THEORY OF DIFFERENTIAL EQUATIONS IN DISCONTINUOUS FIELDS AND ITS APPLICATION TO GALERKIN, MYXED, HYBRID AND OPTIMAL FUNCTIONS FEM

by

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In recent years there has been a renewed interest in discontinuous Galerkin methods and its applications [1]. Related matters are Trefftz, Hybrid and Mixed methods. Also, penalty methods [2], Galerkin/least-squares [3], stabilized methods (SUPG/SD [4] and USFEM [5]), residual free bubbles (RFB [6-9]) variational multiscale (VMS), the partition of unity method (PUM) and nearly optimal Petrov-Galerkin [10]. Finally, the FEM with optimal functions (FEM-OF), recently proposed by Herrera. Most of these methods can be derived using a general theory of partial differential equations in discontinuous functions spaces, which has been developed by Herrera and his coworkers over a long time span. Firstly, it was introduced as an algebraic theory of boundary value problems (*BVP*) and in that form it was capable of supplying a very general framework, which accommodated practically all variational principles for *BVP* known at the time [11]. It also encompassed Trefftz methods and biorthogonal systems of functions [12-14]. Furthermore, this theory also supplies a suitable framework for the development of complete systems of functions and, according to Begehr and Gilbert ([15], p115), it supplies the basis for effectively applying to *BVP* the *function theoretic methods* whose development is due to many distinguished researchers, including Bergman, Vekua, Colton, Gilbert, Kracht-Kreyszig and Lanckau. The Pitman's Advanced Publishing Program devoted a book to it [11], which contains the results that were obtained up to 1984. Afterwards, in 1985 [16,17], a new kind of Green's formulas (*Green-Herrera formulas*) were introduced that are applicable to operators in discontinuous fields. They constitute the *backbone* of the theory of partial differential equations in discontinuous functions that will be explained in this plenary talk. When boundary value problems are formulated in function spaces that contain fully discontinuous members, in order to have well-posed problems, it is necessary to consider *'boundary value problems with prescribed jumps (BVPJ)'*; i.e., problems in which the usual boundary conditions are complemented with certain *'jump conditions'*, at the *'internal boundaries'*. When the *jumps* vanish the standard solutions of boundary value problems (without *jumps*) are recovered. So, for example, for elliptic problems of order $2m$ it is necessary to prescribe the *jumps* of the normal derivatives, up to order $2m-1$, and the problem solutions for *zero jump conditions* are the usual solutions of the classical theory of partial differential equations, which belong to the Sobolev space $H^{2m}(\Omega)$ [18]. Based on this theory, the Finite Element Methods with Optimal Functions (*FEM-OF)* have been derived, which in turn have yielded new orthogonal collocation methods, referred to as *TH-collocation methods* [19, 20]. They possess important advantages with respect to the orthogonal spline collocation methods (*OSCM*), among them: they can be easily and effectively parallel-processed because they are very suitable to be combined with domain decomposition methods (*DDM* [21, 22]). Finally, it must be mentioned that in previous stages of its development the framework that the theory here presented yields, was referred to as Localized Adjoint Method (LAM) [23] and it supplied the theoretical foundations for the Eulerian-Lagrangean LAM (ELLAM, [24]), which was quite effective for treating transport problems, specially advection dominated transport [25]. The generality of the theory must also be stressed, since it is applicable to any linear differential operator independently of its type (elliptic, parabolic or hyperbolic), including the case when the operators' coefficients are discontinuous. In this talk, using it, the numerical methods mentioned at the beginning are revised.

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