Introduction to the
Reconceptualizing Mathematics Program
Developed at San Diego State University

> Mathematics programs for prospective (elementary) teachers must emphasize the intellectual depth of the elementary mathematics curriculum and provide the pedagogical tools to effectively teach this critical material to elementary school children. (Mathematical Association of America-Committee on the Undergraduate Program in Mathematics, p. 41, 2004)

There is today a greater awareness that elementary mathematics is rich in important ideas and that its instruction requires far more than simply knowing the "math facts" and a handful of algorithms. Mathematics courses for teachers must reflect the intellectual depth and challenge of the elementary school curriculum. The Conference Board of Mathematical Sciences (CBMS) recommends that the preparation of the mathematics teachers include courses that develop a "deep understanding of the mathematics they teach," that are designed to "develop careful reasoning and 'common sense' in analyzing conceptual relationships, . . . that develop the habits of mind of a mathematical thinker and that demonstrate flexible, interactive styles of teaching" (CBMS, 2000, pp. 7-8).

We recognize and accept the challenge of presenting mathematics to teachers in a manner that addresses these recommendations. In doing so we provide instruction that will lead teachers of mathematics to *reconceptualize* the mathematics they often think they already know, thus allowing them to develop a deep understanding of the mathematics they will teach. We believe that teachers must know mathematics differently than most people do. Teachers need to know the mathematics they teach in a way that allows them to hold conversations about mathematical ideas and mathematical thinking with their students. A persistent pursuit of explanation is a hallmark of a classroom in which learning is taking place.

Future teachers learn mathematics when they themselves experience doing mathematics—"exploring, guessing, testing, estimating, arguing, and proving—in order to develop confidence that they can respond constructively to unexpected conjectures that emerge as students follow their own paths in approaching mathematical problems" (National Research Council, 1989). This type of confidence is what we seek to instill.

Future teachers must be mathematically proficient if they are to be successful in developing mathematical proficiency in their own students. They must understand that mathematical proficiency is multifaceted, as described by the Mathematic Learning Study Committee (National Research Council, 2001) to include:

- conceptual understanding—comprehension of mathematical concepts, operations, and relations
- procedural fluency—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- strategic competence—ability to formulate, represent, and solve mathematical problems
- adaptive reasoning—capacity for logical thought, reflection, explanation, and justification
- productive disposition—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficiency. (p. 116)

These strands of mathematical proficiency are interwoven and should develop together. We believe this program addresses these competencies in an integrated fashion.

A common axiom is that teachers teach the way they were taught. Prospective teachers are unlikely to demonstrate flexible, interactive styles of teaching unless they have experienced mathematics taught this way. Instructors of the *Reconceptualizing Mathematics* courses, however, may not have experienced such instruction themselves. Thus we provide many forms of instructional assistance to help instructors better understand the mathematics their prospective teachers need to know, to begin to model teaching strategies that these prospective teachers will be expected to use in their own classrooms, and to assist them in many ways throughout the course.

There are four parts to *Reconceptualizing Mathematics: Reasoning about Number and Quantity, Reasoning about Algebra and Change, Reasoning about Shapes and Measurement, and Reasoning about Chance and Data*. At San Diego State we use these materials in four one-semester courses. For two or three semester or quarter programs topics would need to be judiciously selected. We provide timelines that can assist in making decisions about topics to include.

For the instructors of *Reconceptualizing Mathematics* courses we offer:

- Textbook pages that are identical to the students' pages except that in the instructor text marginal notes provide explanations, sometimes answers, and

sometimes suggestions for teaching a particular topic. The margins are left blank in the student version for note taking. This feature has proved to be particularly attractive to both instructors and students.

- Separate instructor notes at the beginning of the instructor version that offer more background and detail than can be placed in margins. At times we suggest ways of approaching a lesson. At times we provide background on particular lessons that can be useful in understanding the point of the lesson, or research about the particular topic.

- *Answers* to all exercises and information in the margins about which exercise answers are not included in the student text.

- Sets of *examination exercises* in Word, to allow cutting and pasting exam items.

- *Videoclips* of interviews of elementary students working on number tasks. The six clips included here are highly motivating. When prospective teachers have a close-up look at how elementary students think about mathematics, they realize that just knowing procedures is not enough—they themselves *must* understand the mathematics, that is, they must reconceptualize the mathematics if they are to develop the deep understanding of mathematics needed to teach well. (For a sample, see the IMAP site at http://www.sci.sdsu.edu/CRMSE/IMAP/)

- *Lists of special materials needed* for each section. Some of the materials can be obtained from the pages for cut-outs in the appendices.

- *Appropriate technology* incorporated into lessons in each part of the book, with instructions on how to use the technology if it is available.

- A *research based curriculum*. The authors have studied the learning and teaching of mathematics for many years, and that work has influenced decisions made while developing these materials.

- A *classroom-tested curriculum*. These text materials have been used and refined in San Diego State University classes for preservice elementary teachers (1500 – 2000 students per year) for eight years, and in our Professional Development Collaborative for six years. The text materials have been available upon request for four years, and many universities and community colleges are now using these

materials. We have requested and received formal reviews from four other institutions, reviews that resulted in revisions.

For both instructors and students, we offer:

- A *Message to Prospective and Practicing Teachers* that lays out expectations of students as participants in learning. This section should be assigned reading on the first day, and should be discussed in class so that students know *your* expectations of them. *This section also provides information on the structure of the book.*

- A *focus on sense-making*, with regular expectations to include explanations and justifications. Students should learn not to expect to be shown "*the* right way," as though imitation is the only way one can learn mathematics and as though there is only one way in which certain tasks can be accomplished.

- *Activities* within lessons that can be undertaken in pairs or groups during class, then discussed as a full class. The activities provide opportunities to explore, guess, test, estimate, argue, and prove. Many of the activities are rich tasks that invite problem solving, which is "not only a goal of learning mathematics but a major means of doing so. Students should have frequent opportunities to formulate, grapple with, and solve complex problems that require a significant amount of effort and should then be encouraged to reflect on their thinking" (National Council of Teachers of Mathematics, 2000).

- *Discussion* items for whole classroom interactions that can, with some guidance from the instructor, lead to rich conversations that provoke learning through sharing of ideas and insights.

- *Numerous Examples* provide needed clarification and opportunities to explore meanings and demonstrate procedures.

- *Think Abouts* intended to invite students to pause and reflect on what they have just read, or extend their thinking about an idea presented.

- A *Take Away Message* for every section that summarizes the major points made and can serve as a way of reviewing for examinations.

- *Learning exercises* for each chapter section that provide opportunities to practice, to consolidate what has been learned from a lesson, to extend and explore ideas in

new ways, to develop new insights and understandings, to connect to previous learning, and at times to foreshadow lessons yet to come. Many of the exercises and discussion questions are rich enough that different solution methods are possible and even likely.

- *Issues for Learning* sections at the end of most chapters that examine some of what is known from research about the learning of one or more topics in that chapter. In many instances, the issues discussed apply to both elementary students and prospective teachers.

- *Examples of children's ways* of doing mathematics. Many times student solutions are highly original and exemplify the reasons why teachers need to understand mathematics and student thinking in deeper ways than people who do not teach.

- *Exemplary answers* for some of the learning exercises serve as models for student answers.

- At the end of most chapter there is a sample of a page from *elementary school textbooks* that relates to content in these materials.

- A *glossary* of all important terms used in the course.

- An *Appendix: A Review of Some "Rules"* in Part I for students who do not remember basic arithmetic that is assumed in this course.

- Discussion of content standards from the National Council of Teachers of Mathematics document: *Principles and Standards for School Mathematics,* 2000.

- An inexpensively produced text to write in and keep as a reference, rather than as a resale item after the final exam.

Alignment Issues: We have attempted to do more than simply reorganize the usual content of mathematics courses for elementary and middle grade teachers. We have selected and developed topics and problems that exemplify mathematical reasoning and problem solving, make mathematical connections clear, clarify and overcome common misconceptions, provide opportunities to communicate mathematically, and promote greater confidence in one's ability to deal with mathematics, in short, to develop all strands of mathematical proficiency. The materials here align with the CBMS document *The Mathematics Education of Teachers* (2001) and with the MAA-CUPM document

Undergraduate Programs and Courses in the Mathematical Sciences (2004). The vision of school mathematics found in the NCTM *Standards* (1989, 2000) documents also guided us as we developed and revised these text materials. In addition, a study of state mathematics frameworks and credentialing requirements show that teachers who know the content in *Reconceptualizing Mathematics* will be prepared to pass credentialing tests for elementary school teachers (assuming that they also know basic algebraic symbol manipulation) and should be prepared to teach state-required mathematics.

The Role of Problem Solving: Both routine and non-routine problems can be found throughout the text materials and the Learning Exercises. Such problems permeate the course, rather than being relegated to sections called problem solving. For example many problems in *Reasoning with Numbers and Quantities* are solved by analyzing the quantitative structure of the problems. In *Reasoning about Algebra and Change* describing graphs qualitatively is fundamental to the notion of Change. In *Reasoning about Shapes and Measurement* there are rich problems that explore and develop spatial reasoning. In *Reasoning about Chance and Data* some of the misconceptions relating to probability are explored. Each part has many other instances and types of problems that are non-trivial and go beyond application of skills.

The Use of Technology: Each of the four parts uses technology in a different way. We highly recommend using this technology, but it is not required in any of the four parts.

Part I: Reasoning about Numbers and Quantities contains and "Issues for Learning" on the appropriate use of calculators in elementary school. Calculators should be available when solving problems where the emphasis is *not* on developing understanding or skills related to computation. For example, calculators would not be used for computational estimation. We also recommend the use of videoclips of children's thinking as a motivational device. Finally, we include a set of applets that were designed for use with this content.

Part II: Reasoning about Algebra and Change. We introduce an applet called "Over and Back" which allows students to explore the important concept of average rate, and we discuss with the instructor the incorporation of the motion detector (to be purchased) as a way to visualize the graphing of motion.

Part III: Reasoning about Shapes and Measurement contains an optional chapter that introduces the *Geometer's Sketchpad* (from Key Curriculum Press). These lessons build

on the content of Part III and provide not only information on making some standard geometric figures, but reflection questions that help solidify understanding of geometric concepts. The set of lessons here were developed by Janet Bowers, who has also designed the online course for *Geometer's Sketchpad* for Key Curriculum Press.

Part IV: Reasoning About Chance and Data is perhaps most dependent on software. We recommend Fathom (a computer program available from Key Curriculum Press) or TI 73 Explorer (a calculator available from Texas Instruments). Some of the statistical work can be done using Excel if neither Fathom or TI 73 is available. Data sets are available via Fathom and Excel, or students can type in data sets. We also recommend the *Illuminations* website applets which should be available to the majority of students. The lessons in Part IV are developed independently of any software so that whatever is available can be used. Appendices explain how to use each of the types of software.

The Design of the Learning Exercises: The format and content of the Learning Exercises are somewhat different than what prospective teachers have experienced in other courses. We know that most students coming into our classes have the belief that a homework exercise takes at most a few minutes to complete. But most good mathematical problems take more time. Schoenfeld, in an often quoted reference, has said:

> We have done a serious disservice to any student who emerges from the classroom thinking that mathematics only applies to situations that can be solved in just a few minutes—and that if you can't solve a problem in a short amount of time, you should simple give up" (p. 160).

In the video study associated with the Third International Mathematics and Science Study (TIMSS) countries that achieved higher scores than in the U.S. showed that in Japan, teachers often focused on one problem, with extensions and explanations, for the entire class period, whereas in the U.S. teachers gave multiple problems or just practice on definitions that did not require much thought (Stigler & Hiebert, 1999). Some of the recent K-12 curricula also sometimes focus on one problem for a relatively long period of time. Prospective teachers need to experience problems that more than a few minutes to solve if they are going to be able to teach from some of the current textbooks.

The primary objective of the Learning Exercises that appear at the end of each section is to provide students with a deeper understanding of the content of that section. At times

this means that some of the learning exercises provide practice needed if a skill is to be developed. Many of the learning exercises, however, require students to make connections with what was previously learned, and to extend what was learned in the section. Thus, some of the learning exercises require considerably more time than one might expect from a single exercise, particularly if the instructor is requiring an explanation of the work.

We have structured the Learning Exercises to provide many opportunities for learning on topics that are essential, some of which take more than a few minutes to complete. We have provided fewer exercises on topics where skill development is not necessary. For example, computing with different bases is intended to help prospective teachers better understand our place-value system and its importance in developing algorithms for computing. But prospective teachers will not be teaching computation in bases other than ten, and therefore do not need to develop expertise in computation with numbers displayed in other bases.

We have, for each section, told the instructor which answers the student does not have available. This information can be used to assign exercises. Instructors will, of course, need to survey the exercises and make decisions on assignments based on their own expectations and on time available.

The Role of Explaining: Students are told that the overall goal of these courses is to come to understand the mathematics deeply enough to participate in meaningful conversations about this mathematics and its applications, that being capable of solving a problem or performing a procedure, by itself, will not enable them to add value to the school experience of their students. But when they are able to converse with their students about mathematical ideas, reasons, goals, and relationships, their students can come to make sense of the mathematics. Opportunities for explaining occur during Activities and Discussions in class, and in Learning Exercises outside of class.

Publication Information: The final textbook will be published in two volumes by Freeman Press. The first will contain two parts: *Reasoning About Numbers and Quantities*, and *Reasoning About Algebra and Change*. The second will contain the other two parts: *Reasoning About Shapes and Measurement* and *Reasoning About Chance and Data*. The parts vary in length. There is a possibility for custom publishing to allow institutions to choose those parts and chapters that best suit their course syllabi.

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