The Effects of Professional Development on the Mathematical Content Knowledge of K–3 Teachers

by

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Abstract

Our overall project goal is to map a trajectory for the evolution of the teaching of elementary school teachers to support teacher preparation and professional development. Here we focus on one aspect of this evolution: mathematical content knowledge. We developed and administered a content- and context-relevant mathematics assessment to 131 prospective and practicing teachers who differed in terms of the number of years they had engaged in sustained professional development focused on children’s mathematical thinking. Overall results indicate that teachers learn mathematics from professional development, even if developing mathematical content knowledge is not the primary goal of the professional development. A second result is that the prospective elementary teachers approached standard algorithms with an orientation toward calculating without conceptualizing; this algorithmic orientation was significantly stronger for the prospective teachers than even for teachers who had yet to begin professional development. A third result is that the choice of the content item is important for revealing differences, and we discuss implications for researchers interested in measuring teacher content knowledge.

During the evolution of teachers’ knowledge, beliefs, and practices over the course of their careers, teachers’ professional development needs also change. In a large-scale study we designed to begin to map a trajectory for the evolution of the teaching of elementary school teachers engaged in sustained professional development, our goal is to contribute to a better understanding of the needs of teachers at different points of their development to enable facilitators to modify professional development to better serve the teachers with whom they work.
We recognize that important aspects of teaching improve with experience, but we argue that teaching experience alone is generally insufficient for teachers to develop the type of expertise necessary to teach mathematics in ways consistent with current recommendations (National Council of Teachers of Mathematics [NCTM], 2000; National Research Council [NRC], 2001). Therefore we grouped teachers on the basis of the number of years in which they had engaged in sustained professional development focused on children’s mathematical thinking instead of their years of teaching experience. We created three K–3 teacher groups (teachers about to begin sustained professional development, teachers who had completed 2 years of sustained professional development, and teachers who had completed at least 4 years of sustained professional development and who had begun to emerge into teacher leaders), and we anchored our data with data on a group of prospective elementary school teachers.

After considering the constructs that should be addressed to learn about these teachers and reasonable ways to measure the constructs across our four groups, we, in our overall project, investigated teachers’ mathematical content knowledge, beliefs, practices, and professional noticing of children’s mathematical thinking (Jacobs, Lamb, Philipp, Schappelle, & Burke, 2007). In this paper we focus upon teachers’ mathematical content knowledge.

**Mathematical Content Knowledge for Teaching**

Teachers are professionals, and one defining characteristic of professionals is that they hold specialized knowledge different from knowledge that laypeople hold (National Research Council, 2000). Those working with teachers parse knowledge for teachers according to their goals; their decisions are often pragmatic, not theoretical. On the one
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hand, a professional developer focused on issues of teaching may see little benefit in drawing distinctions among types of knowledge, such as general pedagogical knowledge (e.g., how to manage a class), mathematical content knowledge (e.g., $20 \div 4$ might be conceptualized as how many $4$s are in $20$), or pedagogical content knowledge (e.g., how primary-grade students’ reasoning about partitive-division contexts differs from their reasoning about measurement-division contexts), but may support teacher growth by addressing the relationships among these types of knowledge when they arise for the teachers. Furthermore, the development of one type of knowledge is often related to others; for example, teachers who learn to focus upon their students’ mathematical reasoning may find that their students raise issues of mathematics content about which the teachers have yet to grapple. Similarly, novice teachers need support in addressing these interrelated knowledge domains, requiring a student-teacher supervisor to simultaneously address different types of knowledge. On the other hand, a mathematician planning a mathematics course for prospective elementary school teachers is more likely than a professional developer or a student-teaching supervisor to focus upon mathematics content and not upon other types of knowledge for teaching. Consequently, a mathematician’s view of the mathematics that teachers need to know has important implications for the way he or she structures the course and, thereby, for the mathematics learned by the students.

Teachers’ knowledge is one of the most studied constructs in education, and fundamental questions for researchers are What constitutes mathematical knowledge? and How can this knowledge be measured? Early attempts to define teachers’ mathematical content knowledge on the basis of the number of mathematics courses
teachers had completed or teachers’ scores on national examinations showed no important relationships between teachers’ content knowledge and their students’ learning (Fennema & Franke, 1992). However, researchers who have refocused the study of teachers’ content knowledge by examining the mathematics that teachers teach found a relationship between teachers’ mathematical content knowledge and their students’ achievement gains (Hill, Rowan, & Ball, 2005).

Two types of mathematical content knowledge teachers need to be prepared to teach for understanding are common content knowledge, the mathematical knowledge teachers are responsible for developing in students, and specialized content knowledge, the mathematical knowledge that is used in teaching but not directly taught to students (Ball, Hill, & Bass, 2005). An example of common content knowledge is the fact that 8 can be written as 008, and an example of specialized content knowledge is the ability to determine which of three multidigit whole-number-multiplication algorithms could be used to multiply any two whole numbers (Hill, Sleep, Lewis, & Ball, 2007).1 These two types of content knowledge are distinguished from pedagogical content knowledge, “the ways of representing and formulating the subject that make it comprehensible to others” (Shulman, 1986, p. 9), including, for example, students’ conceptions. A teacher with pedagogical content knowledge about whole number division would realize that primary-grades children asked to solve a partitive-division problem, such as “How many jelly beans would each child get if we shared 20 jelly beans equally among 4 children?”

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1Educators may disagree about what content knowledge is common and what is specialized. Furthermore, the boundary between common and specialized mathematical content knowledge changes for teachers when they learn more about mathematics and their students’ mathematical understanding. These issues, though important, are beyond the scope of this paper.
generally model the problem by dealing out the 20 among 4 groups and that the answer is the number in each group. In developing our content assessment, we set out to assess common and specialized mathematical content knowledge but not pedagogical content knowledge.

One means by which teachers develop the mathematical content knowledge they need to teach mathematics is through content courses designed for elementary school teachers; such courses are required of prospective elementary school teachers at most colleges and universities. These courses are considered so important that most institutions require more than one course, but even multiple courses cannot sufficiently prepare the prospective teachers for the depth and breadth of mathematical understanding required of teachers who focus their classes so that their students develop the five strands of mathematical proficiency as recommended by the National Research Council in *Adding it Up* (NRC, 2001), described in detail below. Teachers must continue to learn mathematics while they teach. But do they? And if so, what mathematics do teachers learn from teaching? Furthermore, does professional development support the growth of teachers’ mathematical understanding, even if the professional development is focused primarily on something other than enhancing mathematical content knowledge, such as on children’s mathematical thinking? These are the issues that we set out to address with our study.

**Research Questions**

1) Does mathematical content knowledge differ among the following four groups: K–3 teachers who have experienced sustained professional development focused on children’s mathematical thinking for (1) at least 4 years, (2) 2 years, or (3) 0 years but who are
committed to beginning sustained professional development and (4) prospective elementary school teachers prior to beginning their first mathematics course designed for elementary school teachers? If so, what is the nature of those differences? 
2) If differences are found among the groups, what is the relationship between these differences and the types of content tasks used in the assessment?

**Our Stance Toward Assessment of Mathematical Content Knowledge**

The data collection for our overall study was extensive: Participants conducted and videotaped problem-solving interviews with three students; engaged in 1-hour individual interviews with a member of our project team; and completed a variety of tasks over 2 half-days, during three-fourths of which they responded independently to paper-and-pencil and computer tasks. The eight content tasks were administered in four groups interspersed with other assessments over the two half-days. Generally, work on the content tasks was untimed and totaled 1–1.5 hours. We sequenced the tasks to minimize the effect that completing one task might have on participants’ thinking about subsequent tasks. While planning our study, we sought to strike a balance between the goals of assessing individual teachers’ knowledge and maintaining a professional stance toward the teachers, a balance that was difficult to maintain while the teachers completed the individual paper-and-pencil and computer tasks. Professional development for teachers seldom requires individual assessments, so we emphasized during our introduction to data collection that instead of trying to evaluate the teachers, we were striving to understand their perspectives. Furthermore, we developed items amenable to a range of answers so that participants could demonstrate their levels of expertise, and, thus, we did not pose multiple-choice content items. However, even with these
considerations, we were sensitive to the teachers’ experiences during the data collection. Furthermore, our desire to treat the teachers professionally together with our stance toward content knowledge led us to apply two guiding principles to the development of our content items.

Our first guiding principle was to assess content that is deemed important to the K–3 teachers in our study. In 2007, the National Council of Teachers of Mathematics (NCTM) published the *Curriculum Focal Points* as a resource for teachers to “enable students to learn the content in the context of a focused and cohesive curriculum that implements problem solving, reasoning, and critical thinking” (p. 10). The authors highlighted three content areas of focus for each grade level with the recommendation that instruction in these content areas incorporate (a) use of mathematics to solve problems, (b) application of logical reasoning to justify procedures and solutions, and (c) use of multiple representations to support students in making connections among the ideas within and outside mathematics. Topics addressed at the K–3 levels include whole number operations for addition, subtraction, multiplication, and division; base-ten numeration system and place-value concepts; early algebra; measurement; geometry; and introductions to fractions. We addressed a subset of these areas, in particular, place-value understanding, the meanings of operations, and early algebra. We recognized that each of these topics might be assessed using multiple approaches, but we constrained our tasks to those that might arise in the work of K–3 teachers.

Our second guiding principle was to embed content in contexts that arise naturally in the work of teachers, and we generally accomplished this by situating the tasks in the context of children’s mathematical thinking. At first, one might question whether adults
can be challenged with the mathematical content of the primary grades, but our working assumption was that engaging deeply with issues of mathematics teaching and learning at any grade level is complex. For example, consider the Ones Task (Appendix A), designed to assess respondents’ abilities to “unpack” (Ma, 1999) the meaning behind the algorithms commonly used. Although virtually all prospective and practicing elementary school teachers in this country can apply the multidigit addition and subtraction algorithms standard in the United States, many struggle to explain the meanings underlying the differing representations for the regrouped 1s in these two algorithms.

We adopted as our lens for analysis the National Research Council’s (2001) definition of mathematical proficiency, which includes five interrelated strands that, together, comprise proficiency: conceptual understanding, procedural fluency, strategic competence (mathematical problem solving), adaptive reasoning (which includes informally and formally justifying one’s reasoning), and productive disposition.

**Our Content Assessment**

Algorithms are ubiquitous in school mathematics, and four of the eight content items involved algorithms. Two of the items, the Ones Task (Appendix A) and the Time Task (Appendix B), involved standard algorithms, but instead of assessing respondents’ knowledge of the standard algorithms, the tasks were designed to assess understanding of the underlying concepts. The Ones Task required respondents to recognize and explain that the regrouped 1s in the addition and subtraction algorithms represent different amounts, whereas for the Time Task, they needed to recognize that the standard multidigit-subtraction algorithm must be adapted before it can be correctly applied to calculate elapsed time. The other two algorithmic tasks, the Andrew Task (Appendix C)
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and the Doubling Task (Appendix D), required respondents to apply nonstandard algorithms with different numbers, and the Andrew Task also required respondents to provide an explanation of the reasoning a child might have used in his subtraction.

In another approach to assessing content knowledge in this study, we focused upon problem structure. The Division Task (Appendix E) provided a student’s solution to a measurement-division problem in which a total number of flowers (15) was given, a student measured out groups of (5) flowers until all the flowers were grouped, and the answer was the number of groups (3) of flowers. Respondents were asked to predict whether the student would likely apply a similar solution to three problems: a division problem with a different (partitive) structure, a multiplication problem, and a division problem with the same (measurement) structure. We consider the Division Task a content task instead of a pedagogical content task because the item fundamentally assesses whether respondents recognize problem structure, not children’s solution strategies. Similarly, although the Strategies Task (Appendix F) required providing solution strategies, the major criterion for coding responses was whether the respondents recognized that the problem involved a missing-addend, or join-change-unknown (Carpenter, Fennema, Franke, Levi, & Empson, 1999), problem structure.

The Pat Task (Appendix G) shows a child’s complex division strategy; understanding this strategy requires a flexible approach to multiplication, division, and meanings of remainders. The respondent was required to explain the child’s reasoning and to apply the child’s approach to another problem.

Finally, we developed the Decimal Task (Appendix H) as an extension task for the K–3 teachers in our study. Although the relationships among ones, tens, and
hundreds, a topic primary-grade teachers often address by using base-ten blocks, are similar to the relationships among hundredths, tenths, and ones, also addressable with the base-ten blocks, we recognized that few primary-grades teachers teach about decimal fractions. We developed this task to determine whether sustained professional development in which teachers consider place value in depth would enhance their abilities to respond to this extension task.

Content Rubrics

Each of the eight content items was assessed on a 5-point rubric (0–4 scale), with scores of 0 or 1 generally reflecting little mathematical proficiency, scores of 2 reflecting some mathematical proficiency, and scores of 3 or 4 reflecting rich mathematical proficiency. We considered the five strands of mathematical proficiency (NRC, 2001) in developing the rubric for each item, but the application of the strands was unique for each item. We describe this application for one task, the Ones Task (Appendix A, and, for the reader’s convenience, Figure 1), a task we designed to assess teachers’ and prospective teachers’ place-value understandings embedded in two commonly used algorithms. But first, we state a caveat: The five strands of mathematical proficiency (NRC, 2001) are interrelated, and although we reliably coded responses using our 0–4 rubrics, without conducting individual participant interviews, we were generally unable to determine the relative importance of each strand for a participant on a particular item. For example, consider two hypothetical respondents who approached the Ones Task differently but whose responses were both assigned the maximum score. Respondent A had not previously thought carefully about the concepts underlying the regrouped 1s in the

\footnote{For information about how the five strands of mathematical proficiency were applied to rubric development for each of the other seven content items, please contact the authors.}
addition and subtraction tasks but grappled with these ideas during the assessment, whereas Respondent B previously had thought deeply about the underlying concepts and drew upon that understanding when responding. We would consider Respondent A to have displayed, for the Ones Task, skill at approaching a novel problem (*strategic competence*) resulting in sufficient *conceptual understanding* to enable Respondent A to provide a clear explanation (*adaptive reasoning*), and we would consider Respondent A’s willingness to grapple with a challenging and novel task an indication of positive *productive disposition*. We would consider Respondent B to have displayed, for the Ones Task, primarily rich *conceptual understanding*. We could interpret the score of 4 assigned to each of these respondents only as an indication that each displayed a high level of mathematical proficiency for the Ones Task.
Below is the work of Terry, a second grader, who solved this addition problem and this subtraction problem in May.

\[
\begin{array}{c}
\text{Problem A} \\
259 \\
+ 38 \\
\hline
297
\end{array} 
\quad \quad 
\begin{array}{c}
\text{Problem B} \\
429 \\
- 34 \\
\hline
395
\end{array}
\]

Part 1

- Does the 1 in each of these problems represent the same amount? Please explain your answer.

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\text{Space provided for response}

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Part 2

- Explain why in addition (as in Problem A) the 1 is added to the 5, but in subtraction (as in Problem B) 10 is added to the 2.

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\text{Space provided for response}

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Figure 1. Ones Task.

**Conceptual Understanding Applied to the Ones Task**

Conceptual understanding is defined as an integrated and functional grasp of mathematical ideas (NRC, 2001). Conceptual understanding of place-value concepts is required for respondents to identify the differences between the 1s’ values in Part 1 and to explain these differences in Parts 1 and 2. Specifically, for the subtraction algorithm, one needs conceptual understanding to explain the regrouping of 100 from 400 as 10 tens, to combine with the 2 tens in the tens’ column in 429 to make 12 tens, from which 3 tens might be subtracted. (Note, another way to conceptualize the regrouped 1 hundred is to
view the 12 as 120 ones from which 30 ones might be subtracted [Thanheiser, 2005], but we had no such responses.) For the addition algorithm, one demonstrates conceptual understanding in explaining that the regrouped 1 is one 10, which is added to the tens in the tens’ column.

**Strategic Competence Applied to the Ones Task**

*Strategic competence* is defined as the ability to formulate, represent, and solve mathematical problems (NRC, 2001). *Problem solving* has been defined as “what you do when you want to achieve a goal (solve a problem) and you don't have ready access to a reasonably certain path to achieving it (solving it)” (Alan Schoenfeld, personal communication, 2/26/08), or, more simply, as what you do when you don’t know what to do. In coding responses to the Ones Task, we wondered to what extent respondents had previously considered the meanings of the 1s in each of these algorithms, but, as noted in our caveat, we were generally unable to determine from a written response the extent to which the Ones Task was novel for each respondent.

**Procedural Fluency Applied to the Ones Task**

*Procedural fluency* is defined as knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently (NRC, 2001). Because we assumed that all the participants in our study possessed knowledge of when and how to apply the multidigit-whole-number addition and subtraction algorithms, we did not ask them to perform these algorithms, and procedural fluency was not reflected in our scoring rubric for this task.
Adaptive Reasoning Applied to the Ones Task

Adaptive reasoning is defined as the capacity to think logically about the relationships among concepts and situations, including the ability to justify one’s reasoning both formally and informally. To score well on the Ones Task, participants must reason correctly about the meanings of the 1s in both algorithms and must provide a clear explanation of their reasoning, including the multiple meanings of the 1s in terms of reference units. Note that although use of particular terminology was not included in the rubric, clarity of response was included. (Although terminology was generally not a consideration in responses to Ones Task, technical terminology was used in responses to some items. For example, partitive and measurement were included in some responses to the Division Task. Coders did not consider whether respondents used these terms when assigning scores to responses.)

Productive Disposition Applied to the Ones Task

Productive disposition is defined as the tendency to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner and doer of mathematics (NCR, 2001). Because we did not interview respondents, we were unable to code for productive disposition in untimed responses.

Rubric Examples for the Ones Task

Table 1 provides three responses and the rubric score on our 0–4 continuum for the Ones Task (shown in Figure 1). The response scored 0 (#2156) provides evidence only of reasoning instrumentally (Skemp, 1978), that is, applying a procedure without any conceptual understanding. The crossed-out text in Part 1 was crossed out on 2156’s
response form and was interpreted as the respondent’s revising Part 1 to parrot the prompt in Part 2. The response scored 2 (#6362) evinced conceptual understanding and adaptive reasoning on Part 1, but because #6362’s response to Part 2 did not address the question posed, the scores reflected a lack of sufficient conceptual understanding and adaptive reasoning. Note, too, that #6362’s response “Gee I never thought but no” provided an indication that the respondent had not thought about the meaning of the 1s before and, hence, was one of the few responses that provided evidence for strategic competence and for positive productive disposition. The response scored 4 (#5346) evinced conceptual understanding and adaptive reasoning on Parts 1 and 2.

Table 1
Three Responses and Rubric Scores (on a 0–4 Scale) for the Ones Task

<table>
<thead>
<tr>
<th>Rubric score/Participant ID#</th>
<th>Part 1 response</th>
<th>Part 2 response</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 (Respondent 2156)</td>
<td>Not at all. In the a I guess so, they are both adding ten to the second number. After that what you do or how you got the one are the opposite. In Problem a you adding one. In problem B you are adding 10.</td>
<td>Because in A you need you need to carry the 1 in 17 over. But in B you are taking the 1 from the 4.</td>
</tr>
<tr>
<td>2 (Respondent 6362)</td>
<td>Gee I never thought but no. In the first problem you are making one ten from the 17 and carring [sic] it over to the tens. in [sic] the second problem you are taking the hundred and turning it into 10 tens.</td>
<td>Because you are making different values of the base ten system.</td>
</tr>
<tr>
<td>4 (Respondent 5346)</td>
<td>No, in the addition problem it represents a ten because 9 + 8 = 17, but in the sub. problem it represents 100 or 10 tens.</td>
<td>In the addition problem it is a ten from the 17 so we add one more ten to the 50 + 30. In the subtraction problem it’s not 1 more ten it’s 10 tens so we are taking 3 tens away from 12 tens.</td>
</tr>
</tbody>
</table>
Methodology

We used a cross-sectional design to investigate the content knowledge of three groups of practicing K–3 teachers and a group of prospective teachers who were just beginning their studies to become elementary school teachers (see Table 2). In our study, we decided to group teachers according to the length of their engagement in sustained professional development focused on children’s mathematical thinking instead of to their number of years of teaching. Each group of practicing teachers had a mean of 15–16 years of teaching experience, and the ranges in the groups were from 4–5 years of experience to 30–33 years. (We chose to exclude practicing teachers with less than 4 years of teaching experience to avoid the more general issues involved with learning to teach.) We recognize that many important aspects of teaching improve with experience, but we argue that teaching experience alone is generally insufficient for teachers to develop the type of expertise necessary to teach mathematics by responding on the basis of children's mathematical thinking. Instead, we believe that to develop such expertise, teachers need long-term support in the form of sustained professional development focused on children's mathematical thinking.
Table 2  
*Participant Groups*

<table>
<thead>
<tr>
<th>Participant group</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prospective Teachers (PSTs) ((n = 36))</td>
<td>Undergraduates enrolled in a first mathematics content course for prospective elementary school teachers</td>
</tr>
<tr>
<td>Practicing Teachers</td>
<td></td>
</tr>
<tr>
<td>Initial Participants (IPs) ((n = 31))</td>
<td>Practicing elementary school teachers who were about to begin sustained professional development focused on children’s mathematical thinking</td>
</tr>
<tr>
<td>Advancing Participants (APs) ((n = 31))</td>
<td>Practicing elementary school teachers who had been engaged with sustained professional development focused on children’s mathematical thinking for 2 years</td>
</tr>
<tr>
<td>Emerging Teacher Leaders (ETLs) ((n = 33))</td>
<td>Practicing elementary school teachers who had been engaged with sustained professional development focused on children’s mathematical thinking for at least 4 years and who were beginning to engage in formal or informal leadership activities that supported the development of other teachers (e.g., inviting teachers into their classrooms, sharing resources, etc.)</td>
</tr>
</tbody>
</table>

Practicing teachers were drawn from three districts with similar demographics in Southern California. Nearly all the professional development was facilitated by a single individual and drew heavily from the research and professional development project, Cognitively Guided Instruction [CGI] (Carpenter, Fennema, Franke, Levi, & Empson, 1999; Carpenter, Franke, & Levi, 2003). Participation was voluntary and consisted of approximately 5 full-days of workshops per year. Between meetings, teachers were expected to try specific mathematical tasks with their students. The overarching goals of the professional development were to help teachers learn (a) how children think about and develop understandings in particular mathematical domains and (b) how teachers can elicit and respond to children’s ideas in ways that support those understandings. Note
that teachers’ learning content knowledge was not one of the primary goals of the professional development.

**Data Analysis**

Participants completed the content assessment as part of 2 half-days of data collection. The responses were blinded so that coders could not determine a participant’s group. Each content item was scored on a 5-point rubric (0 [low]–4 [high]), and each response was coded by two coders. We achieved, on average, well greater than 80% interrater agreement in coding the content items; disagreements were resolved through discussion.

Because we strove to develop coding rubrics to yield scores in proportion to the understanding reflected in the responses, the content data were treated as interval data. We analyzed scores for the eight content items individually and calculated an overall score as a weighted-total score for the eight content items. For each of the eight content items, we made 6 pairwise comparisons among the four participant groups and a seventh comparison of teachers who had yet to begin the professional development (IPs) with teachers who had engaged in sustained professional development (APs and ETLs).

Planned contrasts were employed for each of the eight items and for the weighted total score.

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3 Five of the eight content items were assigned a weight of 2. The Ones Task was assigned a weight of 3, and the Doubling Task and the Strategies Task were each assigned a weight of 1. The Ones Task was weighted higher because it was considered to be an item that focused most directly upon content knowledge. The Doubling Task was weighted lower because we did not ask for an explanation, and as such, we came to realize after the study began that applying the strands of mathematical proficiency to responses that did not include explanations would be more challenging. The Strategies Task was weighted lower because although the structure of the problem is a content task, a major focus of the task was on children’s mathematical reasoning, knowledge of which measures pedagogical content knowledge. Note that the results of analyzing the overall score with the weighted average are the same as the results of analyzing without the weighted average.
score in the 6 pairwise comparisons among the four participant groups and the seventh pooled contrast (IPs vs. APs & ETLs) with the conservative Holm’s procedure used to maintain the Type I error rate at the .05 level. Because for some comparisons we could not justify a prediction in either direction, we used two-tailed tests in analyzing all comparisons.

Results and Discussion

We highlight three major results from our study of content knowledge. First, teachers learn mathematics through engaging in professional development, even if developing mathematical content knowledge is not the primary goal for the professional development. Second, prospective elementary school teachers approach standard algorithms with an orientation toward calculating without conceptualizing. Third, the choice of the content item is important for revealing differences among teachers and prospective teachers’ mathematical content knowledge. Each of these three findings is discussed below.

Finding 1: Effects of Professional Development on Teachers’ Mathematical Content Knowledge

Figure 2 lists the number of times each participant group’s mean score on a content item was significantly higher than the mean score of another participant group. For example, the 3 in Table 3 shows that the Advancing Participants (APs) scored significantly higher than the Preservice Teachers (PSTs) on three of the eight content items. Note that of the 20 significant differences found on the pairwise content comparisons, the Emerging Teacher Leaders (ETLs) scored significantly higher than
other participant groups 13 times, the Advancing Participants scored significantly higher than other participant groups 5 times, and the Initial Participants scored significantly higher than other participant groups 2 times. Furthermore, whereas the ETLs never scored significantly lower than the other groups, the APs did so 2 times, the IPs 7 times, and the PSTs 11 times. On the basis of these results, we conclude that sustained professional development supported the growth of teachers’ mathematical content knowledge.

<table>
<thead>
<tr>
<th>Group</th>
<th>PST</th>
<th>IP</th>
<th>AP</th>
<th>ETL</th>
</tr>
</thead>
<tbody>
<tr>
<td>PST</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>11</td>
</tr>
<tr>
<td>IP</td>
<td>2</td>
<td>5</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>AP</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>ETL</td>
<td>0</td>
<td></td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>

Total number of comparisons on which a group scored HIGHER than some other group

<table>
<thead>
<tr>
<th>PST</th>
<th>IP</th>
<th>AP</th>
<th>ETL</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>5</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>20</td>
</tr>
</tbody>
</table>

Figure 2. Numbers of times each participant group’s average score was significantly higher than another group’s score and total numbers of comparisons for which each group’s mean score was significantly higher or lower than some other group’s.

All significant differences between participant groups were found on five of the eight items, with no significant differences on the Decimal Task, the Doubling Task, or the Pat Task. (These three tasks will be discussed later.) Table 3 presents the significant results of comparisons by task by participant-group.
We pooled the two participant groups who had engaged in sustained professional development (ETLs and APs) and conducted two-tailed contrasts comparing them to the teachers who had yet to begin sustained professional development (IPs); the results of these comparisons are listed in the last column of Table 3. On four content items, the teachers who had engaged in sustained professional development scored significantly higher than the practicing teachers who had yet to begin. Furthermore, we computed a weighted total of the eight content-item scores (see Footnote #3), and the teachers who had engaged in sustained professional development outperformed the teachers who had not.

These findings provide evidence for the role that sustained professional development plays in the development of teachers’ content knowledge. The three groups of teachers differed in the amount of sustained professional development to which they had been exposed, but they did not differ in the number of years of teaching experience. Thus, teaching experience, alone, does not account for the differences in the teachers’
content knowledge; engagement in sustained professional development provided the teachers with opportunities to grapple with issues of mathematics content, leading to increased mathematical proficiency. The next finding, focused on the prospective elementary school teachers, also highlights the fact that, in general, teaching experience alone did not lead to increased mathematical content knowledge.

**Finding 2: Prospective Elementary School Teachers’ Approaches to Algorithms**

No significant difference was found between the weighted-total scores of the prospective elementary school teachers (PSTs) and the experienced teachers who had yet to begin sustained professional development (IPs) (see Table 3). The mean scores of the PSTs and IPs on the six content tasks for which there were no significant differences between them (Tasks 1, 2, 3, 4, 5, and 7 [see Table 4]) differed little, and on two of the tasks, the Division Task and the Andrew Task, the PSTs’ mean score was higher than the IPs’. Evidently, teaching experience alone had little effect on the development of mathematical content knowledge for most tasks. On two tasks, however, teaching experience supported development of mathematical understanding: the Strategies Task and the Ones Task. The Strategies Task assessed respondents’ knowledge about students’ strategies and the link between strategies and the structure of a missing-addend problem, also referred to as a join change-unknown problem (Carpenter et al., 1999), and the PSTs’ lack of classroom experience affected their understanding. But we were most interested in the PST result on the Ones Task, on which the PSTs’ mean score of 0.31 was by far the lowest of any participant group on any item, with 83.3% (30/36) of the PSTs scoring 0. Although PSTs’ struggles in understanding standard multidigit-addition and -subtraction algorithms have been previously documented (Thanheiser, 2005), when
this result is situated in the context of our study, two noteworthy aspects arise. First, the PSTs’ performance on the Ones Task was far worse than their performance on any of the other three algorithmic-related tasks (Time Task, Andrew Task, and Doubling Task). Second, whereas the Ones Task was by the far the most difficult task for the PSTs, it was not the most difficult task for any of the other three participant groups; in fact, the mean score on the Ones Task for each of the other three groups was about the group’s average score for all the tasks. Why was the Ones Task so much more difficult than any other task for the PSTs? It was the only task the participants could approach in a purely instrumental (Skemp, 1978) way, that is, by focusing solely on the procedure. Because of the past experiences of the prospective teachers, we hypothesize that they responded procedurally, instead of conceptually, to what the 1s represent. We suspect that the PSTs approached each of the other three algorithmic tasks differently than the Ones Task because the Andrew Task and the Doubling Tasks were nonstandard algorithms, not familiar to the respondents, and the Time Task, though based on a standard algorithm, was one that respondents were shown, after their initial response, was not applicable in the context to which it was applied. Note that the IPs performed significantly better than the PSTs on the Ones Task. We conclude that although teaching experience alone did not seem to favor teachers’ content knowledge over PSTs’, even teachers without opportunities to engage in sustained professional development recognize that the algorithms they teach are often not understood by students, and consequently, they recognize that approaching algorithms solely in a procedural way will not support students. We also find noteworthy that the teachers who engaged in sustained professional development performed significantly better than the IPs on the Ones Task.
Evidently sustained professional development provided support for developing rich conceptual understanding of the algorithms, even though algorithms may not be a major focus in the participants’ instruction.

Table 4
Means and Standard Deviations for Each Participant Group on Each Content Task (Scale of 0–4) and on the Overall Weighted Average (Scale of 0–60)

<table>
<thead>
<tr>
<th></th>
<th>PST (n = 36)</th>
<th>IP (n = 31)</th>
<th>AP (n = 31)</th>
<th>ETL (n = 33)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Task 1 (Division)</td>
<td>1.64 (1.33)</td>
<td>1.26 (1.03)</td>
<td>1.87 (1.26)</td>
<td>2.70 (1.24)</td>
</tr>
<tr>
<td>Task 2 (Decimal)</td>
<td>2.06 (1.57)</td>
<td>2.19 (1.52)</td>
<td>2.19 (1.54)</td>
<td>2.30 (1.38)</td>
</tr>
<tr>
<td>Task 3 (Time)</td>
<td>1.86 (1.05)</td>
<td>2.32 (.95)</td>
<td>2.13 (1.09)</td>
<td>2.64 (1.03)</td>
</tr>
<tr>
<td>Task 4 (Doubling)</td>
<td>2.47 (1.72)</td>
<td>2.77 (1.61)</td>
<td>3.16 (1.42)</td>
<td>2.97 (1.53)</td>
</tr>
<tr>
<td>Task 5 (Andrew)</td>
<td>1.67 (1.33)</td>
<td>1.48 (1.29)</td>
<td>2.16 (1.13)</td>
<td>2.55 (1.09)</td>
</tr>
<tr>
<td>Task 6 (Ones)</td>
<td>0.31 (.79)</td>
<td>1.58 (1.26)</td>
<td>2.29 (1.22)</td>
<td>2.67 (1.08)</td>
</tr>
<tr>
<td>Task 7 (Pat)</td>
<td>1.14 (1.46)</td>
<td>1.26 (1.37)</td>
<td>1.48 (1.44)</td>
<td>1.79 (1.62)</td>
</tr>
<tr>
<td>Task 8 (Strategies)</td>
<td>0.89 (82)</td>
<td>1.74 (1.06)</td>
<td>2.45 (1.23)</td>
<td>3.09 (1.26)</td>
</tr>
<tr>
<td>Overall Weighted Average</td>
<td>21.00 (12.67)</td>
<td>26.29 (11.75)</td>
<td>32.16 (10.52)</td>
<td>38.00 (11.37)</td>
</tr>
</tbody>
</table>

Finding 3: Content-Item Choice in Revealing Differences Among Teachers’ and Prospective Teachers’ Mathematical Content Knowledge

This study highlights the relationship between the items used to assess teachers’ mathematical content knowledge and the results found. Although we developed the eight content items specifically for use by K–3 elementary school teachers, no significant group differences were found on the Decimal Task, the Doubling Task, or the Pat Task (Table 4 contains the t-statistic and p-value for each of the contrasts by content item.) The Pat Task was difficult, and although the mean scores increase for the four participant groups when we look across PSTs, IPs, APs, and ETLs, these mean differences were not
significant. The Pat Task was by the far the most difficult item for the ETLs and the APs; this task and the Division Task were equally difficult for the IPs. We think that this task was sufficiently difficult so as to present a kind of floor effect, resulting in no significant differences. The Doubling Task was the easiest task for three of the participant groups and was the second easiest for the ETLs, and we think that the task presented a kind of ceiling effect. The Doubling Task primarily assessed respondents’ abilities to apply a nonstandard approach to another set of numbers, and because we did not ask respondents to explain the mathematics underlying the approach, we think the task did not sufficiently discriminate among respondents’ differing levels of mathematical proficiency as applied to this task. The Decimal Task was designed as an extension task, that is, it included mathematical content related to primary-grades mathematics but generally not addressed until upper elementary school. The Decimals Task had two parts. In the first part, we presented two base-ten-blocks representations for decimals, one using the blocks proportionally and the other using them nonproportionally, and we asked respondents which approach made sense and why. In the second part, we asked the respondents to draw pictures indicating two ways to represent 5.24 using base-ten blocks. We considered the Decimal Task to be an extension because although teachers generally do not address decimal numbers in primary grades, they do address place value and often address multiple representations showing place value by using base-ten blocks. Results for this task indicated no statistically significant differences (see Table 3), and the means were virtually identical among the groups: mean scores of the ETLs, APs, IPs, and PSTs were 2.30, 2.19, 2.19, and 2.06,
respectively. We conclude from results of these three items that the choice of items is important for detecting content-knowledge differences among groups of teachers.

Table 5
*t-Statistic and p-Value for Each Planned Contrast, by Task*

<table>
<thead>
<tr>
<th>Contrast</th>
<th>PST vs. IP</th>
<th>PST vs. AP</th>
<th>PST vs. ETL</th>
<th>IP vs. AP</th>
<th>IP vs. ETL</th>
<th>AP vs. ETL</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Task 1 (Division)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(t)</td>
<td>1.268</td>
<td>-0.773</td>
<td>-3.583</td>
<td>-1.969</td>
<td>-4.695</td>
<td>-2.695</td>
</tr>
<tr>
<td>(p) value</td>
<td>.207</td>
<td>.441</td>
<td>.000*</td>
<td>.051</td>
<td>.000*</td>
<td>.008*</td>
</tr>
<tr>
<td><strong>Task 2 (Decimal)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(t)</td>
<td>-0.375</td>
<td>-0.375</td>
<td>-0.684</td>
<td>0.000</td>
<td>-0.291</td>
<td>-0.291</td>
</tr>
<tr>
<td>(p) value</td>
<td>.708</td>
<td>.708</td>
<td>.495</td>
<td>1.000</td>
<td>.771</td>
<td>.771</td>
</tr>
<tr>
<td><strong>Task 3 (Time)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(t)</td>
<td>-1.832</td>
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<td>-3.129</td>
<td>0.741</td>
<td>-1.220</td>
<td>-1.973</td>
</tr>
<tr>
<td>(p) value</td>
<td>.069</td>
<td>.289</td>
<td>.002*</td>
<td>.460</td>
<td>.225</td>
<td>.051</td>
</tr>
<tr>
<td><strong>Task 4 (Doubling)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(t)</td>
<td>-0.782</td>
<td>-1.784</td>
<td>-1.310</td>
<td>-0.967</td>
<td>-0.496</td>
<td>0.486</td>
</tr>
<tr>
<td>(p) value</td>
<td>.436</td>
<td>.077</td>
<td>.193</td>
<td>.335</td>
<td>.621</td>
<td>.628</td>
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<tr>
<td><strong>Task 5 (Andrew)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(t)</td>
<td>0.613</td>
<td>-1.659</td>
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<td>-2.191</td>
<td>-3.487</td>
<td>-1.262</td>
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<td>(p) value</td>
<td>.541</td>
<td>.100</td>
<td>.003*</td>
<td>.030</td>
<td>.001*</td>
<td>.209</td>
</tr>
<tr>
<td><strong>Task 6 (Ones)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(t)</td>
<td>-4.774</td>
<td>-7.431</td>
<td>-8.988</td>
<td>-2.563</td>
<td>-3.983</td>
<td>-1.380</td>
</tr>
<tr>
<td>(p) value</td>
<td>.000*</td>
<td>.000*</td>
<td>.000*</td>
<td>.012*</td>
<td>.000*</td>
<td>.170</td>
</tr>
<tr>
<td><strong>Task 7 (Pat)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(t)</td>
<td>-0.330</td>
<td>-0.956</td>
<td>-1.828</td>
<td>-0.604</td>
<td>-1.438</td>
<td>-0.825</td>
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<tr>
<td>(p) value</td>
<td>.742</td>
<td>.341</td>
<td>.070</td>
<td>.547</td>
<td>.153</td>
<td>.411</td>
</tr>
<tr>
<td><strong>Task 8 (Strategies)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(t)</td>
<td>-3.163</td>
<td>-5.794</td>
<td>-8.300</td>
<td>-2.538</td>
<td>-4.899</td>
<td>-2.322</td>
</tr>
<tr>
<td>(p) value</td>
<td>.002*</td>
<td>.000*</td>
<td>.000*</td>
<td>.012*</td>
<td>.000*</td>
<td>.022*</td>
</tr>
<tr>
<td><strong>Overall Weighted Average</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(t)</td>
<td>-1.855</td>
<td>-3.913</td>
<td>-6.060</td>
<td>-1.986</td>
<td>-4.022</td>
<td>-2.005</td>
</tr>
<tr>
<td>(p) value</td>
<td>.066</td>
<td>.000*</td>
<td>.000*</td>
<td>.049</td>
<td>.000*</td>
<td>.047</td>
</tr>
</tbody>
</table>

*p value significant after nominal alpha level is adjusted using Holm’s procedure.
Whether we consider the number of times a participant group’s mean scores were significantly higher (and lower) than other groups (see Figure 2), the overall means (see Table 4), or the specific pairwise differences that were significant (see Table 3 or Table 5), the Emerging Teacher Leaders (ETLs) clearly performed best among the four participant groups on the content assessment, and the Advancing Participants (APs) performed better than the Initial Participants (IPs) and the Preservice Elementary School Teachers (PSTs). We believe that results from three tasks together, the Ones Task, the Division Task, and the Strategies Task, tell a story of content-knowledge development. Significant differences were found between the IPs and APs but not between the APs and ETLs on the Ones Task; significant differences were found between the APs and ETLs but not between the IPs and APs on the Division Task; and significant differences were found both between the IPs and APs, and again between the APs and ETLs on the Strategies Task. We conclude from these results that teachers engaged in sustained professional development developed an understanding of the underlying meaning of algorithms within the first two years but that developing a deeper sense of the differences among the meanings of division required additional time. Attention to children’s strategies and how these strategies relate to problem structure appear to develop uniformly throughout sustained professional development. The Strategies Task is the item closest to pedagogical content knowledge (PCK), and although we draw no conclusions on the basis of one task, we find noteworthy that, at least for this task, development of PCK appeared to be constant throughout the years of sustained professional development.
Conclusions

We are engaged in a large-scale study of the effects of professional development on the mathematical content knowledge, beliefs, practices, and professional noticing of teachers and prospective teachers. The content results show that sustained professional development, even if not focused directly on mathematical content knowledge, positively affects teachers’ content knowledge. Furthermore, although changes in teachers’ content knowledge took place during the first 2 years of professional development, additional professional development led to further change. An in-depth understanding of the mathematics of even primary grades is sufficiently complex that it continues to develop over several years of sustained professional development. Furthermore, these are changes that had not occurred for most teachers before engaging in professional development, even though they had taught for many years.

Another finding is that differences are not discernible from all the tasks. In particular, the tasks that fall under the purview of teachers are the tasks that showed gains from professional development. Furthermore, the most difficult tasks for some, such as the Ones Task, showed some of the greatest professional development effects. An important implication for assessing teachers’ content knowledge is that not all items, even those assessing elementary school content, effectively identify differences among teachers. Using items based on content generally addressed at grade levels beyond those the teachers teach may produce results different from those emerging by using challenging mathematical tasks relevant to the teachers’ work with students. The more carefully we select items to address content of the teachers, by the teachers, and for the
teachers, the more likely we are to detect important teacher differences that might lead to enhanced learning by all students.

This study also has implications for those teaching mathematics to prospective elementary school teachers. Algorithms are ubiquitous in elementary school, and attending to the underlying conceptual meaning of algorithms is an important topic addressed in courses for PSTs. Our study shows that instead of asking PSTs to consider the meaning underlying standard algorithms, we may have more success by starting with tasks that are clearly problematic for PSTs, such as considering standard algorithms applied in inappropriate contexts or nonstandard algorithms.
References


Philipp et. al., AERA, 2008, Content Knowledge


APPENDIX A: Ones Task

Below is the work of Terry, a second grader, who solved this addition problem and this subtraction problem in May.

Problem A

\[
\begin{array}{c}
  2 \text{5} \text{9} \\
+ 3 \text{8} \\
\hline
2 \text{9} \text{7}
\end{array}
\]

Problem B

\[
\begin{array}{c}
  3 \text{1} \text{2} \text{9} \\
- 3 \text{4} \\
\hline
3 \text{9} \text{5}
\end{array}
\]

- Does the 1 in each of these problems represent the same amount? Please explain your answer.

- Explain why in addition (as in Problem A) the 1 is added to the 5, but in subtraction (as in Problem B) 10 is added to the 2.
APPENDIX B: Time Task

(This web-based task was administered so that respondents submit the response to Part 1 before responding to Part 2, and they submit the response to Part 2 before responding to Part 3).

Last month the children in a third-grade classroom were given the following problem:

You were on a train that left Los Angeles at 2:54 p.m. and arrived in Phoenix at 7:12 p.m. How long were you on the train?

Matt solved the problem as follows:

```
  6 1
  4 9
  2 5 4
  1 0 1 2
  __________________
    4 5 8
```

4 hours and 58 minutes

Matt explained,
"I couldn't subtract 4 from 2, so I borrowed 10 to make 12; 4 subtracted from 12 is 8. Then I couldn't take the 5 from the 0, so I borrowed 1 from the 7 and put it by the 0. Then I subtracted 5 from 10; that's 5. Then I subtracted 2 from 6, and that's 4. The answer is 4 hours and 58 minutes."

Time Task (Part 1 of 3)
1. What do you think about Matt's solution?

Time Task (Part 2 of 3)
Laverna, one of Matt's classmates, used a different method to solve the same problem that Matt had solved by subtracting. She "counted up" the elapsed time from 2:54 p.m. to 7:12 p.m.

2. If Laverna answered correctly, what was her answer?
Time Task (Part 3 of 3)

3. The class discussed Matt's subtraction solution and LaVerna's counting-up solution. They saw that Matt's answer was 40 minutes too large. Can you explain why Matt's procedure does not make mathematical sense?

4. If Matt uses his same procedure on another travel-time problem that also requires regrouping from the hours, will his answer again be 40 minutes too large?

5. Why or why not?

Jorge revised Matt’s subtraction procedure and used it to calculate the elapsed time from 4:55 p.m. to 8:19 p.m. His answer was 3 hours and 24 minutes, the correct answer.

6. What does the 7 represent in Jorge's revised procedure?

7. What does the 6 represent in Jorge's revised procedure?

8. How many minutes are represented within the oval in Jorge's procedure, shown below?
APPENDIX C: Andrew Task

In March, Andrew, a second grader, solved $63 - 25 = \square$ as shown below.

\[
\begin{array}{c}
63 \\
- 25 \\
\hline \\
40 \\
\hline \\
38
\end{array}
\]

- Explain why Andrew's strategy makes mathematical sense.

- Please solve $432 - 162 = \square$ by applying Andrew's reasoning.

\[
\begin{array}{c}
432 \\
- 162 \\
\hline
\end{array}
\]
APPENDIX D: Doubling Task

In November, Todd, a third grader, solved the following problem as shown below:

The teacher bought 18 boxes of stickers with 27 stickers in each box. How many new stickers does she have?

- Please solve $33 \times 19 = \square$ by using Todd’s reasoning.
APPENDIX E: Division Task
(Sufficient space was provided for full written responses.)

The teacher needs to put 15 flowers in vases. Each vase can hold 5 flowers. How many vases does she need?

Russ, a first grader, solved this problem in February. To solve the problem, he counted out 15 linking cubes. He pulled out a group of 5 cubes and then another group of 5 cubes and then another group of 5 cubes, so all the cubes were gone. Then he counted the number of groups, "1, 2, 3," and said that 3 vases were needed. (Below is his record of his work.)

![Record of work](image)

Please think about the following problems and whether Russ is likely to use similar reasoning when solving them.

Part 1 There are 20 children on the playground. The teacher wants to play a game with 4 teams. How many children will be on each team if each team has the same number of children?

• To solve this problem, is Russ likely to use reasoning similar to the reasoning he used to solve the vases problem? □ yes □ no Why or why not?

• If you answered no, describe one strategy a young child would be likely to use to solve this problem.
Part 2  On Monday, 10 children went to the library and each checked out 3 new books. How many new books did they bring back to class altogether?

- To solve this problem, is Russ likely to use reasoning similar to the reasoning he used to solve the vases problem?  
  
  - yes  
  - no  
  Why or why not?

- If you answered no, describe one strategy a young child would be likely use to solve this problem.

Part 3  A class of 18 children is going to the zoo. Each car has seatbelts for 6 children. How many cars will be needed to transport all the children to the zoo?

- To solve this problem, is Russ likely to use reasoning similar to the reasoning he used to solve the vases problem?  
  
  - yes  
  - no  
  Why or why not?

- If you answered no, describe one strategy a young child would be likely use to solve this problem.
APPENDIX F: Strategies Task

a) Please provide solution strategies—as many as you can—that you might expect children to use to solve the following problem:

*Pablo read 15 pages of his library book on Saturday. The book has 32 pages. How many pages will he have to read on Sunday to finish his book?*

b) Circle the strategy or strategies that you would most likely see first graders using.
APPENDIX G: Pat Task

In May, a teacher provided the following situation in her third-grade class:

*I was at a store, and I saw that chocolate kisses come in bags of 42. I wanted to share these kisses among 7 people. How many kisses would each person get?*

Following are the steps Pat told his teacher he had performed mentally to solve the problem. The teacher’s follow-up questions confirmed that Pat’s steps reflected a deep understanding of the problem situation.

\[
\begin{align*}
4 \times 10 &= 40 \\
\text{That is three 4s too many, so I have 12 left over. } \\
12 + 2 &= 14 \\
14 \div 2 &= 7 \\
4 + 2 &= 6. \text{ So } 42 \div 7 &= 6.
\end{align*}
\]

a) Please explain how each of Pat’s steps makes mathematical sense in this context.

b) Use Pat’s approach to solve 56 ÷ 8.
APPENDIX H: Decimal Task

Some children in Mrs. Kwon’s 4th-grade class have used base-ten blocks (like those pictured) to solve problems with whole numbers. They used a flat to represent 100, a rod to represent 10, and a cube to represent 1.

<table>
<thead>
<tr>
<th>Flat</th>
<th>Rod</th>
<th>Cube</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>10</td>
<td>1</td>
</tr>
</tbody>
</table>

In April, they considered assigning new values to the base-ten blocks to help them think about what decimal numbers mean. They debated two options:

1. Sonjay suggested that they use a cube to represent 1, a rod to represent $\frac{1}{10}$, and a flat to represent $\frac{1}{100}$.

<table>
<thead>
<tr>
<th>Flat</th>
<th>Rod</th>
<th>Cube</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{100}$</td>
<td>$\frac{1}{10}$</td>
<td>1</td>
</tr>
</tbody>
</table>

(2) Caleb said that they should use a flat to represent 1, a rod to represent $\frac{1}{10}$, and a cube to represent $\frac{1}{100}$.

<table>
<thead>
<tr>
<th>Flat</th>
<th>Rod</th>
<th>Cube</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{10}$</td>
<td>$\frac{1}{100}$</td>
</tr>
</tbody>
</table>

Which of these systems would you prefer? Sonjay’s Caleb’s Why do you prefer that system?
Use the three types of base-10 blocks (flats, rods, and cubes) to draw two ways to show 5.24.

What does each type of base-ten block represent in your drawings? (Remember, you need to have each type of block represent the same quantity in both drawings.)

A **flat** represents ____________ in both of my drawings below.

A **rod** represents ____________ in both of my drawings below.

A **cube** represents ____________ in both of my drawings below.

*First representation of 5.24 using base-10 blocks*

*Explanation*

*Second representation of 5.24 using base-10 blocks (with the flat, the rod, and the cube each representing the same quantity as it represented in the first way you chose to show 5.24)*

*Explanation*