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Susan D. Nickerson & Ian Whitacre

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Susan D. Nickerson and Ian Whitacre

San Diego State University

Gravemeijer's (1999, 2004) construct of a *local instruction theory* suggests a means of offering teachers a framework of reference for designing and engaging students in a set of sequenced, exemplary instructional activities that support students' mathematical development for a focused concept. In this paper we offer a local instruction theory to guide the design of a set of instructional activities in support of the development of number sense. We make explicit the goals, assumptions, underlying rationale, and related instructional activities and provide examples from a mathematics content course for preservice elementary teachers. In this way, we contribute to an elaboration of the construct of local instruction theory.

Externally-developed local instruction theories are indispensable for reform mathematics education (Gravemeijer, 2004, p. 108). Gravemeijer's (1999, 2004) construct of a *local instruction theory*, which was developed in the context of design research, suggests a means of offering teachers a framework of reference for designing and engaging students in a set of sequenced, exemplary instructional activities that support students' mathematical development of a focused concept. Gravemeijer (2004) described a local instruction theory with regards to learning goals, instructional activities, and the role of tools and imagery in the envisioned learning route. Gravemeijer illustrated how Realistic Mathematics Education can be used to develop a local instruction theory for an instructional sequence for addition and subtraction up to 100. Like Gravemeijer, we see a local instruction theory as indispensable to the design of instruction.

The paucity of examples in the literature suggested to us that the field might benefit from further illustration and elaboration of the construct of local instruction theory. In this paper, we offer a local instruction theory in support of the development of number sense. This theory evolved out of a review of the literature, together with the results of a classroom teaching experiment, and subsequent iterations of teaching the course and fleshing out our analysis. We first discuss the differences between local instruction theory and hypothetical learning trajectory (Gravemeijer, 1999, 2004; Simon, 1995); next we discuss the literature on number sense. Within the context of our design research in a class for preservice elementary teachers, we provide an example of a local instruction theory to illustrate the relationship between the local instruction theory and resulting hypothetical learning trajectories. A colleague asked us whether we see a local instruction theory as being made up of a sequence of hypothetical learning trajectories, like

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Correspondence should be sent to Susan D. Nickerson, Department of Mathematics and Computer Sciences, San Diego State University, San Diego, CA 92182. E-mail: snickers@sunstroke.sdsu.edu

so many bricks in a row. We do not. Rather, we see a local instruction theory (LIT) as undergirding and informing particular hypothetical learning trajectories (HLTs).

A *hypothetical learning trajectory* consists of learning goals for students, planned instructional activities, and a conjectured learning process in which the teacher anticipates the collective mathematical development of the classroom community and how students' understanding might evolve as they participate in the learning activities of the classroom community (Cobb, 2000; Cobb & Bowers, 1999; Simon, 1995). Hypothetical learning trajectories have been described for a number of teaching experiments in diverse areas, such as linear measurement, equivalence of fractions, and statistics (cf. Gravemeijer, 2004; Jones et al., 2001; Simon & Tzur, 2004; Stephan, Bowers, Cobb, & Gravemeijer, 2003).

The construct of a hypothetical learning trajectory allows for a range of grain sizes. Simon and Tzur (2004) described an HLT in support of students' understanding the quantitative relationships between a fraction and an equivalent fraction whose denominator is a multiple of the original fraction. Gravemeijer, Bowers, and Stephan (2003) described a much broader HLT for the design of early-grade linear measurement instruction in a teaching experiment with two sequences: linear measurement and flexible arithmetic. Our own notion of HLT involves a smaller grain size, comparable to that of Simon and Tzur (2004). Some of the difficulties that people have distinguishing between HLT and LIT may stem from this issue of grain size.

According to Gravemeijer (2004), a *local instruction theory* refers to "the description of, and rationale for, the envisioned learning route as it relates to a set of instructional activities for a specific topic" (p. 107). In Gravemeijer's (1999) view, there are two important differences between an LIT and an HLT: (1) an HLT deals with a small number of instructional activities, while an LIT encompasses a whole sequence; (2) HLTs are envisioned within the setting of a particular classroom, whereas an LIT comprises a framework, which informs the development of HLTs for particular instructional settings. Thus, the distinction between LIT and HLT is two-fold. One distinction is the *duration* of the learning process and the other is the *situatedness* in a particular classroom.

In this report, we describe an LIT, clearly distinguishable from an HLT, that both informed and was informed by a teaching experiment. The aim of the classroom teaching experiment was to foster preservice teachers' development of number sense with a particular focus on *flexible* mental computation (Heirdsfield & Cooper, 2004) and computational estimation.

BACKGROUND

We conducted a classroom teaching experiment in a mathematics course with a focus on number and operations. A goal of the course is that students develop number sense. According to Reys and Yang (1998),

> Number sense refers to a person's general understanding of number and operations. It also includes the ability and inclination to use this understanding in flexible ways to make mathematics judgments and to develop useful strategies for handling numbers and operations. (p. 225)

The development of number sense is a widely accepted goal of mathematics instruction for K-12 students (e.g., Blöte, Klein, & Beishuizen, 2000; National Council of Teachers of Mathematics [NCTM], 2000; Reys, Reys, McIntosh, Emanuelsson, Johansson, & Yang, 1999). In order for

K-12 teachers to facilitate their students' development of number sense, they must exhibit number sense themselves. If we value instructional approaches in which students are asked to reason and communicate their reasoning and have an expectation that teachers capitalize on these opportunities, teachers need the ability to interpret students' reasoning, recognize number sense-based approaches, and respond appropriately (Carpenter, Fennema, & Franke, 1996; National Research Council, 2001; Sowder, 1992, 2007).

Although there are various components of number sense, many of its characteristics are reflected in mental computation and computational estimation (Blöte et al., 2000; Gliner, 1991; LeFevre, Greenham, & Waheed, 1993; Reys, Reys, Nohda, & Emori, 1995; Reys et al., 1999). Flexible mental computation and computational estimation rely on number sense (Sowder, 1992). Number sense can be described as broadly as good intuition about numbers and their relationships (Howden, 1989), but in the classroom teaching experiment we describe here, our instructional focus was on sense-making within a framework of number relations as evidenced by flexible mental computation and computational estimation (Gravemeijer, 2004; Stephan et al., 2003).

A conjectured local instruction theory provided a framework for the integration of support for mental computation and computational estimation into the unit-specific instructional sequences. We found that students by the end of our teaching experiment developed greater flexibility with regard to addition, subtraction, and multiplication of whole numbers, and their understandings of these operations became less bound to the standard algorithms (Whitacre, 2007; Whitacre & Nickerson, 2010). In this paper, we summarize our LIT and illustrate it with examples from our classroom teaching experiment. We see a contribution in illustrating how the LIT provided a framework of reference in the development of our course for preservice teachers.

THEORETICAL PERSPECTIVE

We take a perspective on learning in mathematics classrooms as a constructive process, which takes place as individual students participate in and contribute to the norms and practices of the classroom community (Cobb & Yackel, 1996; Lave & Wenger, 1991). As such, we are interested in understanding individual students' thinking as well as classroom participation structure, social norms, socio-mathematical norms, and practices (Cobb & Yackel, 1996). In adopting such a perspective, we seek to understand a learner's ability to play a role, including his or her ability to anticipate, sense what is feasible within a context, and improvise or adapt accordingly (Hanks, 1991).

As we conceptualized our instructional design, we found it useful to frame the goals and envisioned learning route in terms of Greeno's (1991) environment metaphor. From this perspective, a person's knowledge and activities are seen as situated within a particular domain or environment. Knowing in an environment consists of knowing how to get around, where to find things, and how to use them. Whether the environment is a physical space or a conceptual domain, knowing one's way around requires relating concepts and solving problems.

Greeno (1991) suggested that a key aspect of knowing from this perspective was a person's ability to construct and reason with mental models. If the environment in question is one's physical world, a mental model might look like a visual representation of a space that someone creates in his or her mind to facilitate thinking about maneuvering within that space. Where the

environment is that of numbers and quantities, a person might construct a mental model of a problem situation in order to reason about solving the problem (Greeno, 1991). This kind of problem-solving activity is dissimilar to the application of an algorithm. Greeno's metaphor enables us to describe activity in a domain involving the use of resources and (mental) models— knowing what tools may be available and knowing how to use them. It also helps us to think about attunement to constraints and affordances of these tools.

We were further guided in our instructional design for this class by Realistic Mathematics Education (RME) and its instructional design heuristics: (1) sequences must be experientially real, (2) students should be guided to reinvent significant mathematics for themselves wherein (3) students and the teacher develop models of informal activity, which become models for mathematical reasoning (Gravemeijer, 1999, 2004; Stephan et al., 2003). We envisioned learning as precipitated by a problem-centered, inquiry-based classroom (Richards, 1991).

Experientially real instructional activities are those that facilitate students' engagement in personally meaningful activity. Although the activities may include mathematical modeling of real world scenarios, both real-world story problems and problems devoid of a story context can constitute experientially real situations for students. In RME instruction, student-generated ways of interpreting and organizing their mathematical activities are models of their informal activity that should be supported through local shifts to become models for more formal mathematical reasoning (Gravemeijer, 1999). Thus, students' modeling activities enable them to develop new tools. *Activity* here refers to both mental activity and physical activity with graphs, equations, and other representations (Rasmussen & Blumenfeld, 2007).

We conduct classroom-based research and design instruction in the form of *design research* or *developmental research* (Cobb & Bowers, 1999; Gravemeijer, 1994, 2009). Design research is characterized by the reflexive relationship between instructional design and classroom-based research (Stephan et al., 2003). In day-to-day practice, a teacher's interpretation of classroom activity affects lesson planning for future classes, and a hypothetical learning trajectory is modified by an understanding of the actual learning trajectory. Within this reflexive, iterative process, an LIT informs the development and revision of HLTs (see Figure 1). A retrospective analysis of actual learning trajectories informs the revision of HLTs in subsequent classroom teaching experiments and may lead to revisions to the underlying LIT.

The Design Research Cycle (micro)
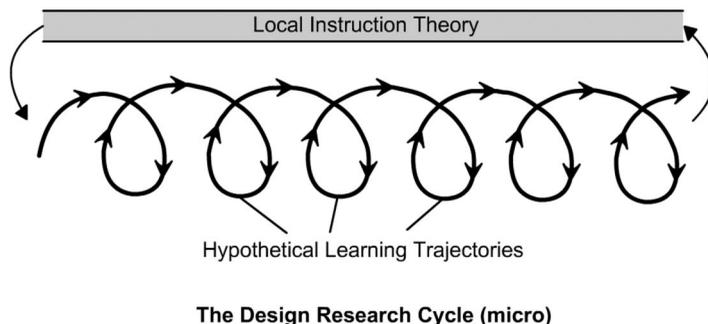
FIGURE 1 LIT as a framework for the development of HLTs.
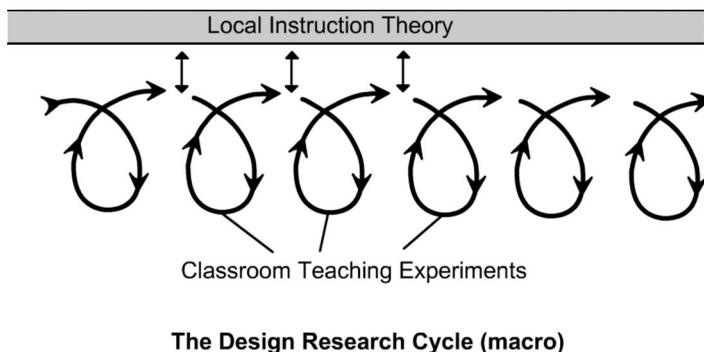
FIGURE 2 The reflexive relationship between LIT and CTEs.

The cyclic activity can be more broadly considered as a sequence of teaching experiments in which previously conducted classroom-teaching experiments (and more broadly, non-classroom teaching research such as interviews) informed a *conjectured* local instruction theory (see Figure 2). An LIT undergirds the development of HLTs for particular classrooms. As the classroom teaching experiments are undertaken, the LIT is empirically tested in multiple settings. This can inform further refinement of the LIT and subsequent development of HLTs.

We began the planning of our classroom teaching experiment and our development of the LIT by first reviewing how researchers describe the characteristics of individuals with good number sense and their recommendations for pedagogy in support of number sense.

TEACHING FOR NUMBER SENSE

Markovits and Sowder (1994) noted that manifestations of number sense stem from a disposition to make sense of numerical situations. *Sense-making* in this context refers to answering for oneself in some meaningful way whether or not a problem-solving approach under consideration is a valid one. Others have described those that have number sense as comfortable and confident with numbers and possessing good intuition about numbers (e.g., Heirdsfield & Cooper, 2002; Howden, 1989; Sowder, 1992).

Greeno (1991) addressed *flexibility* as a characteristic of number sense, giving examples of flexible computation, including one from Hope and Sherill (1987) of the use of the associative property to convert 25×48 to 100×12. Flexibility seems to be the characteristic most commonly associated with skill at mental math (cf. Blöte et al., 2000; Carraher, Carraher, & Schliemann, 1987; Hope & Sherill, 1987; Markovits & Sowder, 1994; Reys et al., 1995; Reys, Rybolt, Bestgen, & Wyatt, 1982; Sowder, 1992). The flexibility exhibited by skilled mental calculators stems from the availability to these individuals of a variety of calculative strategies as well as the possibility of creating new strategies. As Reys and Yang (1998) suggested, number sense includes both the ability and the inclination to use a variety of strategies. The literature describes a number of different strategies used by skilled mental calculators and estimators (Blöte et al., 2000; Hanson & Hogan, 2000; Heirdsfield & Cooper, 2004).

For example, 1010 is the Dutch reference to an addition or subtraction strategy in which numbers are decomposed into tens and ones, which are processed separately and put back together (Blöte et al., 2000).

Deciding how to perform a particular mental computation or estimation requires not only that students have a repertoire of strategies but also that they can choose one that seems appropriate to the problem at hand (Heirdsfield & Cooper, 2004). Individuals with good number sense exhibit facility with a variety of strategies, the selection of which depends on the particular numbers involved. Relative ease and efficiency are considerations in that regard (Hope & Sherill, 1987), and the act of making a choice necessitates some initial reflection (Markovits & Sowder, 1994). This characteristic is one of problem solvers in general, and Schoenfeld (1992) referred to it as *planning*. There are also manifestations in mental math of monitoring and assessing progress as well as acting in response to assessments of online progress (Schoenfeld, 1992). In sum, people who have good number sense tend to exhibit the following characteristics when performing mental math: a sense-making approach, flexibility, planning, and control (Carraher et al., 1987; Markovits & Sowder, 1994; Reys et al., 1982; Schoenfeld, 1989; Sowder, 1992).

While it is helpful for instructors to be aware of the characteristics of number sense and of a variety of mental computation and estimation strategies, does this imply that one should teach to the symptoms of number sense? Some researchers advocate explicit exploration of mental algorithms and suggest that less capable students may need to receive structured instruction in strategies. But a study in which researchers observed classrooms that differed in didactical sequence and instructional strategies suggests that elementary students may not attempt to invent computational strategies if taught a specific method to mentally compute with multi-digit numbers (Varol & Farran, 2007). Many researchers agree that explicit, direct instruction in the use of specific strategies for mental computation or estimation may actually be counter-productive (Greeno, 1991; McIntosh, 1998; Schoenfeld, 1992; Sowder, 1992). One of the central suggestions is that specific mental computation and computational estimation strategies should not be taught directly, even as mental computation is given greater priority in teaching (Case & Sowder, 2000; Heirdsfield & Cooper, 2004; McIntosh, 1998; Reys et al., 1982; Sowder, 1992). This is due to the fact that flexibility, planning, and control are best fostered when students can develop the habit of making choices. As Schoenfeld (1992) pointed out, when strategies are taught directly "they are no longer heuristics in Polya's sense; they are mere algorithms." (p. 354).

McIntosh (1998) wrote that, given time and opportunity, students will invent novel strategies. He suggested that after students have been asked to mentally compute, they should have the opportunity to share and discuss their strategies, highlighting the fact that there may be many valid ways to solve the same problem. Teachers can take advantage of students' spontaneous interest in each other's strategies by encouraging students to try out the strategies shared by peers. Finally, he encouraged an experience of doing mental math in class that is non-threatening and pleasurable. The recommendation that students be given opportunities to invent their own strategies is consistent with the tenets of RME. Ultimately, people need to develop their own lay of the land that involves establishing personal landmarks and getting lost and finding their own way back to those landmarks (Greeno, 1991). Our study used the suggestions of McIntosh (1998) and others as a starting point for pedagogy aimed at the development of number sense.

OVERVIEW

A local instruction theory describes goals, envisioned learning route(s), and instructional activities or plans of action based on underlying assumptions about teaching and learning. The local instruction theory described here is not designed for a particular classroom but it is with a particular classroom culture in mind. We will begin by describing this classroom culture, delineating our goals and envisioned learning routes with rationales for each. We will share some instructional activities not with the intention of offering a portable instructional sequence but, instead, with the idea that the LIT can serve as a framework of reference for others developing HLTs for their own students in classes that have a goal of fostering number sense development.

Classroom Culture

Realizing an inquiry-based classroom requires particular social norms, such as the need to explain one's reasoning and an accompanying expectation that one attempts to make sense of explanations given by others. In addition to explanations, students provide justifications in order to help others make sense. Researchers further describe the importance of an inquiry-based classroom culture wherein students discuss whether a strategy is reasonable, identify its weaknesses and then further strengthen their arguments by considering others' perspectives (cf. Bowers & Nickerson, 2001; Kazemi & Stipek, 2001; Rasmussen & Marrongelle, 2006; Wood, Williams, & McNeal, 2006). Indeed, central to our goals was the need to build intellectual autonomy wherein students have a means of judging the efficacy of strategies (Yackel & Cobb, 1996). Furthermore, the classroom culture should be one in which the teacher attends to the qualitatively distinct ways in which individual students participate and views the students' distinct ways of solving problems as resources on which the teacher and the students can capitalize (Cobb & Bowers, 1999).

Note that we are not suggesting that the classroom culture is a prerequisite condition that must exist before students can begin to develop number sense. Instead, the instructional activities are designed to contribute to the development of these norms. The activities and the norms are interactively constructed. The local instruction theory offers one way in which the classroom culture may be constituted in service of students' development of number sense.

Goals

Our local instruction theory is organized in terms of three major learning goals for students. First, students will capitalize on opportunities to use number-sensible strategies for problem-solving situations both inside and outside the classroom. We define a *number-sensible strategy* as an approach that a student chooses from among a set of possible approaches—the steps of which are meaningful to the individual. A number-sensible strategy characterizes the manner in which the student approaches the task from the perspective of the student. The number-sensible strategy is in contrast to an approach that is algorithmic or nonstrategic from the student's perspective.

Second, students will draw on deep, connected knowledge of number and operations to develop a repertoire of number-sensible strategies. We characterize this deep, connected knowledge of number, in terms of a framework of number relations, as a number space in which a

student understands properties of distance, nearness, and neighborhoods (Lorenz, 1997 cited in Fosnot & Dolk, 2002). In solving problems, students draw on knowledge of the terrain of this number space to select tools. Greeno's (1991) environment metaphor enables us to think about strategies as tools students select based on the particular constraints and affordances of a problem that are salient to the student.

Third, students will reason with models to build on their understanding and flexibly use number-sensible strategies. Our goals would be that students would come to recognize opportunities for solution strategies based in number sense, develop many ways to think about number and operations, and flexibly draw on a repertoire of meaningful strategies. In other words, our three goals related to developing students' sense-making, planning and control, and flexibility—characteristics of individuals with good number sense. One of the most important aspects of an LIT is to articulate assumptions related to each of the previously established goals.

ELEMENTS OF AN LIT

For an LIT to be useful to teachers as a framework of reference, it must be responsive to the socially situated nature of a classroom. Because planning HLTs requires anticipation of how learning may develop around instructional activities, an LIT must not only articulate goals but must also draw attention to assumptions about the mathematical preparation of students. A second important aspect is the rationale for the instructional activities as well as the norms to be negotiated in support of the goals. As part of an elaboration of the goals around which our LIT is organized, we discuss assumptions, envisioned learning routes for instructional activities, and rationales for their inclusion. We also illustrate with vignettes from the teaching experiment how the activities for development of number sense were integrated into our content course for preservice teachers. As we discuss our class in what follows, the students referred to are preservice elementary teachers and the teacher referenced is a mathematics teacher educator.

Tables 1–3 each summarize a goal's envisioned learning route and associated activities. Each table refers to one of the goals of the course. In the first row of Table 1, you see the description of the envisioned learning route for Goal 1. The supportive sequence of instructional activities is described across the second row. Reading the two rows in columns from left to right reveals how the activities and learning route unfold. Table 2 pertains to Goal 2; Table 3 pertains to Goal 3. However, the instructional goals were not linearly enacted. As the following discussion suggests, the goals are pursued somewhat in parallel. As students start to share nonstandard strategies (Goal 1), the class negotiates records of the strategies (Goal 2) and they are asked to reason about whether and how the strategies make sense (Goal 3) (see Figure 3).

Goal 1: Students Capitalize on Opportunities to Use Number-Sensible Strategies

Assumptions

Based on research and our experience teaching other preservice elementary school teachers, we anticipated that when these college students were engaging in computation and estimation, they would initially exhibit an over-reliance on standard algorithms and standard estimation strategies. We anticipated that students would approach mental computation tasks with limited

TABLE 1
Learning Route and Instructional Activities for Goal 1

Goal 1: Envisioned Learning Route	Many students initially rely on standard algorithms	Students use their own nonstandard strategies	Students solve problems mentally in a variety of contexts	Students capitalize on opportunities to use number-sensible strategies
Goal 1: Instructional Activities	Instructor identifies and engineers opportunities for mental computation	Students are invited to use mental computation and to reason quantitatively	Students are invited to carry sense making to solutions with nonstandard strategies	

TABLE 2
Learning Route and Instructional Activities for Goal 2

Goal 2: Envisioned Learning Route	Students may initially name strategies in ways tied to specific examples		Students name strategies according to essential characteristics	Students develop a repertoire of number-sensible strategies
Goal 2: Instructional Activities	Instructor anticipates the nonstandard strategies students might use	Class negotiates records of strategies and initiates practice of naming	Instructor and students negotiate difference and relative efficacy	Instructor and students make strategies objects of discourse

TABLE 3
Learning Route and Instructional Activities for Goal 3

Goal 3: Envisioned Learning Route		Students begin to justify strategies meaningfully		Students use models-of strategies to make sense of nonstandard strategies		Students develop the ability to reason with models
Goal 3: Instructional Activities	Instructor anticipates powerful models for reasoning	Students are asked to explain why the strategies make sense	Instructor and students negotiate use of models of shared strategies		Students use familiar models-of as models for reasoning	
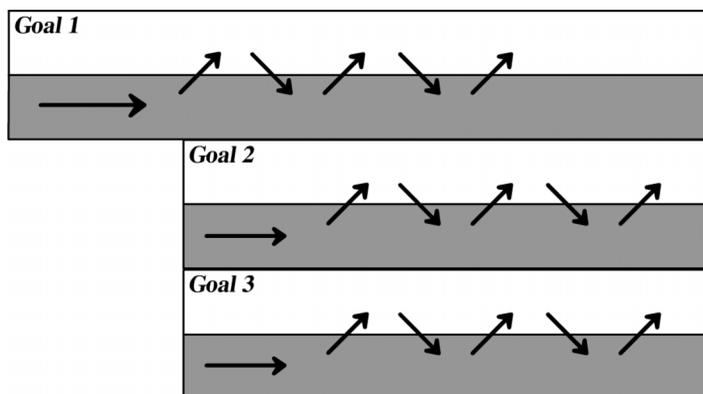
FIGURE 3   Nonlinear path of enacted goals.

options, many using the mental analogue of a standard algorithm (Hope & Sherrill, 1987; Markovits & Sowder, 1994; Reys et al., 1995). Likewise, we anticipated that students who had been taught rounding algorithms would likely approach computational estimation problems with limited, standard ways of rounding and without consideration of magnitude or the use of other benchmark numbers; other students might round the answer after computing (Gliner, 1991; Hanson & Hogan, 2000; Levine, 1982). These algorithms were likely to be utilized without sense-making, in terms of considering the effect of the rounding on the magnitude of the answer.

Envisioned Learning Route and Rationale

In order to support our first goal of students capitalizing on opportunities to use number sensible strategies, it was essential when planning for instruction that the instructor identified opportunities within the curriculum to invite and model the use of number-sensible strategies both for computation and estimation. The students would be invited to use mental computation and computational estimation in problem solving throughout the course, as fit with the particular content of the instructional unit. We wanted students to experience authentic opportunities to use mental computation and computational estimation productively in concert with developing a disposition toward mathematics as a sense-making endeavor as well as confidence in working with numbers. We saw the instructor as a role model for identifying opportunities to use number-sensible computation and estimation strategies as a natural and practical aspect of mathematical activity (Lunenberg, Korthagen, & Swennen, 2007). The instructor needed to guide students in establishing a social norm of sharing sense-based approaches to problem solving. We expected that as students participated in a classroom with a *collective orientation* (Bowers & Nickerson, 2001) toward making sense of numbers, with growing knowledge of the domain of numbers and operations, students would come to recognize opportunities for number-sensible strategies and appreciate the advantages to using them.

Also in planning for instruction, we wanted the students to be presented with opportunities for mental computation and computational estimation that were both grounded and not grounded in a real-world context. Using mental computation or computational estimation based in number

sense in real-world contexts supports recognition of its commensurability with other forms of participation (Cobb & Bowers, 1999; Thompson, 1993). We started with problems posed with a real-world context. The story problems seemed a natural entry point as opportunities for non-standard problem solving as problems in context tend to elicit oral (as opposed to written) computation procedures (Carraher et al., 1987). Note that, whether or not the problems were presented in a real-world context, it was important that students encounter the need to mentally compute or estimate in authentic situations. In the language of RME, this meant that the problems were meant to be experientially real as opposed to exercises designed for the sake of doing mental math. The mental computation and estimation that students did occurred in service of solving a problem at hand rather than being an end in itself.

In an effort to encourage sense making and to discourage an over-reliance on the use of standard algorithms, it was essential for students to learn to reason about relationships among quantities. *Quantitative reasoning* requires an understanding of how quantities in a problem are related; the numerical values of the quantities are considered only after the situation is well understood (Thompson, 1993). The context-based problems provided opportunities for analyzing the structure of a problem with regards to its quantities, thus opening solution possibilities beyond the calculation needed to solve a problem with a standard algorithm and promoting sense making (Bowers & Nickerson, 2001; Smith & Thompson, 2008). Quantitative reasoning draws heavily on everyday experience, focuses on general relationships and making inferences from them, involves expressing quantitative relationships in different ways, and requires and supports the development of a collective orientation toward sense making (Smith & Thompson, 2008).

In sum, our first goal was to have students capitalize on opportunities to solve problems using number-sensible strategies. This necessitated identifying opportunities for mental computation and computational estimation throughout the curriculum, both within and without situations described with a real-world context. The instructor would model and encourage the use of number-sensible strategies. Students would be engaged in quantitative reasoning in context-based problems as a starting place to support a shift from applying school-learned procedures without understanding. Students' nonstandard strategies would be encouraged and a norm developed of sharing solutions and making sense of others' strategies.

Goal 2: Students Develop a Repertoire of Number-Sensible Strategies

Assumptions

Our second goal was for students to develop a repertoire of number-sensible strategies. As suggested earlier, the classroom culture needed to be one in which students shared their mental computation and estimation strategies and made sense of others' shared strategies. Because students were solving some real-world context story problems with quantitative reasoning and because our classes often consist of students educated in countries other than the United States, we expected that the strategies students shared would vary naturally. We expected that initially students would share strategies learned in the country of their youth—some learned meaningfully and others by rote. For example, in the United States, subtraction is usually taught by regrouping, but in other countries children are taught an "equal additions" or other method, potentially new to those students taught in the United States. Initially, we anticipated that the different strategies might be indistinguishable for students except in terms of surface

characteristics. In order to make sense of strategies, it was necessary that students understood key mathematical concepts—fundamental components of number sense.

In preparing to meet the second goal, the instructor needed to identify opportunities wherein the teaching of the mathematics content of the course supported the specific mathematical understandings needed for mental computation and computational estimation grounded in number sense. For example, place value understanding is widely noted as essential for place-value based collection strategies for addition and subtraction to 100 (e.g., Blöte et al., 2000; Gravemeijer, 1999; Sowder & Wheeler, 1989). Likewise, understanding multi-digit multiplication as the sum of partial products was essential to number-sensible computation and estimation of products. Students also need a knowledge of number relationships that includes judging the relative size of numbers, their proximity to decades, benchmarks, and in the case of fractional numbers, proximity to "natural fractions" (Hanson & Hogan, 2000). This includes proximity in terms of nearness relationships as well as an understanding of the multiplicative nature of our number system (Heirdsfield & Cooper, 2004; Markovits & Sowder, 1994). It is important that students understand meaning for operations, the effects of operations, and properties (associative, commutative, and distributive) of number systems (Reys et al., 1999; Rubenstein, 1985; Sowder, 1992; Sowder & Wheeler, 1989). In other words, the content provided *foundations for* as well as *occasions for* mental computation and computational estimation activity with an emphasis on sense making.

Envisioned Learning Route and Rationale

The students would be engaged in instructional activities based on planned HLTs concerning concepts such as place value, meanings for operations, benchmark numbers, and properties of numbers. Experientially real problems would elicit multiple strategies from students. As one aspect of RME's principle of guiding students to reinvent significant mathematics for themselves, the collective mathematical activity needed to include reflective discourse on shared strategies (Cobb, Boufi, McClain, & Whitenack, 1997). This discursive activity is not just a mode of talking about mathematics but, instead, constitutes learning (Sfard, 2002). After collective reflection upon many examples coupled with a growing understanding of foundational concepts, the instructor and students could then negotiate socio-mathematical norms of mathematical difference and relative ease and efficacy of strategies. We conjectured that students would begin to see their problem-specific mental calculations as part of a larger class of strategies with essential characteristics, while constructing a notion of appropriate uses for these strategies (Siegler & Jenkins, 1989). These classes of strategies could be named and would come to be distinguishable from one another. In so doing, students could develop a repertoire of number-sensible strategies. We expected this differentiation would contribute to their development of habits of planning and control (Schoenfeld, 1992).

Goal 3: Students Develop the Ability to Reason with Models

Assumptions

Our third goal was that students would flexibly use models for reasoning. From an RME instructional design perspective, we wanted the instructor and students to develop models *of*

informal activity that could, through a series of local shifts, become models *for* more formal mathematical reasoning (Gravemeijer, 1999). It was essential that the instructor anticipate productive, powerful models for reasoning (Rasmussen & Marrongelle, 2006). By *productive*, we mean that the models should be rooted in imaginable contexts and be flexible enough to be applied at a more general level. We do not suggest there is any one particular appropriate model, but that there are supportive models for a domain. From our perspective, the instructor, in his or her role of advancing the mathematical agenda, must proactively anticipate and capitalize on models *of* student thinking that can be linked to models *for* reasoning. Our college students were familiar with some mathematical models and their initial activity would likely reflect utilization of mathematical models that may be utilized as models for reasoning. Although familiar with the number line, we did not expect them to be familiar with the empty number line model. We anticipated that students would have had prior experience with using multiplication to solve for rectangular area (Simon, 1995). However, rectangular area emerged as a powerful model for reasoning about computation and estimation of products.

Envisioned Learning Route and Rationale

Our third goal was that students would flexibly use new models for reasoning. The instructor and students would negotiate symbolizing models of reasoning, which might then facilitate students' invention of new strategies. For example, one of the more powerful models that has been shown to support the development of key ideas in addition and subtraction is the empty number line, which can be used to model number relationships that include relative sizes and proximity to decades and other benchmarks (Fosnot & Dolk, 2002; Gravemeijer, 1999). The number line model can be used as a *model for* reasoning; students can use the number line model to construct the idea of constant differences and support for related computational strategies.

Rectangular area as a geometric model can be used to model number relationships in multiplication and division. Coupled with an understanding of conservation of area, the rectangular area model can be used to develop multiplication and division strategies and in the case of college students to deepen understanding of associative, commutative, and distributive properties.

Discussion of the LIT in Terms of the Three Goals

The instructional sequences for the LIT for the development of number sense have been summarized briefly in Tables 1–3 and in the previous text in terms of the goals, assumptions, and envisioned learning route.

The following list outlines the general instructional activities, which is the remaining component of the LIT:

- The instructor anticipates opportunities for mental computation and computational estimation within a particular content area. The instructor designs HLTs to support understanding of key concepts.
- The instructor anticipates problem-specific strategies and the mathematics implicit in these strategies. The instructor must ensure there is support for such strategies both in the problem choice and in the understanding of number structure, operations, and properties.

- The instructor anticipates productive models for reasoning in order to productively capitalize on student thinking and make connections to models for reasoning.
- The instructor models identifying opportunities for occasions for the practical, authentic activity of utilizing number-sensible strategies to solve problems.
- Students learn how to use quantitative reasoning in problem solving.
- From the beginning of the course, students are expected to perform computations and estimations mentally and to share the strategies. The instructor guides the negotiation of the social norm that students discuss shared methods with an aim toward making sense of the mathematics.
- The class community collectively negotiates models of students' strategies; these symbolizations are guided in the sense of transformational records (Rasmussen & Marrongelle, 2006).
- The class may begin by referring to strategies with reference to specific examples.
- Students name the strategies in meaningful ways, thus making them objects of reflection. This gradual shift necessitates the maintenance of a cumulative list of strategies.
- Keeping a list enables discussions to shift to what constitutes a different strategy. Thus, criteria need to be negotiated for aspects of a strategy that are essential, as opposed to incidental.
- A robust repertoire of resource strategies emerges from the organization of various examples; names and characterizations are refined as more examples are seen.

EXAMPLES FROM THE TEACHING EXPERIMENT WITH PRESERVICE TEACHERS

We will illustrate aspects of the LIT via examples from a teaching experiment. The vignettes are chosen to illustrate how aspects of the envisioned learning route unfolded and how the LIT related to HLTs.

Vignette 1: Employing Sense-Making

In order to support the first goal of the LIT that students capitalize on opportunities to use number-sensible strategies in problem solving, the instructor anticipated opportunities within the curriculum in which to invite and model the use of meaningful mental computation and computational estimation. Students were invited to use mental computation and estimation in situations with and without a real-world context. Following are examples of each.

Early in the course, students were invited to engage in quantitative reasoning. The class was launched with problems that had rich quantitative structure and multiple possible solution paths. One such problem was the following:

> The last part of one triathlon is a 10K (10 kilometers, or 10,000 meters) run. When runner Aña starts this last running part, she is 600 meters behind runner Bea. But Aña can run faster than Bea: Aña can run (on average) 225 meters each minute, and Bea can run (on average) 200 meters each minute. Who wins, Aña or Bea? If Aña wins, when does she catch up with Bea? If Bea wins, how far behind is Aña when Bea finishes? (Sowder, Sowder, & Nickerson, 2010, p. 9)

Students were encouraged to first focus on the relationships among the quantities, such as Bea's position when Aña starts running or the distance between Aña and Bea. Through collectively solving problems such as this, students came to consider the quantitative structure of a problem independent of the values given in the problem. Once a student understood the relationships among quantities, he or she could use the relationships to find all the values needed. Context-based story problems without an easy algorithm encouraged a sense-making approach to problem solving.

In the course of our teaching experiment, the instructor routinely stopped the discussion in the course of solving a problem to invite students to mentally perform calculations and share their strategies. For example, in the story problem above students were asked, "How long will it take Aña to catch up?" The students focused on the decreasing distance between Aña and Bea. The particular numbers involved in the calculation, $600 \div 25$, are accessible to this population of students and allowed for a wide variety of strategies. For example, one student used his fingers to represent groups of four 25s, counting six fingers to get to 600, and thus finding the quotient as $6 \times 4 = 24$, or 24 minutes for Aña to catch up. The class discussed a few students' strategies for calculating the catch-up time before continuing the discussion of the story problem.

The instructor also invited students to perform mental computations in situations without a real-world context. For example, in working with bases other than base ten, students needed to compute powers of the base numbers, such as computing 8^3 when working with base eight. The instructor asked students to compute this value mentally, anticipating particular strategies they might offer.

As noted earlier, place-value understanding is essential for place-value based addition and subtraction strategies and understanding of multiplication as the sum of partial products. A hypothetical learning trajectory was developed to deepen an understanding of the base-ten place value system. In designing an HLT for place value for our particular classroom setting, we considered our population of college students who typically are coming to understand mathematics they think they already know. Our place-value HLT, briefly summarized, conjectured a learning process in which students develop facility with conceiving of values expressed in various units. We began in a familiar context for college students, money, and asked students to conceive of a number in different monetary units: $10, $1, $0.10. Students developed the notion of the value of number such as 657, not just as 657 ones but also as 65 and seven-tenths tens. The HLT promoted students' understanding of the underlying structure of base ten by experience with numeration systems other than base ten.

As our two examples in the vignette suggest, the instructor found different opportunities to engage students in mental computation and estimation activity. In this way, the LIT informed the development of HLTs necessary for our students' understanding of place value, operations, properties, and so on. Students began sharing nonstandard strategies, discussing their strategies and those of their classmates. This contributed to the development of norms that nonstandard strategies were acceptable and that making sense of the mathematics behind a strategy was important.

Vignette 2: The Evolution of Naming of Strategies

In order to support the second goal of the LIT that students develop a repertoire of number sensible strategies, we planned to make their strategies an explicit topic of discussion. An

understanding of place value, operations, and properties provided foundations for the practice of naming. As students reasoned quantitatively and were invited to share their strategies for solving and computing, the instructor began to refer to strategies by the name of the nominating student. This was done as a means of connecting the distinct discussions of solution methods that were separated by class meetings. The instructor would allude to a method that a student had shared recently by asking (with regard to the calculation at hand) "How might Karen do this calculation?" This naming of strategies (e.g., "Karen's strategy") gave the classroom community a means of productively discussing mathematical difference. Such discussions involved increasing reflection on and reference to previous activity.

As other students utilized what came to be seen as similar strategies, students collectively negotiated descriptive names that differentiated the similar strategies from others. The discussion shifted from naming strategies tied to specific examples to being able to look across many examples to discuss the properties of which they took advantage. For example, one class section chose the name "Break Up, then Make Up" to refer to applications of the distributive property of multiplication over addition (e.g., 15×24 was treated as $15 \times 20 + 15 \times 4$). The named strategies became objects of discourse, enabling discussion beyond the details of a single example and facilitating a focus on the structure of the number system. The practice of naming constituted a vertical shift in mathematizing (Van Den Heuvel-Panhuizen, 2003), which facilitated reflective discourse on strategies. We call named strategies, such as "Break Up, then Make Up," *resource strategies*, as they were used and discussed collectively as tools for problem solving.

In sum, the second goal of the LIT was to have students develop a repertoire of number-sensible strategies. We note the importance of the practice of naming strategies in facilitating students' inclination to see their problem-specific calculations as part of a larger class of strategies. Naming enabled the class discussion to move beyond descriptions of the particular steps that a student performed, so that strategies themselves became objects of discourse. As students developed greater familiarity with the conceptual domain of numbers and operations, they negotiated differences among and relative efficacy of resource strategies.

Vignette 3: Activity with the Empty Number Line Model

The third goal of the LIT was that students would develop the ability to reason with models. As strategies were shared, the class negotiated symbolizations of these strategies that they agreed made sense. For example, when the class was invited to solve a quantitative reasoning problem that involved comparing heights of two pairs of siblings, students determined that they needed to find the difference between 193 cm and 82 cm. Ashley[1] described her method of adding on: "82 plus 100 is 182, then plus 11 more is 193. So, it's 111." The instructor then offered to notate her method on the chalkboard. He wrote the 82 and then asked the students how to illustrate Ashley's mental work. They suggested two "jumps" upward, as in Figure 4a. When the instructor asked how one would know the answer from what had been drawn, students suggested writing numbers to the side, and the instructor made the additions depicted in Figure 4b to notate the measures of each jump.

The strategy of beginning with the subtrahend and adding on to reach the minuend was initially dubbed "Ashley's method" when it was applied to find solutions for other difference

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[1]We use gender-preserving pseudonyms throughout.
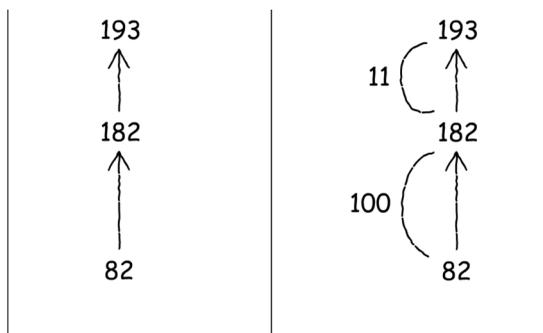
FIGURE 4 (a) Initial notation for adding-on strategy. (b) Elaborated notation for adding-on strategy.
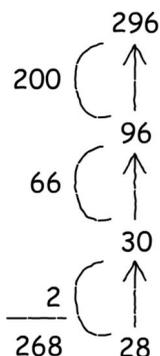
FIGURE 5 Elaborated notation for Ashley's method.

problems. In a subsequent problem (the difference in this case was 296 – 28), a student recalled that Ashley's method involved considering "how many she needed to go up." The instructor followed the suggestion by writing 28 on the board and asking, "What would be a nice number to go up to?" A student suggested an initial jump to 30, which the instructor notated. The instructor and students subsequently negotiated what they perceived to be reasonable jumps, as depicted in Figure 5.

In a subsequent problem (1000 – 729) for which a student suggested an application of Ashley's method, the instructor turned the record on its side and the empty number line replaced the initial, informal notation (see Figure 6) as a shared conventional representation for this strategy. These early symbolizations assumed the role of *transformational records*. That is, while they began as records of student thinking, the symbolizations were "used by students in achieving subsequent mathematical goals" (Rasmussen & Marrongelle, 2006, p. 394). Ashley's method was the first strategy that the class discussed for mental computation of a sum or difference. As more addition and subtraction strategies arose and were discussed, the class negotiated appropriate empty-number-line illustrations for them. In Ashley's method, the minuend and subtrahend
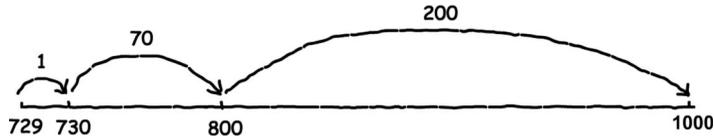
FIGURE 6 Initial introduction of empty number line for 1000–729.

are both represented as locations on the number line. There are other methods in which one of them represents a distance from the location specified by the other (e.g., 1000 is interpreted as a location, and 729 is a distance to be traveled from 1000 in the negative direction). The class agreed on a suitable empty-number-line illustration for each of the addition and subtraction strategies that were discussed in class.

The shift from *model of* to *model for* describes a transition from a phase in which students' strategies are deliberately represented with the model to one in which reasoning with the model enables students to make sense of the mathematics in new ways, which may lead to the invention of new model-based strategies. In this teaching experiment, the strategy that was dubbed "Shifting the Difference" stands out as one that was particularly afforded by the empty number line representation. For example, 142 – 57 becomes 145 – 60 to make the computation easier. On the empty number line, this looks like shifting both locations three units to the right. In post-instruction interviews designed to assess shifts in flexibility in mental computation and computational estimation, some students referred explicitly to the empty number line and others talked in terms of "shifting" the given numbers or making "jumps" when describing their mental computations. It seemed that the strategy of Shifting the Difference, in particular, was one for which students had made new meaning since none of the 13 interview participants used it in the first interview, while seven used it in the second interview (Whitacre & Nickerson, 2010).

The vignettes illustrate the unfolding of the LIT's envisioned learning route and instructional activities in our class for preservice teachers. The LIT articulates goals and focuses attention on our assumptions about the mathematical preparation of our students. The HLTs for deepening understanding of place value, whole-number multiplication and division, and "number neighborhoods" (i.e., magnitude, benchmarks and proximity to decades) are distinctly different from, although informed by, the LIT. Our HLTs, as seen in our brief description for place value, anticipated the collective development of our diverse population of college students who generally lack conceptual understanding of the standard algorithms they use.

HOW AN LIT INFORMS AN HLT

In this section, we illustrate the role played by the LIT in modifying instruction during the teaching experiment. In particular, we will show how it shaped the development of a particular, unanticipated HLT. In so doing, we further illustrate the role that models played in the instruction and provide a detailed, empirical example of the classroom activity related to students' development of number sense with regard to whole-number multiplication. This example also elucidates the relationship between particular curricular content and our broad goal of students' development of number sense.

Vignette 4: Focusing on Partial Products

In the course of a counting combinations problem (which had to do with license plate numbers), the students agreed that the computation of 26×26 was needed. The instructor asked students to compute this product mentally and to share their mental computation strategies. The most common strategy—an invalid one—was to compute $20 \times 20 + 6 \times 6$, ignoring the mixed terms. Some students disagreed with this suggestion, arguing that the result was too small, or simply that it did not agree with the product obtained by using the standard algorithm. Students were challenged to refute the suggested approach in a sense-making fashion; they struggled to do so. The instructor suggested considering rectangular area as a model of multiplication, and a discussion ensued contrasting the representation containing all partial products and the representation showing only the squares 20×20 and 6×6 (see Figure 7), as students had initially suggested. The rectangular area model was relatively informal and enabled students' intuitive understanding of conservation of matter to become a resource (i.e., clearly a 26-by-26-unit rectangle did not have the same area if two of the partial rectangles were removed, as in Figure 7). Rectangular area was used to highlight the role of partial products, and it also seemed to have the potential to become a model for reasoning.

On the next day of class, rectangular area was again used—this time, to settle an estimation (error) question. Students were asked whether 32×80 or 30×83 would be a better estimate (i.e., closer to the exact answer) of 32×83. Most students initially thought that 30×83 would give the better estimate, reasoning additively that 2 is less than 3, so less would be lost. A few considered the partial products and found that 32×80 was actually closer to the exact answer. The instructor again suggested the use of rectangular area as a model of multiplication. He drew two rectangles labeled 32×83 units on the chalkboard and asked students how much would be lost if each of the suggested estimates were used. Students suggested how to cut the rectangles to show the relevant partial products, as shown in Figure 8. Students used these representations to justify the conclusion that 32×80 gave the better estimate.

At this point, students accepted and participated in the use of rectangular area as a model of whole-number products. They had also used representations on the board as evidence in
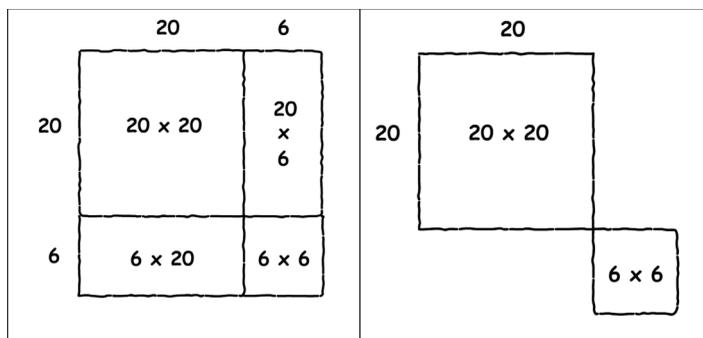
FIGURE 7 Rectangular area model of multiplication as a model of student thinking.
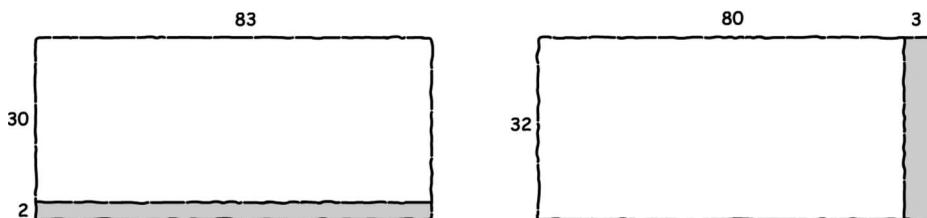
FIGURE 8 Rectangular area model of multiplication is used for computational estimation.

arguments concerning the relative size of products. However, they had not spontaneously used the model in these ways. In terms of the Route to Goal 3 depicted in Table 3, the instructor wanted to facilitate a shift in the role of rectangular area to a model for reasoning. This was accomplished by shifting the focus away from the use of the model itself to represent partial products toward employing it in aid of exploring more challenging questions in comparing partial products.

The instructor designed a Geometer's Sketchpad sketch, which was used the next day of class to engage students in questions around estimation of products. The sketch consisted of a dynamic representation of a rectangle segmented place-value-wise and linked to explicit calculations of the partial products as well as their sum as the total product. The sketch afforded the opportunity for rectangular area to be used quickly and flexibly as a model of products, which allowed for the consideration of many cases in support of students' attempts to reach generalizations. Following up on the estimation discussion from the previous class, the instructor and students began to use the word *weight*, as in "Which weighs more, the 2 (in 32) or the 3 (in 83)?" (In other words, which contributes more to the total area?) The lesson focused on students exploring the conditions under which one number of units would weigh more or less than the other, beginning with cases in which they would weigh the same. In this context, students justified their claims regarding relative weight with reference to partial products. Their words and gestures suggested to us that they began to imagine a rectangle changing dynamically in anticipation of their conjectures being tested with the sketch.

In the lesson concerning weight, students used rectangular area as a model to confirm or refute conjectures; students also employed it as a resource for arriving at their conclusions in the first place. Rectangular area subsequently became the conventional model that students used to represent and justify their mental multiplication strategies. As had been done with the empty number line, the instructor and students negotiated how to appropriately represent the strategies that students suggested. The strategies that arose included various applications of additive distributivity, subtractive distributivity, and associativity. While some of the representations were rather straightforward (e.g., "Break Up, then Make Up"), others required students to imagine new possibilities for change involving a rectangle. Conservation of area contributed to students' understanding of partial products and afforded a common-sense resource for reasoning about transformations of products (e.g., the application of associativity discussed below).

We take as evidence for the shift in the role of rectangular area to a model *for* reasoning, some of the strategies that students invented, and the names that those strategies were given. For

example, a student's strategy for computing 24×15 as 12×30 was initially dubbed "Halve and Double." Later, "Halve and Double" was recast as a special case of "Rearranging Rectangles" since students realized that the original rectangle could be cut into any number of pieces, not just two. Thus, the names and characterizations of resource strategies were also refined in connection to students' uses of models for reasoning.

One student suggested computing 24×15 as 40×9 by recognizing that one could count 3-by-3 squares. There are five 3s in 15 and eight 3s in 24. Thus, there are forty 3-by-3 squares in a 24-by-15 unit rectangle. She made an illustration on the chalkboard like that in Figure 9. Such nonstandard strategies, together with illustrations and language, suggested that rectangular area became not merely a model of students' mental multiplication strategies but a model for reasoning.

As this example illustrates, the LIT informed the development of particular HLTs. In the case of such topics as place value, an HLT was planned ahead of time in connection to a particular curricular unit. In the case of students' understanding of partial products, the instructional trajectory was modified significantly in response to students' thinking. The instructor sought to facilitate shifts in the role of rectangular area in students' activities related to multiplication, toward its use as a model for reasoning. The established usage of rectangular area and students' understanding of partial products appeared to be mutually supportive, each facilitating the development of the other. The realization of this learning trajectory served both our immediate content goals and our broader learning goals with regard to number sense.

As noted earlier, an LIT must draw attention to assumptions about the preparation of students. The above examples of the use of models highlight a difference for instructors of elementary students versus college students. Our students generally had previous experience in their K-12 education with the number line model as well as with the use of multiplication in rectangular area/array contexts. Had these models been entirely novel for them, a more gradual introduction to them would have been appropriate. For example, for elementary students, the empty number line needs to be systematically introduced. Imagery can be developed for the empty number line by associating beads on a bead string (arranged in tens) with a number line. Decade numbers would be progressively introduced with children imagining the other intervening numbers. Finally, the actual empty number line could be introduced. Cobb, Gravemeijer, Yackel, McClain, and Whitenack (1997) described a teaching experiment in a first grade classroom in which students failed to develop the underlying imagery for the empty number
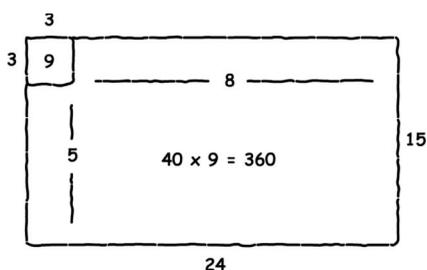
FIGURE 9 Rectangular area model of multiplication as a model for reasoning in new ways.

line. Thus, in applying an LIT, a teacher must be sensitive to the mathematical background of the students in a particular class. By making explicit the relevant assumptions about students' preparation, an LIT can be used in the development of socially-situated HLTs.

DISCUSSION

Our goals for students with regards to developing number sense were (and are) threefold: Students will capitalize on opportunities to use number sensible strategies, will develop a repertoire of resource strategies, and will reason with models to understand and use new number sensible strategies. We have distinguished among a *strategy* or solution path from an observer's perspective, a *number-sensible strategy* from the individual student's (participant's) perspective, and a *resource strategy* from the perspective of the community acting as if the strategies are tools or resources for solving problems.

Our local instructional theory for the development of number sense includes an articulation of goals, assumptions, envisioned learning routes, and instructional activities with rationales for their appropriateness. In articulating and discussing our LIT, we stress the importance of the instructor identifying and engineering opportunities for authentic activity, anticipating nonstandard strategies students might use, and being aware and open to powerful models that can be flexible enough to be used as models for reasoning in new ways. The social norms that need to be developed in order for our goals to be realized include the need to explain strategies and justify them and to make sense of the explanations of others. The collective classroom orientation should be one of making sense of situations (Thompson, Philipp, Thompson, & Boyd, 1994). As the students share strategies and make sense of each others strategies, the practice of naming assists the class in developing a repertoire of resource strategies. The increasing awareness of various strategies and the differences among them, as facilitated by the discourse around students' strategies and by the practice of naming, leads to more evident and productive planning on the part of students. There is a clear difference between a student reporting the particular steps she took to perform a calculation mentally versus a student reporting that she used a resource strategy such as "Break Up, then Make Up," or answering immediately that such a strategy would be a nice choice for the problem at hand. This is evidence of an awareness of the availability of resource strategies and of the habit of selecting one based on the given numbers, both of which are indicators of students building a repertoire of number-sensible strategies.

We have illustrated one instantiation of our LIT, which we believe need not be limited to a course with a focus on number and operations. From our perspective, a local instruction theory is clearly distinguishable from a hypothetical learning trajectory in terms of its generalizability. We illustrated this, in part, by showing an instantiation of a local instruction theory for the elusive topic of number sense, as it was manifested in multiple HLTs. We illustrated this also by pulling out the elements of an LIT that need to be considered regardless of the setting. Thus, in our view, an LIT is not a sequence of HLTs, connected together like a row of bricks. Rather, we see an LIT as functioning to inform particular socially situated HLTs in service of its goals and in light of its assumptions. As stated earlier, in creating an LIT we are not offering a portable instructional sequence but, instead, suggesting that the LIT can serve as a framework of reference for others developing HLTs for their own students in classes that have similar goals.

In order for a local instruction theory to be effectively utilized in the development of HLTs, it should elucidate a set of goals. It is also critical for those offering an LIT to articulate the importance of assumptions about students' preparation and learning that drive the rationale for the envisioned learning route. Likewise, teachers using an LIT to develop HLTs need to consider these assumptions as they seek to apply the LIT to their particular, socially situated classrooms. This provides support, we believe, for teachers to tailor HLTs to their particular classrooms and students. We described an envisioned learning route in support of three interrelated goals, the development of which would be neither sequential nor precisely parallel but coordinated, nevertheless. The goals, assumptions, underlying rationales, envisioned learning routes, and examples of instructional activities together provide support for teachers to modify and develop HLTs for their particular classroom settings.

CONCLUSION

We hope that this example—our conception of a local instruction theory—is useful to the field in fleshing out the construct, especially insofar as it differs from and relates to a hypothetical learning trajectory. In order to support teachers in developing reform-oriented instructional sequences, we need to develop local instruction theories that make explicit their goals, assumptions, underlying rationales, and related instructional activities. This will enable teachers to apply the local instruction theory to the development of instructional activities for their own classrooms. Some important topics, such as number sense, are overarching of the entire teaching sequence. In a case such as this, a local instruction theory is essential to achieving our learning goals. In the spirit of design research, we invite other researchers to contribute to the elaboration of the LIT construct in this and other contexts with the hope of supporting teachers in their difficult work.

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