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Learning by Mapping Across Situations

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Many types of learning require the mapping of information across situations. The proposed organizational framework extends the cognitive study of mappings across problems to include mappings across representations, solutions, and sociocultural contexts. I apply one-to-one, one-to-many, and partial mappings to analyze representative cases that include ones studied within alternative frameworks that have extended the cognitive approach. My objective is to support readers in reflecting on how mappings influence learning through asking new questions and providing an approach for answering those questions.

My motivation for writing this article can be traced back to my reading of Jean Lave's (1988) book *Cognition in Practice: Mind, Mathematics, and Culture in Everyday Life*. I realized it was an important book but did not read it until approximately 10 years after its publication. When I did read it I was surprised to discover that Lave's critique of the cognitive approach to transfer began with one of my own studies (Reed, Ernst, & Banerji, 1974). Like many other academics I would rather have my work critiqued than ignored. I was therefore honored that Lave selected our study to represent the limitations of the cognitive approach.

Lave's book began a lively debate of the relative merits of cognitive (Anderson, Reder, & Simon, 1996) and alternative (Greeno, Smith, & Moore, 1993) perspectives on learning. There have been times when I considered joining this debate, but I have deferred for two reasons. The first is that scholars such as John Anderson, Lynn Reder, and Herb Simon had already strongly defended the cognitive approach. The second is that I thought that alternative approaches to transfer based on the situated cognition and sociocultural perspectives were logical extensions of the cognitive approach. The study of transfer in the 1970s

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focused on puzzles such as missionaries and cannibals (Reed et al., 1974) and the Tower of Hanoi (Hayes & Simon, 1977). In the 1980s studies shifted to classroom problems such as word problems and physics problems and included investigating the important role of expertise in determining how students approached these problems (Chi, Glaser, & Rees, 1982). Studying learning outside the laboratory, which began to gather momentum following the publication of Lave's (1988) book, is a fruitful extension of this earlier research.

These newer alternative perspectives have enhanced understandings of transfer, as revealed by articles in the transfer strand of the *Journal of the Learning Sciences* (Lobato, 2006). However, there are also elements of commonality between cognitive and alternative approaches that could be further developed. Making the argument that different approaches should build on the cognitive perspective requires identifying accomplishments that can be applied to alternative approaches. I argue in this article that cognitive theories of analogical reasoning based on mappings of components (Gentner, 1983; Gentner & Markman, 1997; Holyoak, 2005) form a foundation that can be extended to these other perspectives.

My goal is to propose an organizational framework that analyzes learning and transfer from the perspective of mapping across situations. Studying these mappings provides one approach to finding commonalities in learning across a variety of situations, including those studied within both the cognitive and alternative perspectives.

I selected mapping as the basic unit of analysis because of its generality. I define a *mapping* as a link between two nodes in which the link represents a relation and the nodes represent knowledge states. *Knowledge states* are knowledge about physical or generated aspects of the environment, including concepts, parts of diagrams, mathematical symbols, gestures, and causal explanations. They are distributed between internal and external resources. For instance, a knowledge state may result from attending and encoding part of the physical environment, such as noticing that a diagonal line in a diagram can be used as part of a geometric proof. A knowledge state can also result from constructing a diagonal line in the diagram to use in the proof or from retrieving from memory that the proof includes a diagonal line. Examples of relations are "corresponds to," "is an example of," and "is represented by."

Theoretical formulations of mappings come primarily from studies of analogical reasoning. For instance, there is a mapping between Adolf Hitler and Saddam Hussein in the analogy between Germany's invasion of Austria and Iraq's invasion of Kuwait. The relation connecting the nodes in analogical mappings is "corresponds to" because the mapping identifies corresponding elements in the analogy, such as leaders of the invading countries. Other mappings establish hierarchical relations; a tumor and a fortress are both examples of a central target in variations of Duncker's radiation problem. Still other mappings occur across representations. The letter *a* may represent the number of apples in a word problem or the slope of a line in a graph.

My plan is to begin with a discussion of cognitive analyses of analogical reasoning and transfer because this area has received the most attention in mapping theories. I then summarize the views of theorists who have proposed alternative analyses of transfer. I next propose a taxonomy that examines three different kinds of mappings (one to one, one to many, partial) across four different types of situations (problems, representations, solutions, sociocultural contexts). I then discuss the implications of the taxonomy for analyzing learning. And finally, I review how the mapping framework provides a common theme across different theoretical perspectives.

ANALOGICAL MAPPINGS

The theoretical understanding of mappings has primarily come from the study of analogical reasoning (Holyoak, 2005). Holyoak stated that two situations are analogous if they share a common pattern of relationships among their elements (objects and attributes), even though the elements themselves may differ across the two situations. A target situation serves as a retrieval cue for a potentially useful analogy (the source). Analogical reasoning is used to formulate a mapping—a set of systematic correspondences—to align elements of the source and target. The mapping can result in new inferences about the target, resulting in greater understanding.

Holyoak (2005) used Iraq's invasion of Kuwait as an example. President George Bush's decision to counterattack can be compared to a World War II example in which Hitler occupied Austria. Adolf Hitler and Saddam Hussein are the leaders of the invading forces, Germany and Iraq are the invading countries, Austria and Kuwait are the occupied countries, and Prime Minister Winston Churchill and President George Bush are the leaders of the countries that launch a counterattack (Spellman & Holyoak, 1992). Applying this analogy therefore requires these mappings:

Adolf Hitler → Saddam Hussein
Germany → Iraq
Austria → Kuwait
Winston Churchill → George Bush

Notice that the mappings preserve the *relationships* among the leaders and countries. Adolf Hitler, the leader of Germany, invaded Austria and met a counterattack launched by Winston Churchill. Saddam Hussein, the leader of Iraq, invaded Kuwait and met a counterattack launched by George Bush.

There have been a number of detailed computational models of analogical mappings, including one by Hummel and Holyoak (1997). Mappings in their model are guided by three constraints:

1. *Structural consistency* implies a one-to-one mapping between an element in the source and an element in the target.
2. *Semantic similarity* implies that elements with prior semantic similarity (such as joint membership in a taxonomic category) should tend to map to each other.
3. *Pragmatic centrality* implies that mappings should give preference to elements that are important to goal attainment.

Each of these three constraints is relevant to the mappings that I discuss in this article. My focus is on the structure of mappings, emphasizing those mappings that are important for goal attainment (pragmatic centrality).

A question that arises from the mapping framework is whether mappings are inherent properties of situations or are constructed by the learner. The answer is both, as explained by Newell and Simon (1972) in their book *Human Problem Solving*. Their book provides a detailed documentation of how both the structure of problems (represented by a search space) and peoples' strategies (represented by a problem space) influence problem solving. A *search space* shows the legal moves at each step in solving a problem and is therefore an objective analysis of the problem. A *problem space* describes the choices that the problem solver evaluates while solving the problem. Newell and Simon built simulation models of each solver's problem space by collecting detailed verbal protocols of his or her deliberations while solving the problem.

The search space, however, constrains the kinds of decisions required by the problem solver and determines the effectiveness of those decisions. For example, Herb Simon and I developed a simulation model of how students solved a complex version of the missionaries and cannibals problem, which requires transporting five missionaries and five cannibals across a river (Simon & Reed, 1976). Our goal was to predict, for each of the possible legal moves, the average number of times students made that particular move. We developed a successful model based on the assumption that solving the problem required switching from an ineffective balance strategy to an effective means–end strategy. The balance strategy attempts to place an equal number of missionaries and cannibals on each of the two river banks. It is an effective strategy for satisfying the constraint that cannibals cannot outnumber missionaries on either bank but results in entering a blind alley of the search space. Our strategy-shift model illustrates how both the structure of the problem and strategies adopted by the learners combine to determine effective problem solving.

Some mappings across situations describe inherent properties of situations. Others describe constructions by the learner. However, as the search space constrains the problem space, situations constrain constructions. Situations are the starting point for both cognitive and alternative approaches to transfer. Because my examples illustrate mappings from both perspectives I review the challenges raised by the alternative perspectives before proposing a taxonomy.

EXAMPLES OF ALTERNATIVE PERSPECTIVES

Differences between perspectives has been represented by Alexander (2007) in a two-dimensional space shown in Figure 1. One dimension is whether knowledge exists in the mind or within the environment. The other dimension is whether knowledge is individually formed or socially derived. As noted in Figure 1, various learning perspectives are highly correlated along these two dimensions. Cognitive (information-processing) approaches typically study the learning of an individual within a laboratory setting. Sociocultural and situated approaches typically study how individuals learn from one another outside the laboratory. I refer to the traditional information-processing framework

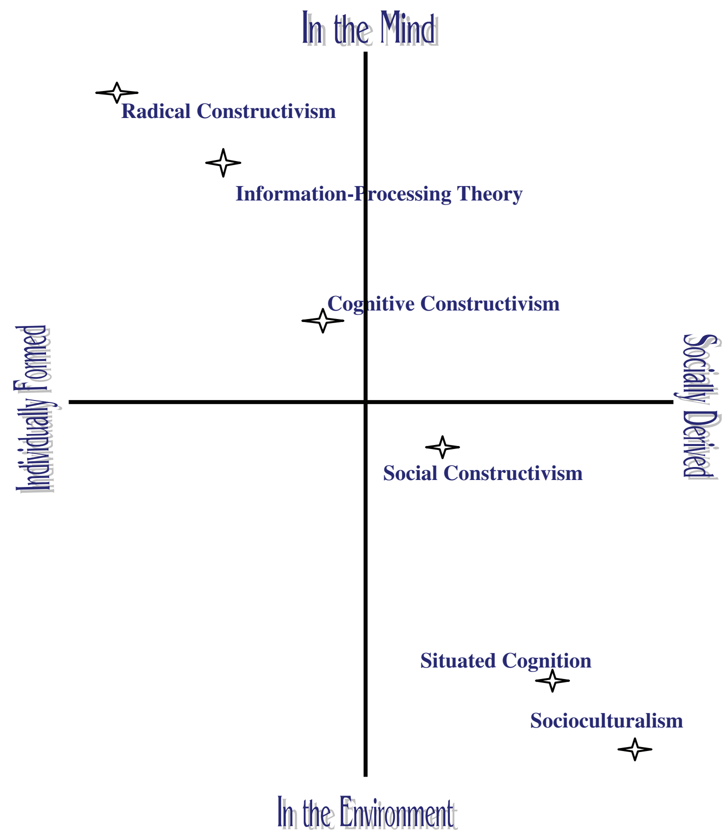


FIGURE 1 A two-dimensional representation of learning frameworks. From Alexander (2007) (color figure available online).

as the *mainstream cognitive perspective* (thus allowing for non-information-processing cognitive approaches) and to other frameworks as *alternative perspectives*.

The following examples present a brief overview of several challenges to the cognitive perspective by theorists working within alternative perspectives. This overview has two purposes. First, it highlights differences between the cognitive and alternative perspectives. Second, it provides cases within the alternative perspectives that I later analyze within the mapping-across-situations framework. Although transfer is multifaceted and there are differences among perspectives, there are also commonalities among transfer situations. The mapping-across-situations framework explores these commonalities and extends them to other learning situations that do not involve transfer.

Situated Perspective: Lave (1988)

The initial major challenge to the cognitive approach to transfer came in Lave's (1988) classic book *Cognition in Practice: Mind, Mathematics, and Culture in Everyday Life*. In her second chapter—"Missionaries and Cannibals (Indoors)"—Lave analyzed four traditional transfer studies that used missionaries and cannibals (Reed et al., 1974), the Tower of Hanoi (Hayes & Simon, 1977), Duncker's radiation problem (Gick & Holyoak, 1980), and electric circuits (Gentner & Gentner, 1983). All of these studies examined students' analogical mapping of structural relations between problems.

Lave (1988) concluded that

> learning-transfer research clearly falls in the functionalist tradition sketched in chapter 1. Its central characteristics include the separation of cognition from the social world, the separation of form and content implied in the practice of investigating isomorphic problem solving, and a strictly cognitive explanation for continuity in activity across situations. All of these *dissociate* cognition from its contexts, and help to account for the absence of theorizing about experiments as social situations and cognition as socially situated activity. (p. 43, emphasis in the original)

The context for this quote is the subsequent discussion of Lave's Adult Math Project, which investigated transfer "outdoors" and in which shoppers used mathematics in the supermarket. The focus of this project was to determine whether mathematics learned in school transfers to the real-world task of using proportional reasoning to determine best buys. The investigators recorded information on the prices and quantities of each grocery item mentioned by the shopper to determine whether the shopper had selected the best buy. A subsequent interview presented 12 best-buy problems at home in which the investigator discussed the solution to be certain that the shoppers had not merely guessed the correct answer.

The Adult Math Project used four tests to measure the shoppers' knowledge of how arithmetic procedures are applied to the integers, decimals, and fractions learned in school. There were no significant correlations between these four tests and the best-buy home test. The only significant correlation with the supermarket purchases occurred for a test of measurement facts. Based on these findings and some other work using the situative perspective, Lave (1988) concluded that "all of the studies demonstrated discontinuities in problem-solving processes between situations, and the uncoupling of math performances from schooling except during tests" (p. 68).

Gibsonian Perspective: Greeno et al. (1993)

In a chapter titled "Transfer of Situated Learning," Greeno et al. (1993) emphasized activities in which a person or a group interacts with objects in a situation. Following Gibson (1966), they used the term *affordances* to refer to relevant properties of objects for supporting activities. Transfer occurs in their framework if the structure of the activity is invariant across the transformation of one situation into the other or, if necessary, a transformation of the activity can be accomplished.

Greeno et al. (1993) contrasted their approach with the symbolic representations used in cognitive theories of transfer. Affordances can be perceived without a symbolic representation of the properties, although symbolic representations are included in their framework when necessary. The interaction of perception and action in their theory mirrors many of the assumptions that were subsequently emphasized in articles on perceptual symbols systems (Barsalou, 1999) and embodied cognition (Wilson, 2002), in which perception and action play central roles in cognition. For example, Greeno et al. argued that "event schemata enable inference of potential states of affairs through a process of perceiving information that specifies the possibility of a transformation in the environment, as perception of an affordance involves perceiving information that specifies the possibility of an action" (p. 107).

Transfer-in-Pieces Perspective: Wagner (2010)

Wagner (2010) took a more cognitive approach to transfer than either Lave (1988) or Greeno et al. (1993) by applying diSessa's (1993) "knowledge-in-pieces" framework to the transfer of learning, but his approach is more compatible with Piagetian constructivism (Piaget, 1970) than with information processing. He used the term *concept projection* as a fine-grained assimilatory structure that allows a person to interpret a situation in a meaningful way and implement a concept within a given situation.

Wagner's (2010) investigation of students' attempts to understand the squash problem in Table 1 can serve as an example. The data were collected as part of teaching interviews during an 8-week summer course on elementary statistics. The problem was part of a unit on the law of large numbers, which states that

TABLE 1
Problems Requiring the Law of Large Numbers: From Wagner (2006, 2010)

The Squash Problem

The game of squash can be played either to 9 points or to 15 points. Christina does not play squash as well as her friend Angela. Is it Christina's advantage to prefer either scoring system over the other when she plays against Angela?

The Post Office Problem

When they turn 18, American males must register for the draft at the local post office. In addition to other information, the height of each male is recorded. The average height of 18-year-old males is 5 feet 9 inches.

Every day for 1 year, 25 men registered at Post Office A and 100 men registered at Post Office B. At the end of each day, a clerk at each post office computed and recorded the average height of the men who had registered there that day. Which would you expect to be true? (circle one)

1. The number of days in which the average height was 6 feet or more was greater for Post Office A than for Post Office B.
2. The number of days in which the average height was 6 feet or more was greater for Post Office B than for Post Office A.
3. There is no reason to think that the number of days on which the average height was 6 feet or more was greater for one post office than the other.
4. It is not possible to answer the question.

larger samples are more likely than smaller samples to be representative of their parent population. Although the law of large numbers generally refers to population averages or proportions, neither value is provided in the squash problem. This absence caused five students to create different concept projections to defend their answers that Christina would have a better chance of winning if she selected the smaller score. Jason and Wilma structured the problem to emphasize performance quality, Peter focused on the relative position of the scores, Mike talked about the outcome of the game, and Tasha spoke about the percentage of points earned.

Wagner's (2006, 2010) transfer-in-pieces framework also differs from the cognitive approach by investigating how multiple sources of information, rather than a single-source problem, influence transfer. I discuss these multiple sources when I later return to Wagner's (2006) research.

Actor-Oriented Perspective: Lobato (2008)

Lobato (2008) has also been influenced by traditional approaches to transfer but proposed that most of the work on transfer by cognitive psychologists has been guided more by their perception of the task than by the learner's perception of the task. To correct this imbalance, she argued that researchers need to rethink transfer. She contrasted the previous work of cognitive psychologists with a view that she labeled *actor-oriented transfer*. Table 2 summarizes the differences.

TABLE 2
Theoretical Assumptions of Actor-Oriented Transfer Compared to Classical Transfer: From Lobato (2008)

Dimension	*Classical Transfer Approach*	*Actor-Oriented Transfer*
1. Definition	The application of knowledge learned in one situation to a new situation	Generalization of learning; "similarity making" is a primary process but generalizing can also involve constructing differences and modifying situations
2. Perspective	Observer's (expert's)	Actor's (learner's)
3. Research method	Experimental methods to identify improved performance between learning and transfer tasks	Ethnographic methods to look for the influence of prior activity on current activity and how learners see situations as similar
4. Research questions	Was transfer obtained? What conditions facilitate transfer?	What relations of similarity are created? How are they supported by the environment?
5. Surface vs. structure	Paired learning and transfer tasks share structural features but differ by surface features	Researchers examine learners' construal of "transfer" settings, acknowledging that a surface feature for an expert may be structurally substantive for a learner
6. Location of transfer	Transfer measures a psychological phenomenon	Transfer is distributed across mental, physical, social, and cultural planes
7. Transfer processes	Overlapping symbolic abstract mental representations (schemes)	Multiple processes, including "focusing phenomena"
8. Metaphor	Static application of knowledge	Dynamic production of relations of similarity
9. Content domain	Mathematics is often treated as a set of procedures and transfer as a decontextualized ability	The conceptual sense that people make of mathematics is central to the nature of transfer
10. Abstraction	Inductive and individualistic—a common property is extracted	Constructive and involves social and individual processes

One distinction is that actor-oriented transfer emphasizes perceived similarities among tasks rather than the expert-defined mappings that have been identified by cognitive psychologists. An implication is that more transfer occurs than has been reported by cognitive psychologists, who are only interested in expert-defined mappings.

It can also be seen from Table 2 that Lobato's actor-oriented view borrows many of the key assumptions of the situated approach (Greeno et al., 1993). Perceived similarities are not limited to symbolic representations but can be influenced by perceptual constraints and prior activities. In agreement with Wagner

(2010), Lobato and Rhodehamel (2010) proposed that the regularities noticed by the learner are not inherent in the situation but are constructed by the learner. In addition, transfer is distributed across social and cultural situations. However, there are also differences in emphasis. Whereas Lave (1988) emphasized the lack of transfer across situations, Lobato (2008) argued that more transfer occurs than is typically reported by those who study it.

An actor-oriented approach to transfer often reveals idiosyncratic ways in which individuals learn. Although some of these may seem random, the nature of learners' generalizations is constrained by sociocultural practices. Teachers and classroom discussions determine what students notice, as revealed by two divergent centers of focus across two mathematics classes (Lobato & Rhodehamel, 2010). One class treated points, lines, and slopes as mathematical objects, whereas another class treated them as physical objects. Students in the first class focused on the slope of a line as an attribute of a relationship between quantities, whereas students in the second class treated slope as a measure of the steepness of a physical object. Sociocultural practices in the classroom are the focus of Engle's (2006) analysis of transfer.

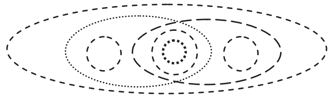
Framing Perspective: Engle (2006)

According to Engle (2006), transfer across contexts can be greatly facilitated if the teacher frames the instruction to encourage transfer. Drawing on the view that tasks are to prepare for future learning (Bransford & Schwartz, 1999), Engle argued that learning environments enhance transfer if "learners come to understand that what they are currently doing is part of a larger intellectual conversation that extends across time" (p. 457).

Engle's (2006) analysis builds on cognitive contributions but goes beyond them in proposing a sociocultural account of transfer in which appropriate framing of discussions is essential for success. Table 3 contrasts framing as bounded events with framing as ongoing activities. Framing learning as an ongoing activity encourages students to share their ideas with an extended audience who would be interested in discussing and building on their ideas. Framing should also make it clear that students learn ideas in order to use them, creating an expectation for transfer.

Each of these alternative perspectives presents a conceptual framework that differs from the traditional cognitive perspective. Although some of these claims may be incompatible with the cognitive approach, others are extensions that provide more comprehensive interpretations of learning and transfer. The mapping-across-situations taxonomy therefore extends the structural mappings studied by cognitive psychologists to encompass a broader range of applications.

TABLE 3
Two Contrasting Ways in Which Learning Situations May Be Interactionally Framed: From Engle (2006)

Frameable Aspects of Learning Situations	*As Bounded Events* 	*As Parts of Open Ongoing Activities* 
Who: Participants	• Fixed set • Just those co-present	• Open, expanding set • Co-present and imagined
When: Temporal horizon	• Set starting point • Set ending point	• Build on past • Project to future
Where: Location	• Narrowly defined • Circumscribed	• Broadly defined • Extendable
What: Topics	• Individual topics • Not connected	• Multiple topics at once • Linked or embedded
How: Roles and practices	• Students positioned more passively as recipients, mouthpieces, etc. • Predefined procedures	• Students positioned more actively as respondents, creators, etc. • Negotiable processes
Why: Purposes	• Single purpose • Fixed in advance	• Multiple purposes • Can be renegotiated

A MAPPING × SITUATION TAXONOMY

This article makes two contributions to the study of mappings across situations. First, it distinguishes among three different kinds of mappings: one to one, one to many, and partial. Second, it shows the application of these mappings to four different kinds of situations: problems, representations, solutions, and contexts (classrooms and cultures). Table 4 shows the 12 cells created by this 3 × 4 classification of mappings and situations.

One type of *mapping* that is emphasized by Hummel and Holyoak (1997) is an isomorphic mapping in which there is a one-to-one correspondence across important elements and relations in the two situations. Those elements and relations that are required to understand one situation uniquely map onto those elements and relations that are required to understand the other situation. In contrast, in a one-to-many mapping, a single element in one situation maps onto more than one element in the other situation. A partial mapping is one in which only some of the

TABLE 4
Examples From the Proposed Taxonomy for Mapping Across Situations

Mapping	*Problems*	*Representations*	*Solutions*	*Contexts*
One to one	Gick & Holyoak (1980, 1983)	Greeno et al. (1993)	Reed (1997)	Nunez & Sweetser (2006)
One to many	Reed et al. (1974)	Greeno & van de Sande (2007)	Chi & Ohlsson (2005)	Legare & Gelman (2008)
Partial	Gentner & Gentner (1983)	Lobato (1996, 2008)	Rittle-Johnson et al. (2009)	Engle (2006)

elements and relations in one situation map onto the other. Figure 2 shows these three types of mappings.

The other dimension in the taxonomy is *situation*. I distinguish among four kinds of situations. The first is mapping across problems. Transfer across problems has been the core of the cognitive approach, typified by its study of establishing an analogy between problems. Information learned from solving one problem can be helpful for solving or understanding a related problem. A second situation is mapping across different representations, such as text, diagrams, simulations, and mathematical symbols. The challenge here is to coordinate the representations to solve a problem. This area is beginning to attract more interest as multimedia provides a need to integrate multiple representations (Reed, 2006). The third situation is mapping across different solutions to the same problem. This case concerns how components of the initial solution are utilized in constructing the second solution and may involve trying to understand another person's alternative solution. The last situation is context. I use *context* in the sociocultural sense to refer to the social dynamics of classrooms (Engle, 2006) and cultures (Legare & Gelman, 2008; Nunez & Sweetser, 2006). Mappings therefore can occur across many situations, including entire search spaces (specifying legal moves and problem states), different solutions to the same problem, varieties of diagrams, words with multiple meanings, curriculum learning trajectories, and opposing cultural beliefs.

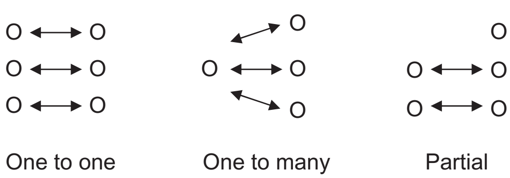


FIGURE 2 Illustration of one-to-one, one-to-many, and partial mappings across situations. The arrows represent the mapping of elements and relations across situations.

One change in my extension of mappings from problems to other kinds of situations is that I often focus on only key aspects rather than on the entire situation. The cognitive analysis of transfer across problems typically represents the mappings across entire problems, but this would become cumbersome, if not impossible, as situations become more complex. Emphasis on key aspects of the situation is an important assumption in Lobato's (2008) formulation (focusing phenomena in Table 2), in which classroom instruction may focus on aspects of a lesson that can either facilitate or inadvertently hinder transfer. Emphasizing key aspects is also consistent with the pragmatic centrality constraint in the Hummel and Holyoak (1997) formulation, in which mappings should give preference to elements that are important for goal attainment.

I construct the taxonomy by discussing mapping across problems, representations, solutions, and context with one-to-one, one-to-many, and partial mappings as subsections in each of these sections. Cases from both the cognitive and alternative perspectives fill the cells. My goal is not to provide algorithms for helping researchers classify their own research into a particular cell. Rather, it is to encourage readers to reflect on how mappings can help provide answers to a variety of questions, such as the following:

- Can instructional techniques that encourage the abstraction and generalization of solutions in the laboratory do the same in the classroom?
- Are simple solutions or complex solutions more effective in supporting transfer?
- Which kinds of scaffolding support linking diagrammatic, symbolic, and textual representations?
- How can mappings facilitate collaborative learning?
- How can mappings represent the replacement of indigenous knowledge by Western knowledge?

MAPPING ACROSS PROBLEMS

Lave's (1988) critique of cognitive psychologists' research on transfer included studies by Gick and Holyoak (1980), Reed et al. (1974), and Gentner and Gentner (1983). Each of these studies used a mapping framework to investigate transfer across problems. Gick and Holyoak (1980) studied whether students would more likely discover the convergence solution to Duncker's radiation problem if they were previously exposed to isomorphic variations of the problem. In contrast to these one-to-one mappings, Reed et al. (1974) examined the consequences of one-to-many mappings in variations of the missionaries and cannibals problem. Gentner and Gentner's (1983) study illustrates partial mappings because they proposed that analogies to circuit problems only partially explained key concepts such as capacitance and resistance. This section contrasts the type of mapping used in these three studies to form a foundation for extending these three types to representations, solutions, and contexts.

One-to-One Mappings Across Problems

Isomorphic problems enable a one-to-one mapping between the components and relations of the two problems so that learners can use the solution of one problem to understand or solve the other problem (Hummel & Holyoak, 1997). The research of Gick and Holyoak (1980, 1983) is perhaps the best known example of this type. Their 1980 study investigated whether more people would discover the convergence solution to Duncker's radiation problem (multiple rays converging on a tumor) if they were first exposed to an isomorphic solution. The analogous story described a military problem in which an army had to be divided in order to converge on a fortress. Using the one-to-one mapping required mapping goals, resources, constraints, and solutions across the two problems:

Use Army to Capture Fortress → Use Rays to Destroy Tumor
Sufficiently Large Army → Sufficiently Powerful Rays
Single Road Constraint → Intense Ray Constraint
Converge on Fortress After Dividing Army → Converge on Tumor After Dividing Rays

More than half of those who read the story and received a hint to use previous information included the convergence solution among their proposed solutions, compared with only 8% of those who did not read the story. But when Gick and Holyoak (1980) omitted the hint to use the story, the number of convergence solutions greatly decreased. Their findings demonstrated that people could use an isomorphic solution when prompted but did not spontaneously recognize the similarity between the two problems.

Gick and Holyoak (1983) subsequently discovered that people were likely to form a general convergences schema if they read and compared two analogous stories before trying to solve the radiation problem. For example, some students read both the military story and a story about spraying foam on an oil fire by surrounding it with many small hoses. Students who described the relation between these two stories were much more likely to think of the convergence solution to the radiation problem than were students who read only a single analogous story. Gick and Holyoak (1983) proposed that the problem solvers formed a more abstract representation of the solution (forces converging toward a central target) that allowed them to overlook the different surface features (oil fire, fortress, tumor). Catrambone and Holyoak (1989) found support for this interpretation by showing that reading two analogous stories without comparing them was not very helpful.

A characteristic of one-to-one mappings that has instructional implications is the facilitation of transfer through schema abstraction. Would these findings generalize from the laboratory to real-world situations as discussed by Lave (1988)? A recent study indicated that they do (Gentner, Loewenstein, Thompson, & Forbus, 2009). The participants in the first experiment were 124 management consultants with approximately 15 years of work experience who were attending a multi-day training session to learn negotiation strategies. They received two written cases that illustrated the principles of a contingent contract. The first described the shipment of boots by an Asian manufacturer to a buyer in the United States. The buyer wanted the manufacturer to pay the additional cost of airfreight because he thought the ship would be late. The two finally agreed that the buyer would pay the airfreight charge if the ship arrived on time but the manufacturer would pay the bill if the ship arrived late. The second case described two brothers who inherited a farm. Ben wanted to sell the rights to the farm's output under a long-term contract for a fixed amount, but Jerry wanted to keep the rights because he thought crop prices would rise. They resolved the conflict by deciding they would either sell or keep the farm depending on whether the price of next year's crop fell or rose.

The comparison group was asked to describe the similarities and parallels in the two negotiations. The separate-case group was asked to describe the solution and its probable success after reading each case. All participants were then asked to think of an example from their own experience that embodied the same principle as the two cases. As predicted from the previous laboratory findings (Catrambone & Holyoak, 1989; Gick & Holyoak, 1983), participants in the comparison group were more likely to retrieve memories from their previous experiences that embodied the principle of a contingent contract. They were also more likely to link the two cases (88% vs. 17%) and make contingent agreements in later face-to-face negotiations (59% vs. 33%). These findings provide a successful demonstration of applying a laboratory paradigm to the classroom.

One-to-Many Mappings Across Problems

Although one-to-one (isomorphic) mappings are the foundation of the cognitive study of transfer, a one-to-many mapping across problems reveals new insights about transfer (Reed et al., 1974). We investigated transfer between two problems that appeared to have similar search spaces in which the problem states and moves in one problem were similar to the problem states and moves in the other problem. One of the problems was the missionary cannibal (MC) problem, in which three missionaries and three cannibals cross a river using a boat that can hold two people under the constraint that cannibals can never outnumber missionaries. The other problem was the jealous husbands (JH) problem:

> Three jealous husbands and their wives, having to cross a river, find a boat. However, the boat is so small that it can hold no more than two persons. Find the simplest schedule of crossings that will permit all six persons to cross the river so that no woman is left in the company of any other woman's husband unless her own husband is present.

We anticipated, based on our perceived similarity of the two problems, that there would be substantial transfer from one problem to the other. Our first experiment found no transfer. Our second experiment found some transfer when students were informed about the mapping between the two problems: Husbands correspond to missionaries and wives correspond to cannibals. However, even with this hint, there was evidence of transfer only from the JH problem to the MC problem.

An explanation for this asymmetrical transfer occurred to us only after we closely examined the mapping between search spaces. There is a many-to-one mapping of moves from the JH problem to the MC problem: Moving a husband corresponds to moving a missionary, and moving a wife corresponds to moving a cannibal. In contrast, there is a one-to-many mapping from the MC to the JH problem because moving a missionary does not specify which husband to move and moving a cannibal does not specify which wife to move. Although all missionaries and cannibals are equivalent, all husbands and wives are not because they are paired with one another. For example, the three circles in Figure 2 representing one-to-many mappings might represent moving husbands A and B, B and C, and A and C. Each of these moves maps onto moving two missionaries, but moving two missionaries does not specify which two husbands to move:

	Husbands A and B
Two Missionaries ↔	Husbands A and C
	Husbands B and C

It should therefore be more difficult to map moves from the MC problem to the JH problem because this mapping does not specify a unique move. Our analysis of the mappings between the two problems therefore provided a theoretical explanation of asymmetrical transfer.

Partial Mappings Across Problems

A third case (Gentner & Gentner, 1983) discussed by Lave (1988) illustrates partial mappings. A similarity between isomorphic and partial mappings is that both are one-to-one mappings between the source and the target. The difference is that one-to-one mappings are sufficient for solving the target problem whereas partial mappings are not. The isomorphic variations of the tumor problem provide the key elements for solving these problems through the convergence solution. In contrast, Gentner and Gentner realized that a single analogy was insufficient for understanding the more complex domain of electric circuits.

The Gentners therefore identified two analogies to help students understand electric circuits—flowing waters and teeming crowds. They predicted that students who used the flowing waters analogy (pressure of water, flow in a pipe) should do well on questions about voltage and current because serial and parallel reservoirs combine in the same manner as serial and parallel batteries. In contrast, students with the teeming crowds model should do better on resisters because of the analogy to gates:

Pressure of Water → Voltage
Flow in a Pipe → Current
Resistance ← Narrowness of a Gate

The results of the first experiment supported their predictions. In the second experiment, the analogy to flowing waters was not as helpful as expected because students lacked knowledge in this area. Gentner and Gentner (1983) concluded that mapping from a source has a limited impact on understanding the target when students do not have a good understanding of the source domain.

Spiro, Feltovich, Coulson, and Anderson (1989) discussed practical implications of partial mappings. They proposed that simple analogies help beginners gain a preliminary understanding of complex concepts but can later block fuller understanding if learners never progress beyond the simple analogy. One consequence is that instructors need to pay closer attention to how analogies can fail. The authors discussed eight possible failures, including misleading properties, missing properties, a focus on surface descriptions, and wrong grain size. Their remedy was to use multiple analogies to convey the complexity of difficult ideas.

The use of multiple analogies is related to the transfer dilemma discussed by D. W. Carraher and Schliemann (2002) after interviewing two fifth-grade students about their understanding of operations on positive and negative quantities. The students' responses revealed that they were wrestling with multiple, competing ideas that included their understanding of number lines, arithmetic operations of addition and multiplication, use of the minus sign, and fractions. There was little evidence of transfer from a single problem, as assumed in many laboratory studies of transfer. Use of multiple sources of knowledge increases the difficulty of identifying mappings from prior experiences, but this increased complexity should not discourage researchers from making the attempt.

An admirable effort to discover the use of multiple sources of information is Wagner's (2006) microgenetic analysis of how an undergraduate student named Maria gradually constructed a more sophisticated understanding of the law of large numbers. The analysis was based on a 2-hr interview followed by a second session that occurred 5 days later. The post office problem in Table 1 was one of the five problems investigated in the study.

Maria's initial approach to the post office problem was based on her belief that smaller samples have higher averages than larger samples. She could not defend this belief and had no idea whether it was correct. To help Maria, during the second session the interviewer/tutor (Wagner) asked Maria whether a sample of 25 or 100 men would be better to determine the average height of men at her university. Maria correctly responded that a sample of 100 men would be more representative and more accurate. However, she initially applied her knowledge of surveys, in which larger samples were better, to incorrectly select the larger sample when the correct answer was the smaller sample (as in the post office problem). Additional interactions with the tutor enabled Maria to understand that because larger samples would more likely result in a sample mean closer to the expected value, smaller samples would more likely result in a sample mean further from the expected value.

Larger Samples Are Accurate → 100
Larger Samples Are Correct → 100
25 ← Smaller Samples Are Variable

Maria used her newly constructed knowledge to see commonalities in problems that she had previously viewed as different. For example, she initially did not see the relation between the post office problem and a spinner problem in which the probability of landing on a blue segment was .70. She later realized that both involved expected values, although one problem stated a mean and the other a proportion. Many other examples in Wagner's (2006) microgenetic analysis of transfer reveal how Maria's knowledge gradually accumulated by incorporating variations in the content of previous solutions. These examples illustrate how partial mappings from solutions can have both negative (always select large samples) and positive (relation between means and proportions) consequences.

These studies demonstrate differences between the cognitive (Gentner & Gentner, 1983; Spiro, Feltovich, Coulson, & Anderson, 1989) and alternative approaches to learning. As shown in Table 2 (Lobato, 2008), the classical approach looks for conditions to facilitate transfer by selecting helpful analogies. The actor-oriented approach interviews students to determine their perceived similarities between current and previous instructional material. Both approaches study mappings across problems but from different perspectives.

A blend of these two approaches occurs when teachers select analogies online that fit the context of student difficulties. Evidence comes from examining a random sample of 25 eighth-grade classrooms that were videotaped as part of the Third International Mathematics and Science Study (Richland, Holyoak, & Stigler, 2004). A definition of *analogy* based on Gentner's (1983) structure-mapping model resulted in the classification of 103 analogies, 86 of which were initiated by the teachers. When students had difficulty, teachers generated analogies with higher surface similarities to make the analogies more transparent. The blend occurs because although teachers selected helpful analogies, their selections were based on their understanding of their students' capabilities.

MAPPING ACROSS REPRESENTATIONS

Mapping across representations occurs when information from diagrams, formal concepts, mathematical symbols, and formulas is coordinated to produce a solution. All of the examples in this section require mapping components of diagrams to formulas, text, or mathematical symbols. The examples were initially developed within the framework of the situational (Greeno et al., 1993), sociocultural (Greeno & van de Sande, 2007), and actor-oriented (Lobato, 2008) perspectives. Greeno et al. (1993) discussed Wertheimer's (1959) classic problem of transforming a trapezoid into a rectangle so that its area can be calculated by using the formula *area* = *length* × *width*. Greeno and van de Sande (2007) analyzed a student's translation of a key phrase in a word problem ("an even border of flowers") into mathematical symbols for solving the problem. Lobato (2008) studied how students used informal formulas of rise/run to find components of a diagram to calculate the slope of a slide. Holyoak (2005) indicated that models of analogical reasoning have generally failed to address how the solution of problems is dependent on the difficulty of perceptually encoding key relations. These studies address this issue.

One-to-One Mappings Across Representations

One-to-one mappings have an advantage over other mappings when all of the solution steps can be transferred to solve another problem. However, only one of multiple solutions to a problem may support a one-to-one mapping. Discovering this solution is therefore advantageous, as demonstrated by Greeno et al.'s (1993) analysis of calculating the area of a trapezoid by transforming it into a parallelogram (Wertheimer, 1959).

As discussed previously, Greeno et al. (1993) argued for a situated approach to transfer in which transfer occurs if the structure of the activity is invariant across situations or if a transformation of the activity can be accomplished. They selected

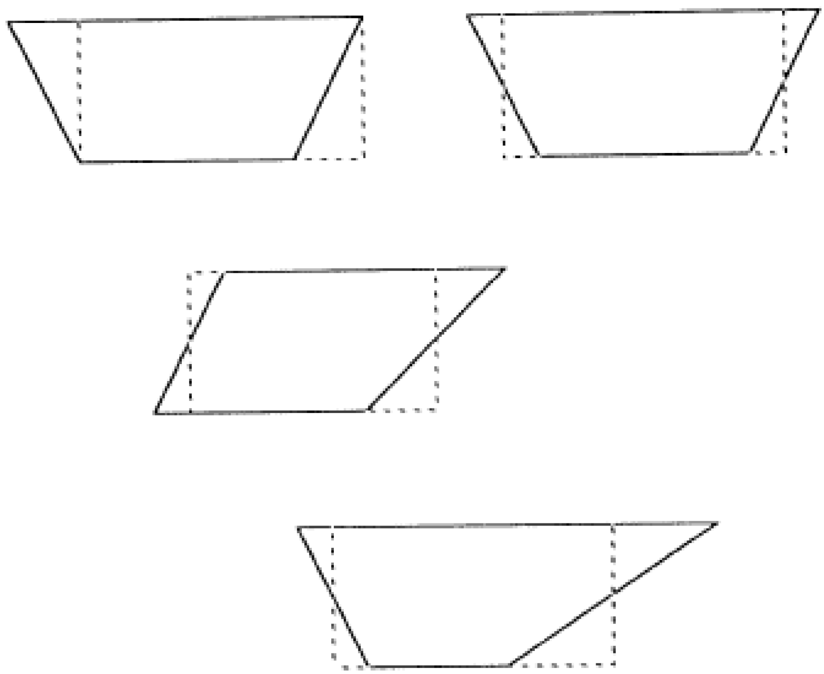


FIGURE 3 Geometric figures used by Wertheimer (1959).

Wertheimer's (1959) investigation of transformations of geometric forms as an example of such an activity. After investigating the students' ability to transform a parallelogram in order to apply the formula for area as the product of the length and width, Wertheimer studied the more complex forms shown in Figure 3. He was interested in analyzing both successful and unsuccessful transfer but did not specify the number of successful solutions for the different transformations.

Wertheimer's task provides a promising paradigm to further investigate the number and kind of transformations that are required to match geometric forms (Hahn, Chater, & Richardson, 2003). For instance, Edwards (1991) has designed a computer match game in which one geometric form must by modified to match another by using geometric transformations such as slide, rotate, pivot, reflect, and flip. In the Wertheimer forms cuts are required before the transformation is applied. Transforming a (vertically) symmetrical trapezoid into a rectangle can be accomplished by cutting one side and sliding it to the other side, as illustrated by the upper left transformation in Figure 3. Notice, however, that although cut and slide transforms a trapezoid into a rectangle, these operations cannot transform the lower forms (asymmetrical trapezoids) in Figure 3 into rectangles.

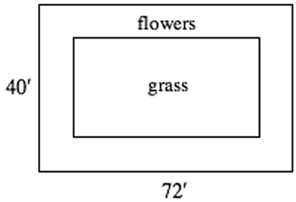
Fortunately, both Wertheimer (1959) and Greeno et al. (1993) presented an alternative transformation that has greater generality. This transformation, shown in the upper right of Figure 3, requires two cuts followed by pivots around a shared point. The two cuts and two pivots can be used to transform all of the forms in Figure 3. A cut and pivot provide one-to-one mappings on the left side of the symmetrical trapezoids to the left side of the asymmetrical trapezoids. The same mappings also apply to the right side of the figures. The advantage of the two cut-and-pivot transformations is that they enable students to calculate the areas of all of the Figure 3 forms by transferring the operations from symmetrical to asymmetrical trapezoids:

Cut Left Diagonal → Cut Left Diagonal
Pivot Left Diagonal → Pivot Left Diagonal
Cut Right Diagonal → Cut Right Diagonal
Pivot Right Diagonal → Pivot Right Diagonal

Readers may wonder why I have discussed this case as a mapping across representations when I have limited my analysis to geometric forms. Edwards's (1991) matching task does involve only transforming geometric forms, but Wertheimer's (1959) task requires coordinating representations. The purpose of his transformations is to change a form into a rectangle so that the formula for a rectangle (area = length × width) can be applied. These geometric mappings are therefore guided by the pragmatic centrality principle (Hummel & Holyoak, 1997), which gives preferences to mappings that are important for satisfying the goal of applying a formula.

An instructional challenge is to provide a scaffolded learning environment that can support a constructivist approach for these kinds of complex learning activities (Hmelo-Silver, Duncan, & Chinn, 2007). As children grow older, they are able to perform more complex transformations of shapes—transformations that are supported by interactive visual learning tools on the National Council of Teachers of Mathematics website (http://www.nctm.org). The Shape Cutter tool was one of three such tools selected by Liang and Sedig (2009) for redesign. They applied two frameworks to guide the redesign. The micro-level framework specifies low-level actions such as composing, cutting, fragmenting, searching, rearranging, and annotating (Sedig & Sumner, 2006). The macro-level framework supports more global pedagogical objectives such as focus, flow, and cognitive offloading to reduce memory load (Sedig & Liang, 2006).

Their redesigned Shape Cutter tool enables learners to find solutions to geometric problems by (a) constructing a polygonal shape, (b) partitioning the shape into smaller polygonal parts, and (c) dispersing and rearranging the parts by using transformational geometry. For example, the hints button could remind students about geometric forms (such as a rectangle and triangle) that have a formula for



Sally's backyard is 40 feet wide by 72 feet long, and it is structured so that a central rectangle of grass is surrounded by an even border of flowers. The area of the border (flowers) is 5/12 of the area of the whole garden. If Sally wants to be sure to walk her dog for .25 mile, how many laps will she have to take around the grass?

FIGURE 4 The garden border problem. From Greeno and van de Sande (2007).

calculating area and suggest approaches to partitioning a shape into parts in which the formulas could be applied.

One-to-Many Mappings Across Representations

The example in this cell focuses on how a student's one-to-many mapping from text to mathematical symbols resulted in a more complex solution to a problem than the one-to-one mapping constructed by the teacher. It discusses Greeno and van de Sande's (2007) use of the garden border problem shown in Figure 4 to illustrate a shift in a teacher's perspective to understand that a student's different approach to a problem can provide an alternative solution. I selectively focus on part of the verbal transcripts to illustrate the important role that mapping across representations played in constructing this perspectival understanding.

The key difference between the two solutions is how the teacher and student use the phrase "an even border of flowers." The teacher represents the width of this border by the unknown variable (such as w) and constructs an equation to represent the area of the inner rectangle by multiplying the length of this rectangle by its width:

$$(72 - 2w) \times (40 - 2w) = 1{,}680$$

This equation follows a previous calculation that the area of the inner rectangle is 1,680 square feet. Because the border is even, $2w$ can be subtracted from both the length and width of the outer rectangle.

However, this one-to-one mapping does not occur for the student (Gillian), who represents the border's width by two variables: $w_1 = (72 - y)/2$ with respect to the length of the outer rectangle and $w_2 = (40 - x)/2$ with respect to the width of the outer rectangle.

$$\text{Width of Flower Border} \rightarrow \begin{array}{l} w_1 = (72 - y)/2 \\ w_2 = (40 - x)/2 \end{array}$$

van de Sande and Greeno's (2011) verbal transcript of the interaction shows how the teacher (Ms. Sanchez) attempted to guide her students toward representing width by a single variable but likely inadvertently influenced them to represent width by two variables. Early in the transcript Ms. Sanchez states that "we need to have a variable to write the equation" and asks students what the variable would be. However, to represent width as a variable she places a question mark to represent the width of the left vertical border and a second question mark question mark to represent the width of the top horizontal border. Her intention presumably is to show students that the question mark is the variable, but it likely encouraged Gillian to represent the border's width by two variables.

Another student (Hannah) understands how this representation could work by stating "Couldn't you use substitution since you have two equations? I mean, you could make another equation x times y equals 1680." Ms. Sanchez informs the class that they now have only one equation, but Hannah later replies, "72 minus y divided by 2 and 40 minus x divided by 2. They're both going to equal the same number." After Ms. Sanchez equates the two expressions on the board, she replies,

> Ok. This is very interesting. This is a completely different way than I looked at it. I only had 1 (holding up one finger) variable when I wrote this, so this is very . . .. this is hard for me to think about.

After further interactions with the class, Ms. Sanchez states,

> I started with the question mark (pointing to the question mark on the diagram) and my question mark was my variable. But, for you guys . . . you did it this way (indicating the dimensions of the inner rectangle) and I don't see why that wouldn't work as well.

She then encourages the class to figure out the values of x and y by using the two equations on the board.

Gillian and Ms. Sanchez discovered that Hannah's beginning of a solution could be completed to solve the problem. This second solution was not as efficient as the one discovered by Ms. Sanchez because it required two equations to solve for the two unknown variables. However, Ms. Sanchez recognized that the two equations provided a learning opportunity for the class to practice solving for two unknown values. I return to the issue of the relative advantages of simple and complex solutions when discussing partial mappings across solutions.

Partial Mappings Across Representations

My previous cases of partial mappings across problems analyzed learning situations in which students needed more than one problem to solve another problem. Each problem provided only part of the solution. Another situation that involves partial mappings is assigning partial credit to a student's solution. Although an instructional problem may have provided sufficient information for solving a test problem, the student's solution is only partially correct.

An example of mappings across representations occurs when words in a definition, such as *rise* and *run* in the definition of slope, are mapped onto a diagram. This example is part of a study that Lobato (2008) analyzed in her chapter on the distinction between cognitive and actor-oriented approaches to transfer. In that study, five high school classes of Algebra I students participated in a 6-week unit on slope and linear functions in which slope was calculated by dividing the rise (vertical change) of a line by its run (horizontal change). The results indicated that 87% of the 139 students subsequently found the slope for previously encountered staircases, but only 40% of the students transferred this knowledge to find the slope of a playground slide.

Lobato interviewed 15 of the students, and some of their identifications of rise and run are shown in Figure 5 (Lobato, 1996). According to my cognitive perspective, many of the students displayed a partial mapping from the staircase to the slide task as revealed by their partial success. For instance, Jarek, Lisa, and Kim correctly identified rise but incorrectly identified run. Rolando correctly identified run but misidentified rise. Latasha hedged her bet by displaying a one-to-two mapping to the transfer task, whereas Nick showed a correct one-to-one mapping if height and base are accepted as synonyms for rise and run. By following a cognitive perspective, I am focusing on correct mappings, much like a teacher would assign partial credit on the basis of how much of the solution is correct.

However, according to the actor-oriented perspective, there are more mappings between the two tasks because most students did identify both rise and run in the slide diagrams. From an actor-oriented perspective, seven of the nine students used a one-to-one mapping of rise and run onto a part of the slide diagram. Nick used a one-to-one mapping of "height" and "base" onto diagram parts. Latasha used a one-to-many mapping of rise and run by mapping each onto two parts of the diagram.

Identifying incorrect mappings can be helpful in redesigning the curriculum to improve transfer. A particularly challenging aspect of the slide task is that the correct line has to be constructed. This is a problem for ecological approaches that argue that all relevant information is in the environment (Gibson, 1966). It also makes a structure-mapping approach more difficult to apply because an element of the target has to be created to establish the mapping. My own bias is to initially

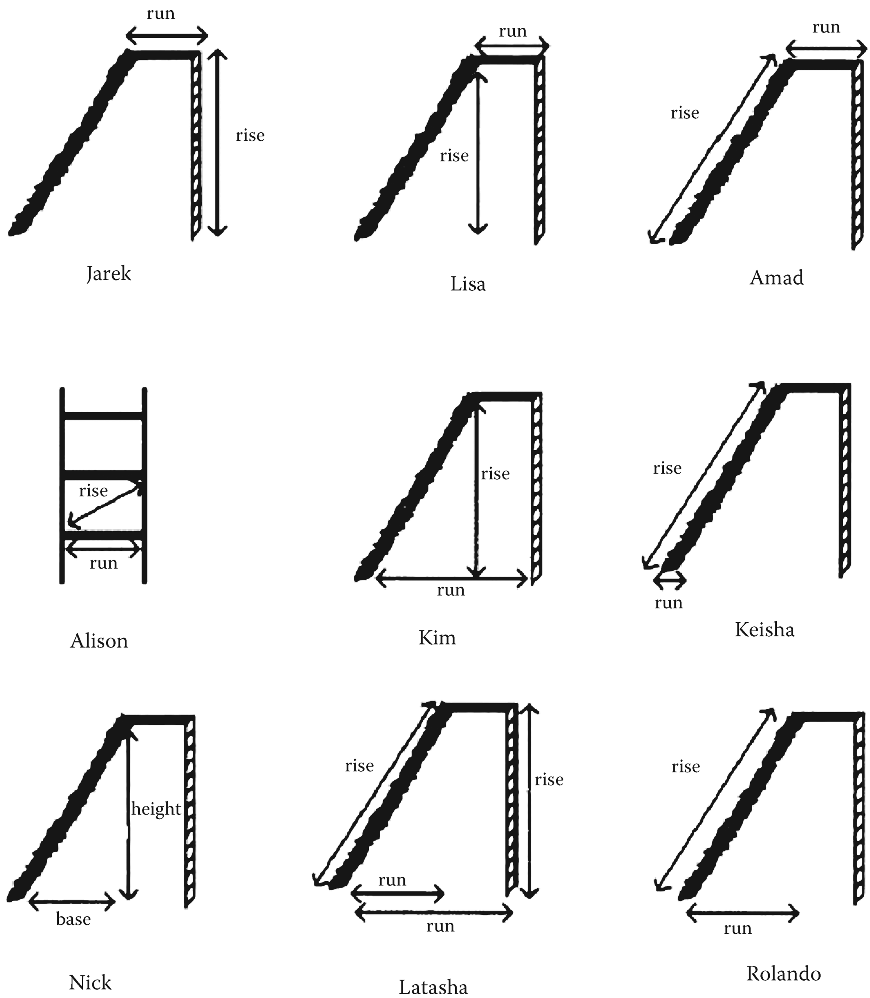


FIGURE 5 Students' identification of rise and run in slides. From Lobato (1996).

present worked examples that include the critical parts of a diagram before asking students to subsequently construct them. However, an alternative would be to challenge students with a difficult assignment to motivate them for future learning (Bransford & Schwartz, 1999).

Another challenge for students is the selection of a vertical line to represent "rise." The word *rise* in my *Oxford Guide to the English Language* includes "upward slope" among its definitions. References to the rise of a hill or to a sunrise may refer to an angle rather than to vertical extent. One-to-many mappings from a word to its definition create the opportunity for misunderstandings. Differences between the popular and technical meanings of the word may have contributed to the frequent labeling of the diagonal line in Figure 5 as “rise” rather than the vertical line.

Although I used partial mappings as a context for discussing partially correct identifications, I used one-to-one mappings to include incorrect identifications and one-to-many mappings from a word to its definitions as a possible contributor to the incorrect identifications. These analyses demonstrate that the application of different mappings depends on the focus within a task. This focus also depends on what is important to the theorist. Lobato (personal communication, March 9, 2011) conceives of the slope examples presented here as instances of one-to-one mappings rather than partial mappings. From an actor-oriented view, she focuses on one’s conceptualization of a situation as *what* is being mapped rather than a correspondence between elements of a formula and parts of a drawing. For example, if Jarek comprehends slope situations as involving stair-steps (i.e., something with visually connected up and over components that suggests climbing in an imagined state of affairs), then he has made a “complete” or one-to-one mapping of his conceptualization of slope from the initial learning to the transfer situation.

MAPPING ACROSS SOLUTIONS

The study of mapping across solutions examines the relation between alternative solutions to the same problem. At one extreme there is no mapping because the new solution is constructed independently of the old solution. At the other extreme the new solution is constructed entirely by transforming the old solution through one-to-one mappings (Reed, 1999). In between are partial mappings in which one solution is more economical than the other because it requires fewer steps (Rittle-Johnson, Star, & Durkin, 2009).

One-to-One Mappings Across Solutions

One-to-one mappings across solutions occur when the second solution is a derivation of the first. For example, I initially represented the solution of a mixture problem as a weighted average to show its relation to other weighted average problems such as average speed (Reed, 1984). If C1 and P1 are the concentration and proportion of the lower concentration and C2 and P2 are the concentration and proportion of the higher concentration, then the concentration of the mixture is

$$\text{Concentration} = \text{C1} \times \text{P1} \;+\; \text{C2} \times \text{P2} = 20\% \times .3 \;+\; 40\% \times .7 = 34\%$$

for an example in which 3 liters of a 20% solution are mixed with 7 liters of a 40% solution.

I subsequently used mixture problems to explore students' reasoning about linear functions (Reed & Evans, 1987). Students saw a graph of the function, such as the concentration of a mixture continuously increasing from 20% to 40% as the proportion of the higher concentration varies from 0 to 1. The instruction focused on three characteristics of the function: range (the concentration is between 20% and 40%), monotonicity (the concentration increases as the proportion of the higher concentration increases), and linearity (this increase is linear).

I initially had no idea if the algebraic and graphical solutions could be related. Although the weighted average equation is not linear, I was able to derive an equation from it that is a linear function of P2:

$$\text{C1} \times \text{P1} + \text{C2} \times \text{P2} \rightarrow \text{C1} \times (1 - \text{P2}) + \text{C2} \times \text{P2} \rightarrow \text{C1} - \text{C1} \times \text{P2} + \text{C2} \times \text{P2} \rightarrow \text{C1} + \text{P2} \times (\text{C2} - \text{C1})$$

Notice that the derivation consists of a sequence of one-to-one mappings from one symbolic expression to another symbolic expression if chunks such as (1 – P2) are included in the mapping. The last expression provides an alternative solution to the problem that represents concentration as a linear function:

$$\text{Concentration} = \text{C1} + \text{P2} \times (\text{C2} - \text{C1}) = 20\% + .7 \times (40\% - 20\%) = 34\%.$$

The y-intercept is the lower concentration (C1) and the slope is the difference between the two concentrations (C2 – C1). The derivation enabled me to see how my two different conceptual representations of mixtures were, in fact, different solutions to the same problem (Reed, 1999).

At a general level such transformations can be represented in a search space in which one problem state is mapped onto (transformed into) another problem state by the application of an operator (Newell & Simon, 1972). Applying operators in a search space can therefore be represented as a sequence of one-to-one mappings between problem states. Arriving at the goal state in most cases will simply solve the problem (such as moving all of the people across the river) rather than generating a new solution. However, viewing a mixture problem either as a weighted average or as a linear function are two very different perspectives that are connected through the derivation of one solution from the other.

Another example of mappings in a derivation occurs when a computational formula, such as for calculating variance, is derived from a conceptual formula based on the definition of a concept. Such cases create an opportunity for students to practice using the means–end analysis heuristic. As described by Newell and

Simon (1972), this is the primary heuristic for transforming an initial state (the conceptual formula) into the goal state (the computational formula) by gradually eliminating differences between the current problem state and the goal state. For instance, in the previous mixture example, the weighted average formula contained both the proportion of the weaker concentration (P1) and the proportion of the stronger concentration (P2). Because the formula for the linear function contained only P2, I had to eliminate P1 in one of the solution steps. Representing P1 as 1 – P2 allowed me to eliminate this difference between the initial state and the goal state. Derivations therefore not only provide a learning opportunity for understanding the relation between two different solutions to the same problem but provide an opportunity for practicing problem-solving heuristics.

One-to-Many Mappings Across Solutions

I hedge in this section because its content is more about one-to-many mappings across concepts than about one-to-many mappings across solutions. My justification is that making connections among concepts supports many of the activities (including constructing solutions) that I discuss in the taxonomy. In addition, my definition of a mapping as a relation between two nodes is the unit for representing conceptual knowledge in semantic networks.

Relations among concepts are frequently represented as semantic networks in which a person's knowledge base consists of many concepts (nodes) connected by relations (links). One way of characterizing learning is through an increase in the number of connections in a semantic network (Chi & Ohlsson, 2005), as demonstrated by comparing a child's knowledge base for 20 familiar and 20 less familiar dinosaurs. Chi and Koeske (1983) constructed a network based on interviews to discover the child's knowledge organization of the 20 less familiar dinosaurs. Many nodes had multiple links, illustrating a one-to-many mapping from one concept to other concepts (attributes or other dinosaurs). There were even more links among concepts in the child's knowledge of the 20 familiar dinosaurs. The clusters were more densely connected and less diffused.

Dense connections among concepts is an ideal that is often not achieved. Tytler and Prain (2009) concluded that children's conceptual understanding is fragmentary (diSessa, 1993) after tracking nine children throughout 7 years of primary school. diSessa's (1993) knowledge-in-pieces framework proposes that novices typically rely on contextually sensitive knowledge for predicting and explaining physical phenomena rather than on systematic theories that integrate these phenomena. My semantic network interpretation of the knowledge-in-pieces framework is that such a network consists of isolated fragments rather than dense connections.

One consequence of isolated fragments is that it makes information more difficult to recall. Consider the following simple story about ants (Kieras, 1978):

(1) The ants ate the jelly. (2) The ants were hungry. (3) The ants were in the kitchen. (4) The kitchen was spotless. (5) The jelly was grape. (6) The jelly was on the table. (7) The table was wooden. My semantic network representation of this story uses a link to connect the two concepts (such as "ants" and "jelly") in each sentence by the verb ("ate"). The organization of the story enables me to attach the information in each new sentence to a node (concept) that already exists in the network. For instance, I can attach "were" and "hungry" in the second sentence as a link to the "ants" node created from the first sentence.

In contrast, the reorganization of the same seven sentences in the order 4, 7, 2, 3, 5, 6, and 1 creates fragments. I cannot attach the information in the second sentence ("The table was wooden") to my network representation of the first sentence ("The kitchen was spotless") because the two sentences contain different information. The same is true of the third sentence ("The ants were hungry"), so my network begins as isolated fragments. Eventually I can make connections among the fragments but, according to Kieras's (1978) findings, the damage has already been done. His participants recalled less information if they read the second story. This simple example shows the importance of constructing curricula that make explicit connections to previous material.

Another method for helping students build densely connected semantic networks is to require them to literally construct the networks through concept mapping. Concept maps identify the relations among concepts and are instructional tools built on the theoretical idea of semantic networks (O'Donnell, Dansereau, & Hall, 2002). Diagramming the connections among ideas is helpful when assessment measures learners' organization of knowledge. Requiring students to construct concept maps of information in a general science text significantly improved scores on short-answer and essay tests but did not significantly improve scores on a multiple-choice test (Holley & Dansereau, 1984).

The organization of knowledge into densely connected semantic networks also supports problem solving. Wagner (2006) adopted diSessa's (1993) knowledge-in-pieces framework to discuss his findings about Maria's difficulties in making connections to prior concepts and solutions. Although solving the post office problem requires the use of the law of large numbers, Maria did not regard this law as relevant because of her fragmented understanding created by linking the law to a particular context.

Another problem-solving example that involves both networks and mappings is the ability of students to select an appropriate diagram (network, hierarchy, matrix) for solving a problem. Novick and Hurley (2001) pointed out that making good selections depends on understanding the differences between these representations. The one-to-many mappings prevalent in networks do not occur in hierarchies when moving from smaller to larger categories. The single path can be represented by one-to-one mappings across categories such as brontosaur → dinosaur → reptile → animal. In a matrix there are no links, so it does not make sense to talk about moving along pathways. In terms of mappings, the one-to-many mappings in networks are replaced by one-to-one mappings for moving up hierarchies and no mappings for moving across cells in a matrix. Understanding these mappings should therefore facilitate transfer of representations across different problems (Novick & Hmelo, 1994).

Partial Mappings Across Solutions

Partial mappings across solutions are relevant to comparing the efficiency of solutions. I previously raised the issue of efficiency when discussing two different solutions of the garden border problem (Greeno & van de Sande, 2007; van de Sande & Greeno, 2009). The single equation constructed by Ms. Sanchez was more efficient than the pair of equations constructed by her students.

Rittle-Johnson et al. (2009) discovered that asking seventh- and eighth-grade students to compare two solutions for solving the same problem was helpful when they had the appropriate prior knowledge. One of the solutions showed a shortcut method. In the following example the first method requires the use of multiplication ($\times$), subtraction (–), and division ($\div$). The second method requires the use of only division and subtraction:

$$\begin{array}{lll@{\qquad\qquad}ll} 3(x+1)=15 & \times & & 3(x+1)=15 & \div \\ 3x+3=15 & - & & x+1=15 & - \\ 3x=12 & \div & & x=4 & \\ x=4 & & & & \end{array}$$

Students who were familiar with one of the two methods typically noticed that one method required fewer steps or was more efficient than the other. Comparing solutions for these students produced flexible knowledge of procedures. In contrast, students who were not familiar with either method benefited more from the sequential presentation of the solutions.

Generating and comparing different solutions to a problem is part of a collaborative problem-solving method that Alan Schoenfeld (1985) used in his mathematics classes at Berkeley. The class often produced three or four different solutions to the same problem. After arriving at a solution, Schoenfeld led a discussion regarding its efficiency and whether some discarded ideas offered promise for an alternative solution.

Solutions with fewer steps are more efficient for solving a problem, but simplicity can be a disadvantage when the solution is transferred to solve a more complex problem. Consider the following problem:

Tina and Wilt are sitting 4 meters apart on a seesaw. Tina weighs 65 kg and Wilt weighs 80 kg. How far from the fulcrum must Tina sit to balance the seesaw?

$$65 \times d = 80 \times (4 - d)$$

The correct equation shows that the product of weight and distance from the fulcrum must be equal for Tina and Wilt for the fulcrum to balance.

This is a difficult problem for students in a college algebra class, so we gave them the solution to an analogous problem to help them solve it (Reed, Ackinclose, & Voss, 1990). Some students received a solution to a simpler problem:

Laurie weighs 60 kg and is sitting 165 cm from the fulcrum of a seesaw. Bill weighs 55 kg. How far from the fulcrum must Bill sit to balance the seesaw?

$$60 \times 165 = 55 \times d$$

The simpler solution provides only a partial mapping to the test problem because Laurie's distance from the fulcrum is stated directly (165 cm) in the example but Wilt's distance is stated as a relation ($4 - d$) in the test problem.

Other students received a solution to a more complex analogous problem:

Dan and Susie are sitting 3 meters apart on a seesaw. Mary is sitting 1 meter behind Susie. Dan weighs 70 kg. Susie weighs 25 kg and Mary weighs 20 kg. How far from the fulcrum must Susie sit to balance the seesaw?

$$20 \times (d + 1) \; + \; 25 \times d = 70 \times (3 - d)$$

This solution contains more information than is needed but it does contain the necessary information for representing distance relations.

Reed et al. (1990) evaluated the hypothesis that example solutions that are simpler than the test problems would be less effective than example solutions that are more complex. The rationale is that applying a more complex solution requires eliminating excess information, which should be easier than generating missing information from the partial mappings. The results from students in college algebra classes supported the predictions. For instance, 13% of the students who received the simpler solution solved the fulcrum test problem, compared to 47% of the students who received the more complex solution. Only 3% of the students in both groups solved the test problem on the pretest.

Students, however, did not show a preference for the more complex solutions when offered the choice of seeing one of the two solutions. Instead, they selected analogous examples on the basis of perceived similarity to the test problems. Examples perceived as more similar to the test problems were not more effective than problems perceived as less similar to the test problems. The mapping of symbols, rather than perceived similarity, therefore determined successful transfer. Instruction on how these mappings influence the effectiveness of examples might improve students' decisions in selecting analogous solutions, as comparisons of alternative solutions to the same problem were helpful for students who had some familiarity with algebra (Rittle-Johnson et al., 2009).

MAPPING ACROSS CONTEXTS

Mapping across contexts occurs in the rich environments studied within the sociocultural tradition. The sociocultural study of transfer and conceptual change has been more social than cultural, so I examine cultural influences to illustrate both one-to-one and one-to-many mappings. The two cases illustrate different ways in which a culture can respond to the introduction of Western beliefs. One-to-one mappings occur when the beliefs of one culture replace corresponding beliefs of another culture (Nunez & Sweetser, 2006). One-to-many mappings occur when beliefs from both cultures are used to construct explanations (Legare & Gelman, 2008). Both of these examples illustrate conceptual change following exposure to another culture. The third case involves students in a fifth-grade classroom who were participating in a unit on endangered species (Engle, 2006). Partial mappings show how they transferred their explanations about whales to other endangered species.

One-to-One Mappings Across Contexts

Cultures become Westernized when their traditional actions are replaced by actions used in Western cultures. The substitution of an action from one culture for an action in another culture represents a one-to-one mapping across contexts. Such substitutions have begun to appear in the Amerindian language spoken by the Aymara in the Andean highlands of South America (Nunez & Sweetser, 2006).

In Western culture the future lies in front, and the past lies behind. However, the words and gestures used in the Aymara language reverse the space–time metaphor. Their future lies behind the body because it is unseen, and their past lies in front of the body because it has been seen. Interviews of nine older (65–84 years old) Aymara revealed that, with a single exception, they always used a forward gesture to refer to the past (Nunez & Sweetser, 2006). However, the interviews of six younger (36–64 years old) Aymara revealed that only one always referred to the past with a forward gesture. The others had mapped the Aymara concepts and gestures onto Western representations:

Past Is in Front → Past Is Behind
Forward Gesture for Past → Backward Gesture for Past
Future Is Behind → Future Is in Front
Backward Gesture for Future → Forward Gesture for Future

These mappings are one to one because a backward gesture in the Aymara language maps onto a forward gesture in the Western culture and a forward gesture in the Aymara language maps onto a backward gesture in the Western culture.

Reversing the meaning of forward and backward gestures is an example of a particular kind of one-to-one mapping that Ross (1987) labeled *cross mappings*. He studied mathematics problems in which he systematically varied the correspondence between objects in the example solution and objects in the test problems. One test problem had identical objects that had the same roles as in the example—for instance, mechanics choosing which cars to service. Another test problem had identical objects but reversed the roles in the example—for instance, the owners of cars choosing among the mechanics. Ross found that using the same objects with identical roles produced much better performance (74% correct) than a neutral condition (54% correct), but using the same objects with reversed roles produced the worst performance (37% correct). The difficulty of cross mappings provides instructors with the opportunity to examine their students' depth of understanding by presenting examples in which mappings are particularly challenging.

One-to-Many Mappings Across Contexts

One-to-one mappings occur when a new procedure replaces an old procedure. The Aymara illustrated this transition by reversing the relation between space and time. In contrast, both indigenous and Western traditions may simultaneously play important roles in determining thinking. Such cases illustrate a one-to-many mapping in which two different sets of beliefs provide a basis for explanations.

Legare and Gelman's (2008) study of children and adults in two Sesotho-speaking South African communities provides a clear case of one-to-many mappings from two belief systems. The participants were 366 adults and 5-, 7-, 11-, and 15-year-olds where both indigenous and Western biomedical knowledge were available. However, in spite of biological knowledge, witchcraft continued to be an important explanatory framework for interpreting illness and coexisted along with the biological explanations. Tribal members did not consider the two frameworks to be inconsistent but rather sources of complementary causal information.

The investigators found three ways in which coexistence survived in explanations of AIDS. The first, juxtaposition, refers to using both frameworks but not in an integrated manner. Examples include "Might be unprotected sex and bewitchment" and "Having so many enemies causes bewitchment, and maybe unfaithful partners cause AIDS too" (p. 635). In proximal/distal explanations, witchcraft provides the "why" (distal) cause and biology provides the "how" (proximal) cause. Examples include "A witch can make a condom weak, and break" and "Witchcraft can make you have sex with someone who has AIDS" (p. 635). Real versus fictitious explanations propose that someone can have an incurable disease that is disguised as AIDS. Examples include "Witchcraft can cause a disease that acts like AIDS" and "The spell was supposed to look like AIDS" (p. 635).

Each of the three kinds of coexistence illustrates an interpretation of an event by combining causes from both cultures:

Explanations of AIDS → Biological Causes
Witchcraft

Legare and Gelman (2008) found that explanations based on witchcraft declined with age during childhood and adolescence but increased again during adulthood. They interpreted these findings as evidence that the explicit information provided in school that witchcraft is not a cause of AIDS does not seem to have a lasting impact into adulthood. There is also evidence that explanations based on witchcraft are not caused by a lack of biological knowledge. Participants understood the biological explanations of AIDS better than the biological explanations of flu but were as likely to use bewitchment explanations for AIDS as for flu.

Partial Mappings Across Contexts

Partial mappings constitute frequent mappings across contexts because much of learning consists of integrating some old (mapped) knowledge with new knowledge. However, the relevance of old knowledge is not always transparent. According to Engle's (2006) framing perspective (see Table 3), transfer across contexts can be greatly facilitated if the teacher frames the instruction to encourage transfer across different instructional units. The framing helps students determine which knowledge should and should not transfer across units.

Engle (2006) studied a fifth-grade classroom that was participating in an endangered species unit under the fostering-a-community-of-learners guidelines (Brown & Campione, 1994). Groups of four to five students focused on a particular animal group and researched why that animal group is, or has been, endangered. Transfer was measured by the sophistication of students' pre- and post-explanations of the endangerment of other species. Engle constructed multicausality diagrams for the group's collective explanations and used cognitive measures, such as those developed by van Dijk and Kintsch (1983) and Chi, Huchinson, and Robin (1989), to measure the causal structure of the explanations.

Analysis of such diagrams revealed a significant increase in the number of causal links per node from 0.94 during preassessment to 1.54 during postassessment. In the postassessment interviews students derived more than one specific result from each habitat feature, often coordinating them. The interviews regarding mudskippers revealed many causes, such as the height of tides, the number of predators, and the amount of food, that were not part of the whale explanation based on birthrate and hunting. The finding that students had generalized their multicausal explanations is consistent with the structure-mapping framework proposed by Gentner (1983), in which higher order relations play an important role in transfer:

Multicausality for Whales	→	Multicausality for Mudskippers
(Birthrate)		(Tides)
(Hunting)		(Food)

Transfer is partial in Engle's (2006) case because the principle of multicausality is transferred but not the specific causes. This partial mapping differs from one-to-one mappings, such as in the convergence isomorphs studied by Gick and Holyoak (1980, 1983). The important components of those solutions map across problems: Using an army to capture the fortress maps onto using rays to destroy the tumor, and the single road constraint maps onto the intense ray constraint. In contrast, causes that threaten whales (birthrate, hunting) do not map onto specific causes that threaten mudskippers (tides, food). This mapping might be considered categorical because causes map across the two situations but not the alignment of specific causes. This partial mapping is productive because the students distinguish between what should and should not be transferred.

Engle's (2006) analysis builds on cognitive contributions but goes beyond them in proposing a situational account of transfer in which appropriate framing of discussions is essential for success. Table 3 contrasts framing as bounded events with framing as ongoing activities. The teacher framed the students' participation by encouraging them to share their ideas with an extended audience who would be interested in discussing and building on their ideas. Framing also made it clear that students were learning ideas in order to use them, creating an expectation for transfer. Framing should therefore increase the likelihood of appropriate mappings.

CONTRIBUTIONS

Cases in the taxonomy include examples from both traditional and alternative perspectives to show that mappings offer one approach for finding commonalities among these different perspectives. The taxonomy and examples make the following contributions.

Types of Mappings

Most of the work on analogical reasoning, particularly as captured in computational models, has focused on one-to-one mappings. The proposed taxonomy distinguishes among three types of mappings—one to one, one to many, and partial—to organize research on learning.

For example, there is a one-to-one mapping from the fortress problem to Duncker's radiation problem (Gick & Holyoak, 1980), a one-to-many mapping from the missionaries and cannibals problem to the jealous husbands problem (Reed et al., 1974), and a partial mapping from the flowing waters and teeming crowds analogies to electric circuits (Gentner & Gentner, 1983).

Types of Situations

Most of the work on analogical reasoning has focused on mappings across problems. The proposed taxonomy extends the study of mappings across problems to include mappings across representations, solutions, and sociocultural contexts. For example, the use of a variable in an equation to represent the width of a garden border (Greeno & van de Sande, 2007) requires mapping across representations, the comparison of two different solutions to a problem (Rittle-Johnson et al., 2009) illustrates mapping across solutions, and reliance on both indigenous and Western interpretations of AIDS (Legare & Gelman, 2008) illustrates mapping across sociocultural contexts.

Geometric Forms

Most of the work on analogical reasoning has focused on mappings across verbal concepts. Cases in the taxonomy include mappings of components in geometric forms. One example is the application of cut-and-pivot operations on diagonals to convert trapezoids into rectangles (Green et al., 1993). Another example is identifying segments in diagrams that can be used to find the slope of a line (Lobato, 1996, 2008).

Multiple Sources of Information

Most of the traditional work on transfer has focused on mappings from a single problem (D. W. Carraher & Schliemann, 2002). Cases in the taxonomy include mappings from multiple sources of information. For instance, Maria used information from several problems to understand the application of sample size to the post office problem in Table 1 (Wagner, 2006).

Constructions

Most of the work on analogical reasoning has focused on mappings across elements that are part of the described problem. Cases in the taxonomy include mappings across elements that are constructed by the learner. For example, students who participated in a unit on why whales could be an endangered species subsequently had to construct possible causes of why mudskippers could be an endangered species (Engle, 2006).

IMPLICATIONS FOR LEARNING

I tried to state some learning implications of the mapping framework as I discussed cases in the taxonomy. I continue this discussion here in my responses to six questions.

Is There Direct Evidence That Mappings Influence Learning?

Yes. Research on the effectiveness of solutions for solving word problems supports the prediction that success in mapping corresponding quantities across problems influences transfer (Reed, 1987). Students received a detailed solution to a mixture problem and to a work problem and then had to construct equations for four variations of each solution. The four variations included two problems that were isomorphic (different story, same solution) and two problems that were similar (same story, different solution) to the examples.

The results showed that students were significantly better in constructing equations for isomorphic test problems than for similar test problems and were significantly better on the work isomorphs than on the mixture isomorphs. A subsequent experiment demonstrated that the amount of transfer was related to students' ability to identify corresponding quantities in the example and test problems, as would be expected by a structure-mapping theory. Students were significantly better at matching corresponding quantities for isomorphic problems than for similar problems and were significantly better at matching corresponding quantities across work isomorphs than across mixture isomorphs.

Is One of the Three Types of Mappings Better Than the Others?

No; the instructional challenge is to maximize the potential of each type of mapping. One-to-one mappings have typically been studied by cognitive psychologists to evaluate whether students can discover the common structure across isomorphic problems (Chi et al., 1982). Facilitating this discovery through comparing solutions is an example of a successful application of a laboratory technique (Catrambone & Holyoak, 1989; Gick & Holyoak, 1983) to the classroom (Gentner et al., 2009).

Partial mappings are useful for describing novices' perceived similarities across problems because their mappings are typically less complete than those of experts (Lobato, 2008). For instance, Figure 5 shows that most students were unable to identify both rise and run for calculating the slope of a slide (Lobato, 1996). Discovering students' perceived similarities provides helpful information to instructors who want to improve their students' perceptions. Partial mappings also contribute to a task analysis to identify how instruction might provide insufficient information for solving a problem (Gentner & Gentner, 1983; Reed et al., 1990).

One-to-many mappings are productive when they create dense connections among concepts in a semantic network (Chi & Ohlsson, 2005). Recognizing that multiple causes can influence outcomes is an example (Engle, 2006). However, the utility of the connections is also important. Believing that witchcraft is one of the causes of AIDS is counterproductive to learning medical explanations of diseases (Legare & Gelman, 2008).

Is Some Instruction More Helpful in Supporting Mappings?

Yes, but answering this question requires measuring the success of the mappings. The Cognitive Tutor Model Analysis tool evaluates students' ability to interpret symbols in equations by mapping them onto their verbal referents in word problems (Corbett, McLaughlin, Scarpinatto, & Hadley, 2000). Students are asked to identify components of the equation from a menu of verbally described quantities. The following mappings show the correct answers from a word problem that asks for the calculation of the total amount of copper in a mixture of high- and low-grade ore:

0.32L	The number of tons of pure copper in the low-grade ore
(91 – L)	The number of tons of high-grade ore
0.82(91 – L)	The number of tons of copper in the high-grade ore
0.16L + 0.82(91 – L)	The total number of tons of pure copper

We used the Model Analysis tool during the final session of an instructional study that compared the effectiveness of three types of worked examples (static table, static graphics, interactive graphics) with a Cognitive Tutor control group that solved the practice problems on the Cognitive Tutor (Reed, Corbett, Hoffman, Wagner, & MacClaren, 2010). The static table condition presented a verbal explanation of the solution and included a table for organizing the quantities of the problem. The static graphics condition represented quantities as bars for ore problems and as stacks of money for interest problems. The interactive graphics conditions enabled students to manipulate quantities in the bars and stacks.

The static table group performed consistently well in identifying the symbolic components of equations, showing the highest accuracy on 18 of the 31 tested components compared to 5 each for the two graphics groups and 3 for the Cognitive Tutor. The advantage of including a table in a verbally described worked example is that a table explicitly links many components of equations to their verbal referents. Although the Cognitive Tutor did not score well on this test, the Cognitive Tutor Model Analysis tool supports tutoring in addition to testing. Students can request advice and receive feedback on mappings when the tool is used in the tutor mode.

Are Mappings Helpful in Planning Learning Trajectories?

One implication of thinking about learning as mappings is that simpler mappings should prepare students for the more complex mappings they will encounter later in the curriculum. An example is the previously cited case from the Third International Mathematics and Science Study (Richland et al., 2004). When students had difficulty, teachers generated analogies with higher surface similarities to make the analogies more transparent.

Another previously cited example discussed the ease of mappings in transferring solutions across problems. Ross (1987) found that using the same objects with identical roles produced much better performance (74% correct) than a neutral condition (54% correct). Using the same objects with reversed roles produced the worst performance (37% correct). The relative ease of these different conditions suggests that mappings in the curriculum should transition from identical roles to neutral roles to reversed roles.

The Gentner and Gentner (1983) study illustrates limitations in applying mappings when students have not received adequate preparation. Students in their second experiment were unable to apply the reservoir analogy to electric circuits because they lacked understanding of the flow of water.

How Do Mappings Support Collaborative Learning?

Collaborative learning typically involves both shared and unshared knowledge among participants in which shared knowledge can be represented as a partial mapping of concepts and relations across concept maps. Although concept maps might be constructed either by the students or by theorists to represent students' knowledge, Engelmann and Hess (2010) provided different concept maps to each of three students in order to experimentally control the content.

The three group members had to determine which pesticide and fertilizer they would use to rescue a fictitious spruce forest. The group received distributed information that consisted of 13 concepts, 30 relations, and 13 task-irrelevant pieces of background information. Each of the three members received an individual concept map containing two unshared concepts, five shared concepts, seven unshared relations, and six shared relations. Each participant also received two shared and five unshared pieces of irrelevant background information. To compile their knowledge and information, the group members created a shared concept map of the provided information and used it to construct a common solution. In the control condition, the participants could only see their individual and the shared constructed concept map. In the experimental condition, the participants also saw the individual concept maps of their collaborators.

The results showed that the experimental groups started significantly earlier in discussing both the pesticide and fertilizer problems. They were also significantly faster in creating both the shared pesticide and fertilizer maps and solved the fertilizer problem significantly sooner. By being provided with external representations of all collaborators' information, they could more easily identify both shared and unshared knowledge. Students in the control condition used the group map mainly to collect all of the information they had in their individual maps and therefore included more problem-irrelevant information. Students in the experimental condition could more quickly find relevant information in their shared and unshared knowledge. Thus, facilitating the mapping of knowledge among students resulted in more effective collaboration.

Does Studying Mappings Lead to New Insights About Learning?

It depends on the reader. Let me describe one of my insights. Both the teacher and I solved the garden border problem in Figure 4 (Greeno & van de Sande, 2007) by constructing a single equation. Although I was awed and inspired by her student's solution of the problem by using two equations, I still believed the simplicity of "my" solution was more elegant.

After multiple revisions of the manuscript I began to value the teacher's request of her students to use the two equations to solve for two unknowns because it should help prepare them for future learning (Bransford & Schwartz, 1999; Engle, 2006). I had previously argued that solutions to more complex problems are more informative for transfer (Reed et al., 1990) but had not yet made the connection to two different solutions to the same problem. Then I also remembered that Wertheimer's (1959) simpler solution for transforming a trapezoidal figure into a rectangle (the upper left diagram in Figure 3) did not transfer to asymmetrical trapezoids. It was the more complex solution in the upper right diagram that transferred.

Although I continue to value simple solutions for their economy, as do Schoenfeld (1985) and Rittle-Johnson et al. (2009), I have new respect for how a more complex solution to a problem can help prepare students for future learning and transfer. I hope that readers' reflections about the many cases reported in this manuscript will lead to new insights about learning. My goal is not that investigators will learn how to classify their study into a particular cell in the taxonomy but that they will be more aware of how mappings are involved in learning. These analyses, of course, may differ from my own, but discussing these differences using a common unit of analysis should facilitate comparisons.

COGNITION IN PRACTICE REVISITED

Lave's (1988) *Cognition in Practice* was one of the motivators for producing this taxonomy, so I would like to revisit it by referring to two reviews on the 20th anniversary of its publication. Lave's Adult Math Project and some other sociocultural articles on transfer have emphasized differences across situations, such as how mathematics used outside the classroom (shoppers, street sellers, dieters, tradesmen) differs from mathematics taught in the classroom. In their review of *Cognition and Practice*, Greiffenhagen and Sharrock (2008) argued that instead of saying that there are two different kinds of mathematics, one could alternatively say that the same kind of mathematics is used differently in different contexts. This more moderate framing encourages looking for both similarities and differences between school and everyday mathematics.

Greiffenhagen and Sharrock (2008) argued that studies of everyday mathematics tend to create strong contrasts but often do not specify the exact nature of the implied contrast. For instance they pointed out that many calculations in Lave's (1988) best-buy problems involved doubling or trebling rather than the more complex calculations required in school. When faced with more complex calculations, shoppers simply abandoned them. Scores on complex calculations on school exams would therefore not be predictive of finding best buys, although arithmetic knowledge is used on the simpler estimates.

D. W. Carraher (2008) also supported a detailed analysis of both the similarities and differences between school and street mathematics. Although his research group was initially tempted to conclude that they were witnessing a different mathematics when they studied young street vendors in Brazil (T. N. Carraher, Carraher, & Schliemann, 1985), it soon became apparent to them that the claim was too broad. Nonetheless, the mental algorithms of street vendors were not the same as the algorithms taught in school. For example, in subtracting 58 from 253, a vendor might first decompose 58 into 53 + 5, subtract 53 from 253, and then subtract 5 from 200. Street vendors might therefore use arithmetic rules such as grouping and commutativity in different ways than taught in the classroom.

This more detailed analysis of Lave's (1988) Adult Math Project is consistent with a partial mapping of arithmetic procedures from the classroom to applications outside the classroom. Arithmetic knowledge is used, although in somewhat different ways than how it was taught in the classroom. A fundamental difference, however, emphasized in *Cognition in Practice* is the importance of estimation outside the classroom. I also had the impression when I began giving estimation problems to university students in the 1980s (Reed, 1984) that I was the first one to ask them to estimate answers to problems. Students seemed surprised that an instructor would ask for an approximate, rather than an exact, answer.

Times have changed. Project 2061 of the American Association for the Advancement of Science recommended that students receive practice on estimation as part of their instruction on problem solving (American Association for the Advancement of Science, 1993). The standards proposed by the National Council of Teachers of Mathematics (2000) also include estimation as an important skill in learning numbers and operations. In response to these initiatives, our Animation Tutor project provides instruction on estimating answers to word problems before introducing techniques for calculating exact answers (Reed, 2005; Reed & Hoffman, 2010).

These studies, analyses, and calls for curriculum changes indicate that there are (increasing) similarities between school and everyday mathematics. D. W. Carraher's (2008) concluding remarks acknowledge this trend:

> If over time the expression *Everyday Mathematics* drops from usage, I would be neither surprised nor disappointed. Eventually the field needs to become absorbed into the mainstream traditions of research in mathematics education. However I would be disappointed if it is remembered only for its descriptive and proscriptive aspects, without recognizing the contributions to research, theory, and the cultural context of learning and thinking. (p. 31)

CONCLUSION

The proposed organizational framework examines how learning occurs through mappings across situations. The definition of a *mapping* as a relation between two knowledge states enables the study of mappings across many different situations that vary in grain size from mapping between two concepts or problem states to mapping between entire explanatory frameworks or different problem solutions.

One advantage of the mapping framework is that it should contribute to Gentner's (2010) prediction that knowledge representation will regain some its former ground as a key component of work in cognitive science. She cited the demand for more powerful Web-searching methods as one reason for its predicted comeback. Gentner argued that knowledge representation has played a diminished role as Gibsonian psychology, situated cognition, and embodied cognition have increasingly influenced mainstream cognitive science. The mapping framework, particularly in its application to mapping across representations, offers one approach to linking semantic knowledge to perception and action.

Another advantage of the mapping framework is that it should aid in sharpening the commonalities and differences between the cognitive and alternative perspectives. My emphasis has been on searching for commonalities, such as illustrating how understanding someone else's perspective or knowledge states can be analyzed through mappings across knowledge states. These commonalities will hopefully contribute to an integration of different perspectives. However, some of the claims made by the cognitive approach may be incompatible with the claims made by the alternative perspectives. Arguments directed at the limitations of the mapping framework should be valuable in identifying these incompatibilities.

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