for Mitchell's claim was that having more vectors than dimensions means the set has to be linearly dependent. He also provided a warrant, "In order for it to be 1–1, it has to be linear independent." In this argument, we see a shift in the function played by the statement "having more vectors than dimensions implies the vectors are linearly dependent." On day 6, this was a claim that needed to be justified. On days 9 and 20, we see that this statement functions as data for new claims, indicating that "having more vectors than dimension implies the vectors are linearly dependent" is a normative way of reasoning and hence functions as-if shared in this particular classroom community per criterion 2.

Mathematical conceptions analysis In this section, we address the question, what conceptions do individual students bring to bear in their mathematical work? Mathematical conceptions analysis may occur by analyzing the utterances of individual students in small group or whole class discussion, by analyzing student written work, and also by interviewing students outside of the classroom setting. It is beyond the scope of this paper to give a detailed analysis of individual student conceptions that makes use of all data sources, let alone all concepts in linear algebra. However, we highlight particular mathematical conceptions of span and linear (in)dependence that students brought to bear in episodes 1–3 as well as in the argument on day 20 that was highlighted in the section "Classroom mathematical practice analysis." For our mathematical conceptions analysis, we draw on Wawro and Plaxco's (2013) characterization of students' conceptions of span and linear (in)dependence in terms of travel, geometric, vector algebraic, and matrix algebraic.

By episode 1, the students had already spent 3 days of class developing notions of span and linear independence and dependence in a way that cultivated numerous conceptions. As an example, consider Justin's statements in utterances 1–5. He argued that three vectors in $\mathbf{R}^2$ would be dependent "no matter what … they don't have to be multiples." The reference to multiples indicates a vector algebraic conception in that it refers to the operation of scalar multiplication as a way to compare vectors. He went on to discuss what happens if two vectors are "not on the same line," invoking a geometric conception. He then concluded with language indicating a travel conception, "we can always get to a point where we can get back on the third vector."

Other students in the focus group also exhibited travel and vector algebraic conceptions in episode 1. In utterances 15–18, Abraham ("you can get anywhere"), Justin, and Giovanni ("and then take it back") each used travel language to articulate parts of the argument they were developing. In utterances 25–28, Kaemon, Aziz, and Abraham each generalized their ideas using vector algebraic language. For example, Abraham said, "If the vectors exceed the dimension, you're saying that it can't be linearly independent?" In this case, Abraham stated the problem in terms of a set of vectors without reference to a graph or traveling.

Students in the focus group often used the language of travel to formulate their ideas, but they could also state them more abstractly in terms of vectors. For example, when asked to state their group's generalization in the whole class in episode 2, Justin stated "if you ever have more vectors than you have dimensions, it's always going to be linearly dependent" (35), using vector algebraic language. In episode 3, when the teacher asked someone to restate the generalization in their own words, Justin stated, "If you have more vectors than dimensions, you'll always be able to return to your original position" (39) converting part of the sentence to a travel conception of linear dependence.

At least one student from another group was more geometrically focused. In (49), Lawson drew vectors on the board explaining, "Graphically think about it. You say you have a vector over here, and then plus a vector that's not parallel to it." He continued with other graphical language but also used vector algebraic language of adding vectors and "multiply it by some scalar." In addition, he expressed traveling conceptions that the vectors are going to "come across this other vector." He ended his white board drawing and explanation with, "That's how we graphically got it."

On day 20, the class investigated the notions of one-to-one and onto transformations and how those concepts were related to ones with which they were already familiar, such as span and linear independence. Mitchell used the symbolic language of transformations to claim that a transformation T: $\mathbf{R}^m \rightarrow \mathbf{R}^n$ could not be one-to-one if m was less than n. When the teacher asked Abraham why this was true he said, "Basically because you have more vectors than dimensions, then it has to be linear dependent, and so in order for it to be 1 to 1, it has to be linear independent." In doing so, he connected his vector algebraic way of talking about linear dependence and independence with the new symbolic language of linear transformations. He also implicitly brought to bear matrix algebraic ideas in that the vectors he refers to are the column vectors of a matrix representing that transformation.

Individual participation in mathematical activity In this section, we address the question, how do individual students contribute to the mathematical progress that occurs across small group and whole class settings? In the previous sections, we detailed a normative way of reasoning that emerged over the 3 days as well as different mathematical conceptions of span and linear dependence that students brought to bear in their work. The emergence of normative ways of reasoning and different conceptions of key ideas requires actors to generate and sustain the dialogue that shapes and is shaped by the emergent ways of reasoning and individual conceptions. We characterize the participants' roles using constructs from Krummheuer's framework of production design (author, relayer, ghostee, and spokesman) and recipient design (conversation partner, co-hearer, over-hearer, and eavesdropper).

In the small group work detailed in episode 1, Justin was the author of both the claim and a justification for any three vectors in $\mathbf{R}^2$ being linearly dependent. In line 1, he initially stated his claim. Aziz, who was the one writing the group's ideas on their white board that day, asked Justin for clarification (utterance 2). This statement positioned Aziz as a relayer of Justin's idea in (1); furthermore, Justin was Aziz's conversation partner for that statement, whereas the other members of the focus group were co-hearers. For the first portion of (3), in which Justin clarified his claim by saying, "For dependent, as long as you have three vectors," Aziz was his conversation partner. For the remainder of (3) and all of (5), though, when Justin clarified his claim from (1) and added justification for the claim (thus extending his role as an author), all other members of the focus group served as conversation partners.

Moving to the whole class discussion in episode 2, Justin presented an argument to the class justifying why any set of three vectors in $\mathbf{R}^2$ would be linearly dependent. This explanation was consistent with the one he gave on day 4 within small group discussion (see utterances 1–5 and 14–20). As such, we assign him the role of "author" within the production design category of our individual participation construct. Moreover, because Justin was at the front of the class addressing everyone, we assign the role of "conversation partner" to every class member.

The second portion of episode 2, in which Abraham encouraged Justin to share the generalization of "If the number of vectors exceeds the number of dimensions, the set is linearly dependent," is quite interesting with regards to the participation construct. In (29), Abraham told Justin "you should bring up *your* more vectors" (emphasis added), which implies that Abraham saw Justin as the author of that idea. In episode 1, however, both Aziz (68) and Kaemon (80) authored at least the seeds of this generalization. Indeed, the first time it appeared in its full form was when Aziz wrote it out on their group's whiteboard (27). Thus, the only production design roles that Justin could be assigned in episode 2 are spokesman or relayer. In (32), Justin pushed Abraham to share their group's generalization with the class (i.e., pushing him to be a relayer or spokesman), but Abraham resisted because he "didn't come up with it" (33). That is, Abraham was aware he was not an author of that idea and resisted what he seemed to assume would present him as such. Finally, in utterances 35 and 37, Justin took an opportunity to share his group's generalization with the whole class. The formulation of how he stated the generalization make Justin a spokesman rather than a relayer because he did not use the exact wording that was on his group's whiteboard.

Within episode 3, when the teacher asked for a volunteer to state generalization #4 in his or her own words, Justin was the first to speak. Because of the nature of the teacher's question (38), this positioned Justin to be a spokesman (39), and we posit that the teacher was his conversation partner and the rest of the class members were co-hearers. The teacher's next statement (40), however, positioned Nate as her conversation partner when she directly asked him if Justin's restatement resonated with him. Next, the teacher pushed for a justification as to why this statement might be true. She treated all members in the class as conversation partners by addressing them all and allowing time for any student to respond. When none did, she chose both Jerry and Saul as conversation partners, which placed the remaining members of the class as co-hearers. If either Jerry or Saul had provided a justification, they would have been considered an author within that production design.

When the class struggled with justifying generalization #4, the teacher had the class engage in exploring examples of three vectors in $\mathbf{R}^2$. In (49), Lawson presented his group's justification for why a set of vectors was linearly dependent. He concluded his explanation with, "that's how graphically we got it," which indicates that Lawson felt some sense of ownership of the justification. As such, we assign the role of author to Lawson for this utterance. Again, because he was addressing the entire class during his utterance, we assign the role of conversation partner to the remaining class members.

Within (50), Aziz took on the role of author. Utterance (50), at a quick glance, seems similar to Justin's explanation in small group work on day 4 (see 14–20); however, Aziz's utterance was more focused on the relationship between span and linear dependence than Justin's or Lawson's arguments were. As such, we assign the role of author to him. This type of progression in explanation also resonates with Justin's explanation in (52), which moved beyond the case of three vectors in $\mathbf{R}^2$ to a generic explanation that generalized from $\mathbf{R}^3$ to any dimension. This justification was original to Justin within that classroom community, so he is considered an author. Without prompting, Aziz stated, "I'd like to add on to this …" (52); this action of adding on to a previous utterance to express a new idea is consistent with the role of ghostee. Finally, because both Justin and Aziz addressed the entire class, the teacher and other students are considered conversation partners.

Disciplinary practice In this section, we address the question, what is the mathematical progress of the classroom community in terms of the disciplinary practices of mathematics? These practices may include defining, algorithmatizing, symbolizing, and theoremizing. In this paper, we highlight the classroom community's mathematical progress in terms of theoremizing.

Theoremizing consists of activity related to both conjecturing and proving. Theoremizing includes aspects of conjecturing, such as noticing relationships between mathematical entities and proposing statements that capture those relationships. It also consists of activities involving making arguments and justifications for or against proposed statements. The level of rigor of these arguments can vary in form depending on the audience. For example, theoremizing in the linear algebra example foregrounds the argumentation necessary to explore the validity of a proposed statement while backgrounding the care and detail involved in creating a rigorous, formal proof.

An analysis of the linear algebra data in terms of the disciplinary practice of theoremizing reveals the following aspects of students' mathematical work: engaging in a mathematical setting, observing relationships, clarifying and refining stated relationships, arguing for (or against) claims, generalizing, and justifying generalizations. Taken as a whole, these various activities progress from work with particular examples in a particular setting to creating and justifying generalized statements and hence characterizes the mathematical progress of the classroom community in terms of the disciplinary practice of theoremizing.

To initiate theoremizing, students engaged in a problem situation in which they constructed examples or struggled to construct examples of sets of vectors with certain properties. As students began to realize under what conditions these examples are or are not possible, they made initial conjectures that eventually led to theorem-like statements. For example, in episode 1, when discussing three vectors in $\mathbf{R}^2$, in utterance (1), Justin said, "The way I think of it … For linear dependent, it can be any three you want." Similarly in (8), Aziz said, "So this linearly independent, it should be no solution, right?" This initial conjecture was put forward to the group as a genuine conjecture (i.e., the truth status of the statement is unknown). As the conversation continued, students asked for clarification or expressed agreement or disagreement. For example, after Justin's initial conjecture, Aziz asked in (2), "For linearly dependent or independent?" After Aziz's initial conjecture, Giovanni pondered, "No, there would be a solution." Students then clarified or refined their conjectures and constructed initial arguments for their claims. In utterances (14)–(16), Justin argued for his claim referring back to the magic carpet problem in which the two given vectors enabled one to "go anywhere in this entire plane."

Beyond the initial conjecture and argumentation referring to a specific setting, for example, three vectors in $\mathbf{R}^2$, theoremizing includes students moving to generalize their initial conjecture. For example, Kaemon referred to the example of a different setting, four vectors in $\mathbf{R}^3$, and Aziz wrote a generalization to any in $\mathbf{R}^m$ on the white board, "If the number of vectors exceeds the number of dimensions, the set is linearly dependent." During day 5, the focus group's argument for the conjecture in $\mathbf{R}^2$ and the generalization to any in $\mathbf{R}^m$ were presented to the whole class with the latter being labeled as generalization #4 by the instructor. In individual reflections written at the end of day 5, both Abraham and Giovanni from the focus group expressed uncertainty about generalization #4. Abraham was concerned about only seeing it for a few examples and stated, "it might be right but it has not been completely justified." Giovanni's concern with generalization #4 focused more on his own understanding of the related issues of linear dependence and independence. Later, on day 6, Justin presented a more comprehensive justification for the generalized conjecture. In this way, we see that theoremizing continued through a process of increased rigor as students pushed to understand why a theorem is true or to be fully convinced of its veracity.

4 Coordinating analyses

In the previous sections, we illustrated the types of results that are possible within each of the four constructs of the expanded interpretive framework. While each of the four constructs are informative in their own right, we add power to this analysis with a discussion of first steps toward combining and coordinating across the four analyses. Indeed, Prediger et al. (2008) argue that such networking strategies and methods are sorely needed, and they describe the benefits that such a coordination or combination affords. For instance, they state that "developing empirical studies which allow connecting theoretical approaches" may further the scientific discipline of mathematics education research by allowing us "to gain an increasing explanatory, descriptive, or prescriptive power" (p. 169).

At a minimum, our four constructs provide an opportunity to analyze the same phenomenon from four distinct points of view—as if one were gazing at the same object from various vantage points in order to capture many qualitative nuances about the object. To illustrate this, we consider a portion of the transcript from episode 3 and discuss how that same data were analyzed with each of the four constructs. This type of coordination is consistent with the call of Prediger et al. (2008) for "studying the same phenomenon or common piece of data from different theoretical perspectives" (p. 173)—as a method for deepening insights on the same phenomenon.

In (38), the teacher drew attention to generalization #4, which had been developed as a conjecture during small group work the previous day. The creation of this generalization is an example of the disciplinary practice of theoremizing because the students were observing mathematical relationships and creating conjectures regarding those relationships. As the teacher asked students to unpack the meaning of this generalization, Justin (39) offered, "If you have more vectors than dimensions, you'll always be able to return to your original position", and Nate (43) agreed. Within the classroom mathematical practice analysis, the first normative way of reasoning detailed was, "A set of vectors being linearly dependent means the same thing as being able to return to your original position." When noticing the two collective level analyses in conjunction with each other, we see that as students engaged in the mathematical work of justifying a generalization (one aspect of what constitutes theoremizing), a previously established normative way of reasoning (that linear dependence means being able to get back to your original position) was employed in the service of that justifying activity. From the individual mathematical conception construct, Justin's rewording (39) of the generalization as "being able to return to your original position" was consistent with the travel conception of linear dependence because it captured notions of "getting to" or "moving to" locations in the vector space. Within the construct of individual participation, we saw that the teacher's question in (38) positioned Justin to be a spokesman (39), and her request to have Nate comment (40) positioned him as a conversation partner.

In addition to using various combinations of the four constructs to more fully interpret students' mathematical progress, there exist multiple ways in which coordination across the four constructs is possible. For instance, one could choose an individual student within the classroom community and trace his/her utterances for the ways in which they contributed to the emergence of various normative ways of reasoning and/or disciplinary practices. Alternatively, when considering a normative way of reasoning, a researcher could investigate who the various individual students are that are offering the claims, data, warrants, and backing in the Toulmin schemes used to document the normative way of reasoning. How do those contributions coordinate with those students' production design roles within the individual participation construct? For instance, does a student ever utilize an utterance that a different student authored as data for a new claim that he is authoring, and in what ways may that capture or be distinct from other students' individual mathematical conceptions? We also imagine ways to coordinate across the two individual constructs as well as across the two collective constructs. For example, how do patterns over time in how student participation in class sessions relate to growth in their mathematical conceptions? Are different participation patterns correlated with different mathematical growth trajectories? In what ways are particular classroom mathematical practices consistent (or even inconsistent) with various disciplinary practices? Finally, future research could take up more directly the role of the teacher in relation to the four constructs.

We anticipate that future work will more carefully delineate methodological steps needed to carry out the various ways in which analyses using the different combinations of the four constructs can be coordinated. Indeed, we view this report as a first step in developing a more robust theoretical-methodological approach to analyzing individual and collective mathematical progress.

5 Concluding remarks

This paper lays out a set of four constructs that allows researchers to analyze individual and collective mathematical progress. These constructs come out of and expand Cobb and Yackel's (1996) interpretive framework. Our expansion allows for further nuance into both collective and individual mathematical progress as well as more possibilities in comparing across individual and collective perspectives. The parsing of the collective into classroom mathematical practices and discipline practices allows for tracking both the progress of normative ways of reasoning about specific mathematics, such as notions of linear independence, and the progress of more general practices of the discipline, such as theoremizing. The two individual constructs focus on the mathematical progress of individuals. This progress is parsed into two categories. One focuses on how individuals participate in creating and interacting within the collective practices, while the other individual construct focuses on the mathematical conceptions of individual students.

The expanded interpretative framework has the potential to be a valuable pragmatic tool for curriculum developers and teacher-researchers. One aspect of classroom-based design research is that analysis and instructional design should feed into and inform each other. Drawing on the work of Cobb (2003), Stephan and Akyuz (2012) suggest that "one way to assess the viability of an instructional sequence is to document both the classroom mathematical practices and individuals' ways of participating in and contributing to them" (p. 439). For instance, Stephan and Akyuz found that analyzing classroom mathematical practices provided valuable feedback toward refining an instructional theory on integers. We further posit that consideration of mathematical progress through the construct of disciplinary practices and individual mathematical conceptions would also be valuable to design-based research. In Wawro et al. (2013), we provide detail regarding ways that research on students' mathematical progress informed the creation and refinement of an instructional sequence in linear algebra that was part of the larger research project from which we drew data for this report. For instance, during the creation stage of a task sequence, we considered how engagement in a specific disciplinary practice (defining) might complement our learning goals for a task sequence. Also, in addition to consulting the literature on students' mathematical conceptions during the creation stage of a task sequence, we collected data on individual student thinking (via videorecording class sessions and conducting individual interviews) to assess the extent to which the task sequence was conducive to actualizing student learning goals. The expanded interpretative framework is beginning to aid us in combining analyses in a coordinated way, but clearly more work is on the horizon.

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