Journal of Educational Psychology
2012, Vol. 104, No. 3, 636–646

0022-0663/12/$12.00 DOI: 10.1037/a0027181

Finding Similarities and Differences in the Solutions of Word Problems

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This study extends the Rittle-Johnson and Star (2009) research agenda of identifying when solution comparisons are effective by combining their quantitative approach with the qualitative descriptive approach advocated by Lobato (2008). In Experiment 1 university students described similarities and differences between detailed solutions of arithmetic or algebra word problems so we could determine whether such comparisons support learning of the combine and compare schema that represent semantic components of these solutions. Comparison of arithmetic solutions improved the selection of arithmetic equations to model word problems but did not transfer to algebra equations. Comparisons of the more complex algebra solutions did not result in either learning or transfer. A more scaffolded variation of the arithmetic comparisons in Experiment 2 replicated the Experiment 1 findings (learning but no transfer), but students identified more schematic components in their comparisons. Comparing a pair of arithmetic solutions followed by a pair of algebra solutions in Experiment 3 replicated the previous findings of improvement on arithmetic, but not algebra, problems. The results connect reported similarities and differences to learning (Lobato, Ellis, & Munoz, 2003) and support the conclusion that complexity can limit the potential benefits of solution comparisons (Rittle-Johnson, Star, & Durkin, 2009).

Keywords: comparisons, schema, solutions, word problems

People's ability to solve problems is related to their ability to determine the similarities and differences among solutions to those problems. For instance, better problem solvers are more capable of categorizing arithmetic word problems (Silver, 1981) and physics problems (Chi, Glaser, & Rees, 1982) based on similar solutions rather than on similar story content. Studies on schema abstraction have asked students to compare similarities of solutions to encourage them to find a common structure in (isomorphic) problems. These comparisons can increase the likelihood of transfer to other problems that have the same solutions such as the convergence solution for variations of Dunker's radiation problem (Catrambone & Holyoak, 1989; Gick, 1986). Other instructional methods have emphasized finding differences, such as the contrasting-cases method, which requires distinguishing between examples and non-examples of a concept (Schwartz & Bransford, 1998). Application of this method to mathematical expressions that either do or do not represent real-world quantities resulted in improved ability to construct equations for algebra word problems (Reed, 2006).

Comparing similarities and differences between problem solutions has typically facilitated learning, but little is known about which types of comparisons are most effective (Rittle-Johnson & Star, 2009). Although comparisons of solutions are generally regarded as beneficial, Rittle-Johnson and Star (2009) argued that designing effective instructional interventions requires a better understanding of what should be compared.

Rittle-Johnson and Star (2009) therefore investigated the effectiveness of asking students to report similarities and differences for solutions of algebra equations such as $5(y + 1) = 3(y + 1) + 8$. Some comparisons involved two solution methods for the same equation. Other comparisons involved solutions to different problem types such as comparing the solution of the previous equation with the solution of another equation that had a different structure: $3(h - 2) + 5(h - 2) = 24$. Still other comparisons involved equivalent solutions such as comparing the solution of the first equation with the solution of another equation that had the same structure: $10(x + 3) = 6(x + 3) + 16$. Students benefited most from comparing different solution methods to the same problem.

Purpose of the Study

Our study continues the Rittle-Johnson and Star (2009) research agenda of documenting when comparisons of similarities and differences among solutions results in improved problem solving. It consists of three experiments that used a pretest–instruction–posttest design. Both the pretest and the posttest contained four arithmetic and four algebra word problems. Instruction in Experiments 1 and 2 required students to indicate the major similarity and the major dissimilarity of pairs of either arithmetic or algebraic solutions. Experiment 3 included a condition in which students compared arithmetic solutions with algebraic solutions.

Appendix A shows one of the algebra pairs consisting of two problems that have different equations for the same story content. The structure of the different equations resembles the structure of

This article was published Online First February 13, 2012.

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This research was supported by National Science Foundation Grant SBE-0354420 to Carnegie Mellon University. Any opinions, findings, conclusions, or recommendations expressed herein are those of the authors and do not necessarily reflect the views of the National Science Foundation or Carnegie Mellon University.

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the different problem types in the Rittle-Johnson and Star (2009) study. However, our students had to select an equation to represent the problem, whereas Rittle-Johnson and Star's students had to solve the equation.

Another purpose of our study was to discover the similarities and dissimilarities perceived by the students. Lobato (2008) has argued that these perceived similarities and dissimilarities often differ from the ones perceived by experts. Our perception is that the main similarity between the two solutions in Appendix A is that both require the addition of distances. The main dissimilarity is that one solution requires adding a number to the unknown rate (r) and the other requires subtracting a number from the unknown rate. Requiring students to report similarities and differences between solutions informs us whether their perceptions match our perceptions. The next section describes how our perceptions are based on the schematic components that form the building blocks of arithmetic and algebra word problems. We argue that learning these schematic components is necessary both to solve arithmetic word problems and to facilitate transfer from arithmetic to algebra word problems.

Similarities and Differences Between Arithmetic and Algebra Word Problems

The relative influence of similarities and differences between arithmetic and algebra word problems should determine the amount of transfer between these two classes of problems. A claim that might have a high probability of going unchallenged is that algebra word problems are more difficult than arithmetic word problems because they involve algebra. Not surprisingly, there is abundant evidence to support such a claim. Much research in mathematics education is concerned with the transition from arithmetic to algebra, as illustrated by a special issue of *Mathematical Thinking and Learning* (Nathan & Koellner, 2007). These and many other articles document the many challenges that students face because of differences between arithmetic and algebra.

Students learning algebra must limit the equal sign to a statement of equality, select arithmetic operations that may be the opposite of those used in arithmetic, and distinguish among different uses of letters as mathematical symbols (Kieran, 1992). Although letters are used in algebra to represent an unknown value, they are also used in mathematics to represent labels ($1y = 3f$), constants (π, e, c), generalized numbers ($a + b = b + a$), varying quantities ($y = 9x - 2$), and parameters ($y = mx + b$; Philipp, 1992). MacGregor and Stacey (1997) presented evidence that difficulties in learning to use algebraic notation have several origins including intuitive reasoning about unfamiliar notation systems, analogies with symbol systems used in other school subjects or in everyday life, interference from new learning in mathematics, and poorly designed teaching materials.

This focus on differences, however, may cause investigators to overlook the similarities between arithmetic and algebra word problems. One knowledge component that is shared by both arithmetic and algebra problems is the addition and subtraction of quantities. Riley, Greeno, and Heller (1983) proposed a taxonomy for simple arithmetic word problems that consisted of (a) *change* problems, in which either the starting amount, the amount of the change, or the resulting amount is unknown; (b) *combine* problems, in which either the total amount or the amount in one of the subsets is unknown; and (c) *compare* problems, in which either the referent, the compared quantity, or the differences between the two quantities is the unknown. Kintsch and Greeno (1985) theorized that learning to solve addition and subtraction word problems requires learning to represent the problems by set schema such as the start, transfer, and result sets for change problems; subset and superset for combine problems; and large set, small set, and difference for compare problems. Subsequent work by Derry (1989) and Marshall (1995) extended this analysis to multistep arithmetic problems (see Reed, 1999, for more detail).

Shared schematic components between arithmetic and algebra word problems create the possibility of transfer between the two types of problems. They also create the possibility of common sources of difficulty. For example, Lewis and Mayer (1987) found that errors in arithmetic word problems were more likely to occur when the language used to describe a compare relation was inconsistent with the arithmetic operation. We evaluate whether this inconsistent language effect applies to both arithmetic and algebra word problems by analyzing whether the word *slower* increases the use of subtraction and the word *faster* increases the use of addition in both types of problems.

Experiment 1

In Experiment 1 we investigated whether comparing similarities and differences between pairs of solutions to arithmetic word problems would improve students' ability to select correct equations for arithmetic word problems and transfer this ability to algebra word problems. A second objective was to investigate whether comparing similarities and differences between pairs of solutions to algebra word problems would improve students' ability to select correct equations for algebra word problems and transfer this ability to arithmetic word problems. A third objective was to investigate students' perceived similarities and differences when comparing solutions to word problems and relate their perceptions to their ability to select correct equations for solving these problems.

Method

Participants. The participants were 120 students in two sections of a cognitive psychology course taught by the first author. Ten students had taken a college calculus course, 66 students had taken a remedial algebra course, and 44 students had taken no college math courses except for introductory statistics. The experiment occurred during lectures on problem solving and was unrelated to topics discussed previously in the course. Students received extra credit for participating, and the final page of the instructional–test booklets described the purpose of the experiment.

Materials. We created corresponding arithmetic and algebra problems that differ in whether an equation contains an unknown variable. For example, the first problem in Table 1 does not require the use of symbols to represent an unknown value because the answer can be calculated by performing only arithmetic operations. In contrast, the equation for the second problem contains an unknown value in which the slower speed is represented by the letter r.

We transformed the arithmetic problems into algebra problems by asking for a value of rate rather than for a value of distance.

Table 1
Example of Corresponding Arithmetic and Algebra Test Problems

Bill runs in a long-distance race. He runs 1.8 hr at a fast speed and 0.7 hr at a slow speed of 7 mph. How far does he run if the fast speed is 1 mph faster?

1. 1.8 hr × 8 mph + 0.7 hr × 7 mph = miles?
2. 1.8 hr × 6 mph + 0.7 hr × 7 mph = miles?
3. 1.8 hr × 8 mph − 0.7 hr × 7 mph = miles?
4. 1.8 hr × 6 mph − 0.7 hr × 7 mph = miles?

Bill runs 20 miles in a long-distance race. He runs 1.8 hr at a fast speed and 0.7 hr at a slow speed. How fast is the slow speed if the fast speed is 1 mph faster?

1. 1.8 hr × $(r+1)$ mph + 0.7 hr × r mph = 20 miles
2. 1.8 hr × $(r-1)$ mph + 0.7 hr × r mph = 20 miles
3. 1.8 hr × $(r+1)$ mph − 0.7 hr × r mph = 20 miles
4. 1.8 hr × $(r-1)$ mph − 0.7 hr × r mph = 20 miles

Although the task required selecting among four equations, rather than producing a numerical answer, the values used in the arithmetic and algebra problems are consistent. For instance, the value of 7 mph in the arithmetic example is the answer to the algebra problem, and the value of 20 miles in the algebra example is the answer to the arithmetic problem.

Table 2 shows the eight arithmetic problems. We generated the problems and the four alternative equations by independently varying whether the relation between the two rates and the relation between the two distances requires addition or subtraction. In addition, there was a consistent and an inconsistent language effect for each of these four problem types.

The eight algebra problems follow the change shown in Table 1 and therefore require the same set schema as the arithmetic problems. Deciding the mathematical operation for rate requires the correct use of a compare schema. Deciding the mathematical operation for distance requires the correct use of a combine schema when distances are added and a compare schema when distances are subtracted.

Students received test booklets that consisted of four pretest problems on each of two pages followed by a pair of problem solutions on each of two pages. Half the students received solutions to arithmetic problems, and half the students received solutions to the corresponding algebra problems. Problems 2 and 3 formed the first set of paired solutions (as shown for the algebra problems in Appendix A), and Problems 5 and 7 formed the second pair. The major similarity was the addition of distances in the first pair and the subtraction of distances in the second pair. The major dissimilarity for both pairs was the different arithmetic operations for representing rates.

Two pages of posttest problems followed the solutions. The pretest and posttest each consisted of one page of arithmetic problems and one page of algebra problems. The order was counterbalanced across booklets because the arithmetic problems preceded and followed the arithmetic solutions, and the algebra problems preceded and followed the algebra solutions.

The four problems on each test page were counterbalanced for the addition and subtraction of distances, the addition and subtraction of rates, and the use of consistent and inconsistent language. In addition, pre- and posttest problems were counterbalanced by creating an equivalent set of eight arithmetic and eight algebra problems based on the problems shown in Table 2.

Procedure. The counterbalancing of pre- and posttest problems and the inclusion of either arithmetic or algebraic solutions resulted in four variations of the test booklets that were randomly distributed among the students in the two class sections. The written instructions explained that the task required selecting an equation that could be used to solve each of eight distance–rate–time problems. The instructions also informed students to carefully study the solutions because we are evaluating how helpful the solutions are for selecting equations for a new set of eight prob-

Table 2
Correct Equations for Eight Arithmetic Problems

1. Bill runs in a long-distance race. He runs 1.8 hr at a fast speed and 0.7 hr at a slow speed of 7 mph. How far does he run if the fast speed is 1 mph faster?
   1.8 hr × 8 mph + 0.7 hr × 7 mph = miles?
2. Mary and Sue ride their bicycles toward each other and meet for a picnic, after Mary rides for 3.5 hr and Sue rides at 8 mph for 2 hr. How far apart were they if Sue rode 2 mph slower?
   3.5 hr × 10 mph + 2 hr × 8 mph = miles?
3. A race car driver drove in a race. She finished after driving at a fast speed of 143 mph for 2 hr and for 1 hr at a reduced speed. How far did she drive if the slow speed was 30 mph slower?
   2 hr × 143 mph + 1 hr × 113 mph = miles?
4. A salesman and his regional manager must drive to meet. The salesman drove for 6 hr at 59 mph, and his manager drove for 3 hr. If the salesman drove 5 mph faster than his manager, how far apart were they?
   6 hr × 59 mph + 3 hr × 54 mph = miles?
5. Jean can go further when she runs for 1 hr than when she walks at 1 mph for 2 hr. If she can run 5 mph faster than she can walk, how much further can she run.
   1 hr × 6 mph − 2 hr × 1 mph = miles?
6. Experimental car Alpha traveled further than experimental car Beta by going 12 hr at 123 mph on a tank of gas, compared with 11 hr for Beta. How much further did Alpha travel if Alpha was 3 mph slower?
   12 hr × 123 mph − 11 hr × 126 mph = miles?
7. A boy scout took 8.5 hr to hike up a mountain and 3.5 hr at a speed of 4 mph to hike down the mountain. The trail up the mountain was longer than the trail down the mountain. How many miles longer was the trip up the mountain if he was 2 mph faster hiking down the mountain?
   8.5 hr × 2 mph − 3.5 hr × 4 mph = miles?
8. A boat travels away from a dock and is pursued by a police boat that leaves the same dock 11 hr later. After traveling for 2 hr at 24 mph the police boat closes the gap. How far behind was the police boat if the first boat was 20 mph slower?
   13 hr × 4 mph − 2 hr × 24 mph = miles?

lems. The experimenter told the participants that they should spend 3 min on each test page and that he would tell them when to turn the page. Students had 4 min to compare each pair of solutions. The data from 10 students (seven in the arithmetic group and three in the algebra group) were not analyzed because they did not answer all the questions, leaving 56 students in the arithmetic group and 54 students in the algebra group.

Results

Learning and transfer. Table 3 shows the percentage of equations correctly selected during the pretest and posttest by students who compared either arithmetic or algebra word problems. A Test (pretest, posttest) × Problem (arithmetic, algebra) × Language (consistent, inconsistent) repeated measures analysis of variance (ANOVA) evaluated these three within-subject variables.

Each of the three variables produced a significant main effect. Students improved from the pretest (43% correct) to posttest (49% correct), $F(1, 108) = 9.75$, $p = .002$, $\eta_p^2 = .083$; were more successful on arithmetic problems (56% correct) than on algebra problems (36% correct), $F(1, 108) = 69.32$, $p < .001$, $\eta_p^2 = .391$; and did better on consistent language problems (58% correct) than on inconsistent language problems (33% correct), $F(1, 108) = 137.25$, $p < .001$, $\eta_p^2 = .556$.

Although the Problem × Test interaction was not significant, $F(1, 108) = 1.24$, $p > .05$, $\eta_p^2 = .011$, we analyzed improvement from pre- to posttest to evaluate prior hypotheses about arithmetic and algebra learning. The arithmetic group selected correct equations for 53% of the arithmetic problems on the pretest and 64% of the arithmetic problems on the posttest, $F(1, 55) = 6.92$, $p = .011$, $\eta_p^2 = .112$. The algebra group selected correct equations for 37% of the algebra problems on the pretest and 40% of the algebra problems on the posttest, $F(1, 53) = 0.67$, $p > .05$.

Neither of the groups demonstrated significant transfer. The arithmetic group selected correct equations for 31% of the algebra problems on the pretest and 35% of the algebra problems on the posttest, $F(1, 55) = 1.26$, $p > .05$. The algebra group selected correct equations for 50% of the arithmetic problems on the pretest and 57% of the arithmetic problems on the posttest, $F(1, 53) = 2.08$, $p > .05$. To examine the causes of these modest amounts of learning and transfer, we analyzed students' perceived similarities and dissimilarities of the problem solutions.

Perceived similarities and dissimilarities. Table 4 lists the most frequently reported similarities and dissimilarities for the two pairs of problems and two types of solutions (arithmetic and algebra). The results support Lobato's (2008) claim that students often perceive similarities that differ from those perceived by experts. Although we hoped that students would report that distances are added in the first pair of problems and subtracted in the second pair, we found that they often listed more generic similarities such as both problems require the same steps, involve distances, or multiply rate by time. Although these similarities are correct, they do not aid in distinguishing among the four alternative equations on the pretest that required determining whether distances (and speeds) should be added or subtracted. Fewer than half of the 56 students who received the arithmetic solutions and the 54 students who received the algebra solutions reported the addition (Pair 1) or subtraction (Pair 2) of distances as the major similarity between the two solutions.

The 47 students who correctly reported the subtraction of distances in the second pair selected the correct equations on 51% of the pre- and posttest problems, compared with 41% for the 63 students who did not report the subtraction of distances, $F(1, 106) = 8.27$, $p < .01$, $\eta_p^2 = .072$. There was no interaction with the solution, $F(1, 106) = 1.49$, $p > .05$, $\eta_p^2 = .014$. Thus, students who correctly reported the major similarity between a pair of either arithmetic or algebraic solutions were more successful in selecting the correct equations on the tests.

Although the solution type did not have a substantial effect on reported similarities, it did have a large effect on reported dissimilarities. Addition versus subtraction of speeds was the most frequent difference reported for the algebra solutions but virtually ignored for the arithmetic solutions. The most frequent difference reported by students who received arithmetic solutions was that a single object moved in one problem and two objects moved in the other problem. A likely cause of this discrepancy is that, as illustrated in Table 1, the arithmetic operations are salient in the algebra solutions ($r+1$, r) but hidden in the arithmetic solutions (8, 7). Making these operations more salient in arithmetic problems (7 + 1 rather than 8) might increase focusing on these relations in the arithmetic solutions.

Table 3
Percent Correct Responses on Pretest and Posttest

Solution	Arithmetic		Algebra	
	Pretest	Posttest	Pretest	Posttest
		Experiment 1		
Arithmetic	53	64	31	35
Algebra	50	57	37	40
		Experiment 2		
Distance	40	50	33	31
Rate	45	62	43	41
		Experiment 3		
Unmixed	50	77	36	35
Mixed	72	67	35	37

Note. Students in Experiment 1 compared either arithmetic or algebra solutions. Students in Experiment 2 compared only arithmetic solutions in which paired similarities were based on either distances or rates. Students in Experiment 3 compared either arithmetic solutions followed by algebra solutions (unmixed) or arithmetic solutions with algebra solutions (mixed).

Discussion

Comparing two pairs of solutions served the dual purpose of encouraging learning of the solutions and providing diagnostic data of the perceived similarities and differences between the two solutions. Both arithmetic and algebra word problems were challenging for students, particularly when there was an inconsistent language effect. The large inconsistent language effect supports and extends the Lewis and Mayer (1987) findings based on extensive relations (*more*, *less*) to the intensive relations (*faster*, *slower*) used on our study.

The diagnostic data revealed that we did not find the perceived similarities and differences that we had hoped to find, which

Table 4
Reported Similarities and Dissimilarities Between the Two Solution Pairs in Experiment 1

	Pair 1		Pair 2	
Variable	Arithmetic	Algebra	Arithmetic	Algebra
Similarities				
Add distances	*13*	*16*	0	0
Subtract distances	0	0	*21*	*26*
Same steps	10	6	12	9
Involve distances	14	6	14	8
Multiple Rate × Time	10	12	2	2
Dissimilarities				
Add vs. subtract rate	*1*	*24*	*1*	*26*
One vs. two objects	21	4	10	5

Note. Arithmetic refers to the 56 students who compared arithmetic problems, and Algebra refers to 54 students who compared algebra problems. Responses reflecting combine schema (add distances) or compare schema (rate, subtract distances) are shown in italics.

nonetheless supports Lobato's (2008) claim that similarities perceived by experts are often not shared by others. However, those students who did report that both problems in the second comparison required subtraction of distances were more successful in selecting correct equations on the tests. As argued by Lobato (2003), the diagnostic data provided by the actor-oriented approach has an advantage over traditional methods of learning and transfer by providing more information that is useful in design experiments to improve instruction.

Students who compared arithmetic problems typically did not report as a difference the addition of speed in one problem and the subtraction of a speed in the other problem. Including these operations in the arithmetic equations should increase reporting this difference, as occurred for the algebraic equations. Showing the arithmetic operations for speed would also increase the structural similarity (Holyoak & Koh, 1987) between the arithmetic and algebraic solutions. In addition, providing feedback following the comparisons should help direct students' attention to schematic similarities and differences. Experiment 2 builds on the findings of Experiment 1 to implement these changes in instruction.

Experiment 2

The primary goal of Experiment 2 was to increase transfer from arithmetic to algebra word problems by making the arithmetic operations more salient and by providing feedback on similarities and differences between pairs of arithmetic problems. The large discrepancy between arithmetic and algebra problems in reported differences for adding or subtracting rates may have been caused by the implicit (hidden) operations for arithmetic and the explicit (shown) operations for algebra. As discussed by Goldstone, Landy, and Son (2010), perceptual processing can play an important role in problem solving, including equations, so making arithmetic operations explicit for the arithmetic solutions should increase reporting these operations. In addition, the instruction provided feedback about the similarities and differences between each pair of problems.

A secondary goal of Experiment 2 was to examine whether comparisons of similarity and dissimilarity differ in effectiveness. In Experiment 1 the two problems were similar within each pair because distances were added in the first pair and subtracted in the second pair. The two problems were dissimilar within each pair because rates were added for one problem and subtracted for the other. Experiment 2 included a second condition in which similarities involved rates (subtracted in the first pair and added in the second) and dissimilarities involved distances (added for one problem and subtracted for the other). This second condition enabled us to contrast similarity with dissimilarity comparisons for learning correct arithmetic operations.

Method

Participants. The participants were 95 students from two sections of a cognitive psychology course taught by the first author. Nineteen students had taken a college calculus course, 43 students had taken a remedial algebra course, and 33 students had taken no college math courses except for introductory statistics. The experiment occurred during lectures on problem solving and was unrelated to topics discussed previously in the course. Students received extra credit for participating and were later informed about the purpose of the experiment.

Materials. The solutions differed from the arithmetic solutions in Experiment 1 because of the inclusion of plus and minus signs to show the arithmetic operations for speed. For example, the four instances of $r - 30$ in the algebraic solution of the race car problem in Appendix A had a comparable $143 - 30$ representation (rather than 113) in the arithmetic solution. Another change from Experiment 1 was the inclusion of feedback to describe the major similarity and dissimilarity between the two problems, as shown in Appendix B.

In addition to the distance similarity condition in Experiment 1, the rate similarity condition contained pairs of problems in which both problems required either subtracting rates (first pair) or adding rates (second pair). There were different arithmetic operations for distances in each of these pairs. The same four problems were paired differently to create these two conditions. Others aspects of the test booklets (instructions, test problems) were the same as in Experiment 1.

Procedure. The two instructional conditions (distance similarity and rate similarity) and the counterbalancing of pre- and posttest questions resulted in four instructional–test booklets that were randomly distributed to students. The procedure was identical to the procedure in Experiment 1 except that students had 1.5 min to study the feedback following their comparison of each pair of solutions. The data from 19 students were not analyzed because they did not answer all the questions, leaving 39 students in the distance similarity condition and 37 students in the rate similarity condition.

Results and Discussion

Learning and transfer. Table 3 shows the percentage of equations correctly selected during the pretest and posttest for students who made either distance similarity or rate similarity comparisons. A Test (pretest, posttest) × Problem (arithmetic, algebra) × Language (consistent, inconsistent) repeated measures ANOVA evaluated these three within-subject variables. In addition, group (distance similarity, rate similarity) was included as a between-subjects variable.

Each of the four variables produced a significant main effect. Students improved from the pretest (40% correct) to posttest (46% correct), $F(1, 74) = 6.10$, $p = .016$, $\eta_p^2 = .076$; were more successful on arithmetic problems (49% correct) than on algebra problems (37% correct), $F(1, 74) = 16.51$, $p < .001$, $\eta_p^2 = .182$; and did better on consistent language problems (52% correct) than inconsistent language problems (35% correct), $F(1, 74) = 42.36$, $p < .001$, $\eta_p^2 = .364$. In addition, the group that compared rate similarity problems (48% correct) was more successful than the group that compared distance similarity problems (39% correct), $F(1, 74) = 5.01$, $p = .028$, $\eta_p^2 = .063$.

However, the two groups did not differ in learning, as shown by the lack of a Group × Test interaction, $F(1, 74) < 1$, $p > .05$. The distance similarity group improved from 37% correct on the pretest to 41% correct on the posttest. The rate similarity group improved from 44% correct on the pretest to 52% correct on the posttest. The pretest difference indicates that students in the rate similarity group were more proficient at solving word problems.

The only significant interaction among the four variables was the Test × Problem interaction, $F(1, 74) = 10.20$, $p = .002$, $\eta_p^2 = .121$. Studying solutions to the arithmetic problems resulted in improvement on these problems from 43% correct on the pretest to 56% correct on the posttest. However, there was no transfer to algebra problems in which students selected to correct equation on 38% of the pretest problems and on 36% of the posttest problems.

The Experiment 2 results therefore replicated the Experiment 1 findings of greater accuracy on (a) arithmetic, (b) consistent language, and (c) posttest problems. They also replicated the lack of transfer from arithmetic solutions to algebra problems.

Perceived similarities and dissimilarities. Table 5 lists the most frequently reported similarities and dissimilarities for the two pairs of problems and two groups (distance similarity and rate similarity). The data reveal that the distance similarity group did reasonably well in identifying the similarities in distance relations. Fifteen of the 39 students reported that distances are added in the first pair, and 26 students reported that distances are subtracted in the second pair. Identifying the relation between distances was the most frequently mentioned similarity for both the first and second pairs.

However, the data also reveal that the 37 students in the rate similarity group often failed to identify the similarities in rate relations even though these relations were made more explicit by including arithmetic operations (8 + 2 rather than 10). The number of students who reported rate similarities increased from two to 12 following feedback on the first pair, but both of these numbers were considerably less than the number of students who reported distance similarities. Rather than report rate similarities, these students often reported that both problems involved distances or multiplied rate by time.

One possible reason for the greater salience of distance similarities is that the solutions began with an explanation of the relation between the two distances and then continued with a top-down hierarchical description (Reed, 1987). Distances were represented as the product of rate and time, followed by inserted values for rate and time as the final step. A second possible reason for the emphasis on distance similarities is that the mathematical operation for combining distances is spatially separated from the other symbols in the equation. Two spaces surrounded the plus or minus sign between the two distances, which also had a more central location in the equation, whereas no spaces surrounded the plus or minus sign for adding or subtracting a number from rate:

$$2 \text{ hr} \times r \text{ mph} \;\; + \;\; 1 \text{ hr} \times (r-30) \text{ mph} = 400 \text{ miles}$$

Our use of two spaces around the arithmetic symbol between distances is consistent with the Landy and Goldstone (2007) finding that spacing can facilitate applying arithmetic operations in a correct order. They found that college students were faster and more accurate in calculating the answer to “2 + 3×5 = ?” than the answer to “2+3 × 5 = ?” Although the answer is 25 for both equations, only the spacing in the first equations supports the standard order that multiplication precedes addition. Our findings suggest the spacing might also influence which arithmetic operations are more salient.

In contrast to the failure of the rate similarity group to report similarities, both groups performed moderately well in identifying dissimilarities. This was particularly evident in the second pair in which 21 of 39 students in the distance similarity group correctly identified the difference between adding and subtracting rates and 25 of the 37 students in the rate similarity group correctly identified the difference between adding and subtracting distances (see Table 5).

The greater consistency in reporting differences than similarities may be caused by fewer differences in the solutions. As indicated in Tables 4 and 5, reported similarities included that both problems involve the same steps, involve distances, or multiply rate by time. Such similarities are too generic to be useful for selecting the correct equation. In contrast, the only major competing dissimilarity was that there were two moving objects in one problem and one moving object in the other. The modified design in Experiment 2 greatly reduced reporting this irrelevant difference.

Table 5
Reported Similarities and Dissimilarities Between the Two Solution Pairs in Experiment 2

	Pair 1		Pair 2	
Variable	Distance	Rate	Distance	Rate
Similarities				
Add distances	*15*	0	0	4
Subtract distances	2	0	*26*	2
Add rate	0	0	0	*12*
Subtract rate	0	*2*	1	0
Same steps	5	1	1	0
Involve distances	4	15	6	2
Multiple Rate × Time	6	9	0	12
Dissimilarities				
Add vs. subtract rate	*10*	9	*21*	6
Add vs. subtract distances	2	*12*	7	*25*
One vs. two objects	4	2	3	0

Note. Distance refers to the 39 participants in the distance-similarity group, and Rate refers to the 37 participants in the rate-similarity group. Responses reflecting combine schema (add distances) or compare schema (rate, subtract distances) are shown in italics.

Experiment 3

Experiment 2 made progress in enhancing students’ abilities to focus more on the arithmetic differences that distinguish among the four types of equations. However, transfer to algebra word problems still proved elusive. In Experiment 3 we therefore abandoned the transfer paradigm to investigate whether instruction would increase students' performance on both arithmetic and algebra problems if they studied solutions to both types of problems.

In the previous experiments, students compared either two pairs of arithmetic solutions or two pairs of algebra solutions. In Experiment 3 they compared either a pair of arithmetic solutions followed by a pair of algebra solutions (unmixed group) or two pairs consisting of one arithmetic solution and one algebra solution (mixed group).

Method

Participants. The participants were 73 students from two sections of a cognitive psychology course taught by the first author. Twelve students had taken a college calculus course, 31 students had taken a remedial algebra course, and 30 students had taken no college math courses except for introductory statistics. The experiment occurred during lectures on problem solving and was unrelated to topics discussed previously in the course. Students received extra credit for participating and were later informed about the purpose of the experiment.

Materials. The test booklets followed the same format as the test booklets in Experiment 2 except that for the unmixed group the second pair of instructional problems consisted of two algebra problems instead of two arithmetic problems. Students in the mixed group received an arithmetic solution for the first member of each pair and an algebra solution for the second member of each pair. Another change from Experiments 1 and 2 is that each of the two pre- and posttest pages consisted of two arithmetic and two algebra problems rather than four arithmetic problems on one page and four algebra problems on the other page. The second pretest page contained the four problems that received solutions, allowing for immediate feedback as in the first two experiments. Other aspects of the test booklets (instructions, feedback) were the same as in Experiment 2.

Procedure. The two instructional conditions (unmixed and mixed solutions) and the counterbalancing of pre- and posttest questions resulted in four instructional–test booklets that were randomly distributed to students. The procedure was identical to the procedure in Experiment 2. The data from 16 students were not analyzed because they did not answer all the questions, leaving 28 students in the unmixed condition and 29 students in the mixed condition.

Results and Discussion

Learning and transfer. Table 3 shows the percentage of equations correctly selected during the pretest and posttest for students who made either unmixed (arithmetic or algebra) or mixed (arithmetic with algebra) comparisons. A Test (pretest, posttest) × Problem (arithmetic, algebra) repeated measures ANOVA evaluated these two within-subject variables. In addition, group (unmixed, mixed) was included as a between-subjects variable.

Students showed marginal improvement from the pretest (48% correct) to posttest (52% correct), $F(1, 55) = 3.75$, $p = .058$, $\eta_p^2 = .064$, and were more successful on arithmetic problems (66% correct) than on algebra problems (36% correct), $F(1, 55) = 83.49$, $p < .001$, $\eta_p^2 = .603$. There was no difference between the two groups, $F(1, 55) < 1$, $\eta_p^2 = .005$.

The major finding was revealed in the Test × Problem × Group interaction, $F(1, 55) = 9.91$, $p = .003$, $\eta_p^2 = .153$. Studying solutions to the unmixed problems resulted in a substantial improvement on arithmetic problems from 50% correct on the pretest to 77% correct on the posttest. This improvement on arithmetic problems and the lack of improvement on algebra problems replicates findings in Experiments 1 and 2 (see Table 3). However, students in the mixed group who compared arithmetic with algebra problems did not improve on either the arithmetic or algebra problems. The high pretest score of 72% correct may have contributed to the lack of improvement, but the posttest score declined slightly to 67% correct. In addition, the reported dissimilarities between arithmetic and algebra problems revealed a distracter in this comparison.

Perceived similarities and dissimilarities. Table 6 lists the most frequently reported similarities and differences for the two groups across the two pairs. The unmixed group initially did better in reporting the similarities involving distance operations, but both groups improved following feedback. Twelve of the 28 students in the unmixed group, but only three of the 29 students in the mixed group, reported the addition of distances for the two problems in the first pair. However, feedback following the first pair resulted in a substantial increase for both groups in reporting the arithmetic relation between the two distances in the second pair.

Reporting dissimilarities was more problematic for the mixed group. Eight students in the unmixed group reported addition versus subtraction of rates for the first pair, and 20 students reported this difference for the second pair. In contrast, only one student in the mixed group reported this difference for the first pair, and only seven students reported this difference for the second pair.

Students in the mixed group were instructed not to report that only one problem requires algebra when comparing differences

Table 6
Reported Similarities and Dissimilarities Between the Two Solution Pairs In Experiment 3

	Pair 1		Pair 2	
Variable	Unmixed	Mixed	Unmixed	Mixed
Similarities				
Add distances	*12*	*3*	0	0
Subtract distances	0	0	*17*	*21*
Same steps	5	8	2	3
Involve distances	10	8	5	1
Multiple Rate × Time	0	4	0	2
Dissimilarities				
Add vs. subtract rate	*8*	*1*	*20*	*7*
One vs. two objects	21	4	10	3
Calculate rate vs. distance	0	10	0	15

Note. Unmixed refers to the 28 students who compared arithmetic followed by algebra problems, and Mixed refers to 29 students who compared arithmetic with algebra problems. Responses reflecting combine schema (add distances) or compare schema (rate, subtract distances) are shown in italics.

between arithmetic and algebra solutions. None of the 29 students mentioned this difference, but 10 of the students initially stated that one problem required the calculation of rate and the other problem required the calculation of distance. This difference in goal did not occur in the unmixed problems because arithmetic problems required the calculation of distance, and the algebra problems required the calculation of rate.

Even more students (15) reported this difference in goal on the second pair. The feedback following the first pair stated:

> The first problem requires subtracting a number from the stated speed (12 mph) because the other speed is 3 mph slower. The second problem requires adding a number to the unknown variable (*r* mph) because the other speed is 8 mph faster.

Although this feedback was intended to focus students' attention on addition and subtraction in the compare schema, the more salient differences between arithmetic and algebra problems limited its effectiveness.

General Discussion

Learning and Transfer

The major objective of our study was to investigate the effectiveness of comparing solutions for either arithmetic or algebra word problems and relate the perceived similarities and differences to success in selecting correct equations. The results of all three experiments found a consistent learning effect from comparing arithmetic word problems as revealed by the gains made by the arithmetic group in Experiment 1, both groups in Experiment 2, and the unmixed group in Experiment 3. The only condition that did not produce learning occurred for the mixed group in Experiment 3 in which students compared arithmetic solutions with algebra solutions.

The gains that occurred for arithmetic problems were not accompanied by gains for algebra problems. Those groups that compared algebra solutions (the algebra group in Experiment 1 and the unmixed group in Experiment 3) did not improve on the posttest. In addition, none of the learning gains for arithmetic word problems transferred to algebra word problems.

Our findings that students improved on the posttest after comparing arithmetic, but not algebra, word problems is consistent with the influence of prior knowledge on the effectiveness of the comparison method. Although Rittle-Johnson and Star (2009) initially found that comparing two solutions for the same algebra equation was effective, Rittle-Johnson, Star, and Durkin (2009) subsequently found that students who were less familiar with algebra benefited more from sequential study of the two solutions.

A challenge for comparing worked examples of algebra word problems is that each example contains a lot of information (see Appendix A). Although these solutions could be summarized, the elaborated solutions used in this study are more effective than solutions that provide less information (Reed, Dempster, & Ettinger, 1985). For instance, the inclusion of tables in the worked examples helps students construct equations for algebra word problems (Reed & Ettinger, 1987) and connect the symbols in the equation to the quantities mentioned in the text (Reed, Corbett, Hoffman, Wagner, & MacLaren, 2012).

The dilemma is that although such complexity is helpful when studying a single solution, making the task more demanding by comparing two solutions may overwhelm students. An example of a counterproductive outcome by making the task more demanding was adding units to the quantities in the worked examples. Undergraduates in a Statistical Methods in Psychology class who studied worked examples similar to those in Appendix A constructed fewer correct equations when they used units (such as *tanks/min*) to guide their constructions (Reed, 2006). In contrast, students benefited from prior instruction in which they selected which of two mathematical expressions (such as *.2 tanks/min* × *4 min* or *2 tanks/min* + *4 min*) represented a quantity that had a real-world referent. This simple intervention before they studied worked examples helped them construct equations in which the terms of the equations represented real-world quantities.

Another study by Reed (1989) found that comparing the solutions of two isomorphic word problems was not more helpful for constructing an equation to a third isomorphic problem than studying the individual solutions. Reed proposed that the difficulty of verbally describing a generic schema for the problem isomorphs may have contributed to the failure to replicate the schema abstraction found for variations of Dunker's radiation problem (Catrambone & Holyoak, 1989; Gick & Holyoak, 1983). Our findings indicate that the greater complexity of the solutions to algebra word problems may also have limited the effectiveness of comparing isomorphic solutions in Reed's study.

Perceived Similarities and Dissimilarities

The Experiment 1 results revealed that students' described similarities and differences were often too generic (involve distances) or too superficial (one vs. two objects) to help them learn the combine and compare schema for distances and rates. Displaying the arithmetic relations involved in rate (8 + 2 rather than 10) and providing feedback in Experiments 2 and 3 increased the reported addition and subtraction of distances and rates—a finding that is consistent with other research (Fisher, Borchert, & Bassok, 2011; Goldstone et al., 2010; Kellman, Massey, & Son, 2010) on how varying the physical structure of equations can either facilitate or impede learning. However, these increased reports did not result in improved posttest scores for the algebra word problems.

Reed's (2011) analysis of learning as mapping across situations (problems, solutions, representations, sociocultural contexts) provides a framework for examining the successful increase in noticing important similarities and differences across solutions while failing to transfer this information. Finding the similarities and differences between solutions can be based solely on the physical structure of the equations. Reporting that distances are added in the two problems in Appendix A but rates are subtracted in the first problem and added in the second problem requires only comparing the final equations.

The lack of transfer suggests that students did not connect the physical similarities and differences in the equations with the underlying schematic components. This connection requires mapping across representations from text descriptions to mathematical terms in equations. These mappings are particularly difficulty when the word *faster* requires subtraction and the word *slower* requires addition. We found a large inconsistent language effect for both arithmetic and algebra problems.

The combined importance of attending to both physical and mathematical properties of problems is incorporated into Lobato et al.'s (2003) concept of "focusing phenomena," which directs students' attention toward useful mathematical properties when a variety of information competes for their attention. The importance of attending to appropriate physical properties is evident from difficulties students encountered in identifying the appropriate horizontal and vertical distances in a diagram of a slide that would enable them to calculate its slope (Lobato, 2008). However, understanding slope also depends on learning that its calculation depends on the ratio, and not the difference, of these two quantities (Lobato et al., 2003).

As pointed out by Rittle-Johnson and Star (2009), the comparison method requires careful support to be effective. Their classroom instruction incorporated explicit comparison prompts, explanation prompts, and peer collaboration to support productive explanations. Further elaboration of our own comparison instruction is needed to make certain that perceived similarities and differences in equations are linked to combine and compare schema in the text.

Pedagogical Implications

We believe there is a productive analogy between learning to solve algebra word problems and learning to read. LaBerge and Samuels (1974) proposed that learning to read requires many component skills such as analyzing the features of letters, combining the features to identify letters, converting the letters into sounds for pronouncing words, understanding the meaning of individual words, and combining the meaning of words to understand text. By analogy, solving algebra word problems requires the learning of many component skills such as translating verbal statements into mathematical symbols, making adjustments for reversed language translations, combining symbols to form terms in the equation, and determining which terms to equate. LaBerge and Samuels argued that the ability to acquire complex, multicomponent skills depends on the ability to avoid cognitive overload by efficiently performing the component skills.

An important distinction in cognitive load theory is the differentiation of extraneous load that is reducible by instructional design and intrinsic load that is not reducible by instructional design (Moreno & Park, 2010). The transition from arithmetic to algebra word problems involves intrinsic load because students must learn new algebraic techniques, but also likely involves unnecessary extraneous load because many students attempt this transition before they are proficient in solving arithmetic word problems. As indicated in Table 3, university students' success in selecting the correct arithmetic equation among four alternatives ranged from 40% to 72% correct on the pretests.

A task analysis of algebra word problems reveals that they share the same change, combine, and compare schema as arithmetic word problems. We therefore believe that the transition from arithmetic to algebra word problems should build upon students' understanding of compare, combine, and change schema that form the semantic components of arithmetic word problems (Derry, 1989; Kintsch & Greeno, 1985; Marshall, 1995; Riley et al., 1983). To use the reading analogy, expecting students to solve algebra word problems before they have become proficient in solving arithmetic word problems is like expecting students to understand sentences before they have become proficient in understanding individual words.

Our results underscore the importance of successful research on improving schematic understanding of arithmetic word problems in elementary school. Fuson and Willis (1989) reported success in using diagrams to help second graders represent combine, change, and compare problems. Another study that used diagrams to support Marshall's (1995) schema-based approach found that third-grade children performed better on solving arithmetic word problems than children who received more generic problem-solving strategies (Jitendra et al., 2007). More recently, Ng and Lee (2009) have used a similar approach (called the model method) in which Primary 5 Singapore children constructed diagrams to represent key information in the problems. In addition, conceptually rewording two-step change problems to clarify part–whole relations helped third, fourth, and fifth graders solve these problems (Santiago, Orrantia, & Verschaffel, 2007). Another study found that fourth and fifth graders who learned to change their representation from a whole–part schema to a comparison schema transferred this knowledge to new situations (Gamo, Sander, & Richard, 2010). It is hoped that such findings will encourage schema-based instruction in elementary schools, providing a firmer foundation for subsequently solving more complex problems such as algebra word problems and physics problems (Sherin, 2001).

Whether such techniques are also effective remedial techniques for college students is an unresolved question, because research has focused on schematic learning in elementary school. If successfully applied in the elementary classroom, there would be reduced need for subsequent remedial instruction.

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(*Appendices follow*)

Appendix A

Paired Comparison of Algebraic Solutions

A race car driver drove in a 400-mile race. She finished after driving 2 hr at a fast speed and for 1 hr at a reduced speed. What was the fast speed if the slow speed was 30 mph slower?

This problem can be solved by adding the distances traveled at the two speeds:

$$\text{Distance}_{\text{fast}} + \text{Distance}_{\text{slow}} = 400 \text{ miles.}$$

The distance traveled at the fast speed is determined by multiplying Time by Rate:

$$\text{Distance}_{\text{fast}} = \text{Time}_{\text{fast}} \times \text{Rate}_{\text{fast}} = 2 \text{ hr} \times r \text{ mph,}$$

where r is the unknown variable (fast speed).

The distance traveled at the slow speed is determined the same way:

$$\text{Distance}_{\text{slow}} = \text{Time}_{\text{slow}} \times \text{Rate}_{\text{slow}} = 1 \text{ hr} \times (r-30) \text{ mph.}$$

To summarize:

Speed	Time (hr)	Rate (mph)	Distance (miles)
Fast	2	r	$2 \times r$
Slow	1	$r-30$	$1\times(r-30)$

$$\text{Distance}_{\text{fast}} + \text{Distance}_{\text{slow}} = 400 \text{ miles}$$

$$2 \text{ hr} \times r \text{ mph} \ + \ 1 \text{ hr} \times (r-30) \text{ mph} = 400 \text{ miles}$$

Mary and Sue live 50 miles apart. They ride their bicycles toward each other and meet for a picnic, after Mary rides 3.5 hr and Sue rides for 2 hr. How fast did Sue ride if Sue rode 2 mph slower?

This problem can be solved by adding the distances traveled by Mary and Sue:

$$\text{Distance}_{\text{Sue}} + \text{Distance}_{\text{Mary}} = 50 \text{ miles.}$$

Sue's distance is determined by multiplying Time by Rate:

$$\text{Distance}_{\text{Sue}} = \text{Time}_{\text{Sue}} \times \text{Rate}_{\text{Sue}} = 2 \text{ hr} \times r \text{ mph,}$$

where r is the unknown variable (Sue's speed).

Mary's distance is determined the same way:

$$\text{Distance}_{\text{Mary}} = \text{Time}_{\text{Mary}} \times \text{Rate}_{\text{Mary}} = 3.5 \text{ hr} \times (r+2) \text{ mph.}$$

To summarize:

Person	Time (hr)	Rate (mph)	Distance (miles)
Sue	2	r	$2 \times r$
Mary	3.5	$r+2$	$3.5\times(r+2)$

$$\text{Distance}_{\text{Sue}} + \text{Distance}_{\text{Mary}} = 50 \text{ miles}$$

$$2 \text{ hr} \times r \text{ mph} \ + \ 3.5 \text{ hr} \times (r+2) \text{ mph} = 50 \text{ miles}$$

1. What is the major similarity between the two solutions?
2. What is the major dissimilarity between the two solutions?

Appendix B

Paired Comparison of Arithmetic Solutions With Feedback

The previous pair of problems with correct equations is shown below. Read the feedback to the questions about the similarity and dissimilarity of the two solutions.

A race car driver drove in a race. She finished after driving a fast speed of 143 mph for 2 hr and for 1 hr at a reduced speed. How far did she drive if the slow speed was 30 mph slower?

$$2 \text{ hr} \times 143 \text{ mph} \ + \ 1 \text{ hr} \times (143-30) \text{ mph} = \text{miles?}$$

Mary and Sue ride their bicycles toward each other and meet for a picnic, after Mary rides for 3.5 hr and Sue rides at 8 mph for 2 hr. How far apart were they if Sue rode 2 mph slower?

$$2 \text{ hr} \times 8 \text{ mph} \ + \ 3.5 \text{ hr} \times (8+2) \text{ mph} = \text{miles?}$$

1. What is the major similarity between the two solutions?

Both problems require adding two distances. The total distance traveled by the race car driver is determined by adding the distance traveled at the fast speed to the distance traveled at the slow speed. Similarly, the distance traveled by Mary and Sue is determined by adding the distance traveled by Mary to the distance traveled by Sue.

2. What is the major dissimilarity between the two solutions?

The first problem requires subtracting a number from the stated speed (143 mph) because the other speed is 30 mph slower. The second problem requires adding a number to the stated speed (8 mph) because the other speed is 2 mph faster.

Received May 6, 2011
Revision received November 28, 2011
Accepted January 10, 2012 ■