

An Investigation of Negative-Number Reasoning: The Case of Violet

Jessica Pierson Bishop

Lisa Lamb

Randolph Philipp

Bonnie Schappelle

Ian Whitacre

San Diego State University

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## Abstract

Here we present part of a cross-sectional research study, wherein more than 150 K–12 students and specialized adults will be interviewed to (a) identify and categorize students' conceptions of integers and operations on integers and (b) identify developmental trajectories of students' integer understandings. In this illustrative case study, primary analyses are from a series of interviews with a 2nd-grade student, Violet. Violet invoked conceptions of number as both cardinal and ordinal and utilized a motion metaphor that enabled her to solve problems such as  $3 - 5 = \square$ ,  $-9 + 5 = \square$ , and  $\square + 5 = 3$ . However, her conceptions that addition makes larger and subtraction makes smaller overrode her capacity to solve  $5 + \square = 3$  or  $5 - \square = 8$ . We provide a possible trajectory to classify Violet's current understandings.

Key words: Integers, children's thinking, negative numbers, trajectory, case study

## An Investigation of Negative-Number Reasoning: The Case of Violet

Understanding negative numbers is complex. Although experts hold multiple understandings, metaphors, and contexts that enable them to move flexibly among interpretations of numbers as quantities, directed magnitudes, comparisons, and movements, most students and many adults face various difficulties when reasoning about negative numbers. Even in contexts in which directed magnitudes make sense (such as losses and gains), concepts of inverses and negation can be confusing and difficult to interpret. Furthermore, although decontextualized negative numbers may be manipulated by following simple rules, most adults and children have difficulty imagining -2 apples or jobs.

Many students we have interviewed echo these difficulties when trying to make sense of negative numbers. One second grader explained, “Negative numbers aren’t really numbers. They’re just acting like other numbers except there’s a minus in front of them” (Violet). Although negative numbers exist for Violet, she, like many early mathematicians, struggles to interpret them meaningfully. Bhascara I, a 7th-century mathematician, stated, “People do not approve of a negative absolute number” (Gallardo, 2002; Kline, 1980). As late as the 17<sup>th</sup> century, Descartes rejected negative solutions as false because they represented something less than nothing. Unsurprisingly given mathematicians’ struggles, extending one’s numerical domain from  $N$  to  $Z$  is conceptually challenging.

### Methods

#### *Study Design and Participants*

Here we present part of a cross-sectional research study wherein more than 150 K–12 students and specialized adults will be interviewed to (a) identify and categorize students’ conceptions of integers and operations on integers and (b) identify developmental trajectories of

students' integer understandings. In this paper we leverage cross-sectional data, but our primary analyses are from a series of interviews with one second-grade student, Violet. After conducting 65 pilot interviews, we chose to focus on this child as an illustrative case study (Yin, 2009) because Violet had no classroom instruction on negative numbers and we suspected that her strategies might reveal children's natural intuitions about negative numbers that interviews with older children might not.

We interviewed Violet three times over four months because during her initial interview (a) she knew that negative numbers existed and could reason with and about them; (b) her strategies and approaches to problems varied depending on the context (e.g., number-line and motion models vs. a formal approach relying on generalizations and properties); (c) she was articulate; (d) we wondered why she found some problems easy, yet other seemingly similar problems quite challenging; and (e) many of her responses were representative of other students' strategies.

#### *Data Collection and Analysis*

We developed tasks specific to integer reasoning to be used in problem-solving interviews.<sup>1</sup> The 45–60-minute videotaped interviews were conducted during the school day. We began interview analysis with a process of open coding (Strauss & Corbin, 1998), focusing on solution strategies and underlying ways of reasoning about number and operations. We used the constant-comparative method to identify emergent, distinguishing themes and features of Violet's and other students' reasoning about integers.

#### Findings

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<sup>1</sup> We report here on a subset of the interview tasks, the open number sentences.

During our interviews with Violet, we asked her a series of open number sentences (see Table 1 for these problems and her responses).

Table 1. *Violet's Interview 1 Responses*

<b>Problem</b>	<b>Correct Responses</b>
$3 - 5 = \square$	“Well my brain was thinking that 3 minus 3 is zero and that 0 minus 2, because 3 plus 2 is 5, um, that must be negative 2.”
$-9 + 5 = \square$	Violet used a counting strategy saying, “Negative 9 plus 1 [puts up a finger] would be negative 8. Plus 2 is negative 7 [puts up 2 <sup>nd</sup> finger], plus 3 is negative . . . negative 6 [3 <sup>rd</sup> finger], plus four is negative 5 [4 <sup>th</sup> finger], and plus five is negative 4 [5 <sup>th</sup> finger].”
$\square + 5 = 3$	Violet used trial and error on the number line, starting at -5 and counting up 5 places to end at 0, then starting at -4 and ending at 1. After starting at -3 and ending in the wrong place, she answered, “Negative 2” and counted up 5 places to end at 3.
<b>Problem</b>	<b>Lack of or Incorrect Responses</b>
$5 + \square = 3$	“There’s no way to do that because you have to do 5 minus something. If you add to that, say 5 plus 2, that would be 7 not 3.”
$5 - \square = 8$	Violet changed the minus sign to a plus sign and wrote 3 in the box. When asked if it was possible to solve the original problem, she said, “I don’t think so.”
$5 + \square = 2$	Violet: I’m just thinking this [pointing to the addition sign] has to be a minus for this to be possible. Interviewer: Is there any number that could go in the box for it to be possible? Violet: Not that I know. (Pause) Well if that [points to 5 in $5 + \square = 2$ ] were -5, that could be possible. (Violet uses a number line to solve $-5 + \square = 2$ ): Negative 5 plus 1[points to -4], 2 [points to -3], 3 [points to -2], 4 [points to -1], 5 [points to 0], 6 [points to 1], 7 [points to 2]. So 7.
True/False: $6 + -2 = 6 - 2$	“False. Because that’s a negative 2 [pointing to -2 in $6 + -2$ ], and you can’t add with negative numbers. And that’s [pointing to the 2 in $6 - 2$ ] a 2.” After she said that $6 + -2 = 4$ was false, she was asked whether adding negative numbers is impossible. Violet answered, “The way that I think, yes.”

Violet solved the first problem involving negative numbers,  $3 - 5 = \square$ , by *jumping to zero*, a sophisticated version of a counting-back strategy. Much as children use decade numbers

in incrementing strategies to solve problems involving regrouping (e.g., jumping to 30 to solve  $33 - 5$ ), Violet used the *friendly number* of zero, but when decomposing 5 into 3 and 2, she crossed zero, not 30. This strategy provides insight into her initial reasoning about inverses because she intuitively used the property  $a - a = 0$  in jumping to zero.<sup>2</sup>

We contrast Violet's response with responses of other 6- and 7-year-olds<sup>3</sup> who claimed that the problem could not be solved. Niki said, "Three minus 5 doesn't make sense because 3 is less than 5," whereas Callie said, "Three minus 5 is zero. You have 3 and you can't take away 5, so take away the 3 and it leaves you with zero." The difficulty for Callie, Niki, and others seems to be their single (but, given their mathematical experiences, unsurprising) interpretation of problems of the form  $x - y$ , in which  $y > x$ . One second grader provided insight into these responses in explaining, "Zero is nothing, but negative is more nothing" (Rachel). If zero means *nothing*, taking something from nothing in the world of countable objects is impossible, just as Niki claimed.

Responses like Niki's seem to indicate an understanding of numbers as cardinal and representing a quantity or number of objects. Violet, too, views numbers as cardinal, stating, "Negative numbers aren't really numbers. ... We don't really count with them, and there's no negative-1 cube (shows a unifix cube)." Yet Violet successfully solved this problem, not because taking something from nothing became acceptable to her but because she leveraged other representations and meanings of number. Specifically, she relied on the ordinal nature of numbers. In addition to countable objects, numbers are unique locations or positions that exist in a sequence for Violet. Precisely because numbers are well ordered, she can extend her counting-

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<sup>2</sup> We claim not that Violet had a complete understanding of inverses (namely that  $a + -a = 0$ ) but that her intuitive use of  $a - a = 0$  may serve as a precursor to this more formal notion.

<sup>3</sup> Ten of the 18 (56%) first- and second-graders interviewed solved  $3 - 5 = \square$  correctly.

back strategy beyond zero, as seen in her response to  $3 - 5 = \square$  as well as to  $-9 + 5 = \square$ . We suspect that viewing numbers as positional and sequential might be particularly useful in helping students overcome the confusing notion of taking something from nothing so they can reinterpret  $3 - 5 = \square$  in a new context (Bishop et al., in press).

Now consider problems Violet solved incorrectly or described as unsolvable (lower section of Table 1). Violet was unable to solve  $5 + \square = 3$  or  $5 + \square = 2$ .<sup>4</sup> Because both problems contradict the widely held generalization that addition makes larger, we suspected that open number sentences of this form might be particularly problematic. Similarly, we expected the open number sentence  $5 - \square = 8$  to be challenging because it contradicts the notion that subtraction makes smaller. For young children whose numerical domain is almost exclusively positive, such generalizations about addition and subtraction are sensible. In fact, Violet has good company: The Greek mathematician Diophantus in the 3<sup>rd</sup> century described the equation  $4 = 4x + 20$  as “absurd” because the 4 units on the left were less than the 20 units that were added (Gallardo, 2002); even the Father of Algebra was bothered by problems in which the sum was smaller than either addend.

Given Violet’s insistence that  $5 + \square = 3$  was unsolvable, we anticipated a similar response to  $\square + 5 = 3$ ; in both problems the sum is less than the given addend, and the latter problem is a start-unknown problem, typically harder than a change-unknown problem (Carpenter et al., 1999). Surprisingly, Violet not only engaged with the start-unknown problem but also correctly solved it by interpreting adding and subtracting as motion and using a number-line model.

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<sup>4</sup> Four of the 18 (22%) first- and second-graders interviewed solved  $5 + \square = 3$  correctly.

In the symbolic representation of an addition problem, one might treat the addends as the *same* in some sense; thus, we expected the placement of the 5 in  $5 + \square = 3$  and  $\square + 5 = 3$  to be less important than the fact that the sum of 3 is less than the addend. However, Violet approached these problems quite differently and appeared to have different interpretations of the problems themselves. Upon closer inspection, the open number sentence  $\square + 5 = 3$  is consistent with the idea of addition makes larger in Violet's motion context. Consider the location of the unknown within a motion context in terms of the CGI problem-type framework (Carpenter et al., 1999).

$$\boxed{\begin{array}{c} \text{Starting} \\ \text{Point} \end{array}} + \boxed{\text{Change}} = \boxed{\begin{array}{c} \text{Ending} \\ \text{Point} \end{array}}$$

Violet interpreted the open number sentence  $\square + 5 = 3$  as asking her to start at some unknown place, move right 5 units, and end at 3. In fact Violet said, "You're adding when you go this way (points right), and you're subtracting when you go this way (points left)." When one starts at -2 and adds 5, the resultant sum is, in fact, larger than the starting point of -2, thereby satisfying the addition-makes-larger idea. Within the number-line model, the second addend of 5 is treated not as an amount but as a change or movement. Consequently, it need not be smaller than the sum.

The problem  $5 + \square = 3$ , however, proves difficult to interpret even within a motion context. Even if Violet had overlooked the contradiction to addition makes larger and engaged with the problem using a number line, she would face a conundrum: If addition means to *move right*, how can one start at 5, move right, and end to the left of the starting point? This seeming contradiction is the number-line analogue to the addition-makes-larger contradiction.



Violet's responses indicate that a context of motion, similar to the real-life contexts of CGI problems, may support differential treatment and interpretations of the two addends. Further, Violet, like many K–4 students we interviewed, was unable to imagine a kind of number she could add and move left or add to yield a smaller sum. To be clear, Violet *could* add a negative number but *only* as the first addend. The existence of a solution, much less a negative solution, to  $5 + \square = 3$  was beyond her frame of reference. She had no model to help her make sense of *double signs* (e.g., in  $6 + -2$ ) or the corresponding situations necessitating their use. When asked explicitly to interpret and solve problems with such signs, Violet staunchly maintained, “You can’t add with negative numbers [pointing to  $-2$  in  $6 + -2$ ].” She provided no interpretation and hypothesized no meaning for adding a negative but instead insisted that adding negative numbers is impossible in her way of thinking.

#### Discussion

We draw from Violet's responses to present a possible trajectory of integer reasoning that extends through addition and subtraction of integers. The bulleted points are major integer *milestones* that, presumably, build on and encompass previous points/understandings.

- Objects *below* zero exist (these might be shapes, negative hallways, all zeros, or random [unordered] numbers).
- Negative numbers exist as a well-ordered numeric set (i.e., as locations or positions).  
This understanding is likely related to ideas about what numbers are and how to place/locate numbers on a number line (e.g., a number is *how many* [cardinal view] or a number-as-location [an ordinal view]).
- Negative numbers exist as results of operations (e.g., solutions to problems such as  $3 - 5 = \square$ );

- Negative numbers exist as a first operator or *starting point* as (a) an unknown in problems such as  $\square + 5 = 2$  or (b) a given in problems such as  $-3 + 4 = \square$  (in which a negative number appears as the first addend);
- Negative numbers exist as a second operator or *change* in problems such as  $5 + \square = 2$ , in which the unknown is the negative number, or in problems such as  $6 + -2 = \square$ , in which the negative number appears as the second addend. Note that for negative numbers to exist as a second operator, students must develop meanings for *double signs*.

Violet appears to have achieved all but the last milestone. We hypothesize that for Violet (and students who reason similarly) to reach this final milestone, she may have to either adopt a new metaphor (with her current motion metaphor, for addition and subtraction she can move only right and left, respectively) or expand her motion metaphor by conceiving of addition as moving either right *or left* as determined by the second operator. That is, she would have to reason that addition of a negative number entails moving to the left on the number line, a notion that extends beyond her current set of mathematical experiences and conceptions. Because our work is ongoing, we anticipate identifying multiple trajectories of integer reasoning that children take on their way to more sophisticated and flexible ways of reasoning.

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