Unpacking A Conceptual Lesson: The Case of Dividing Fractions

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Abstract

A conceptual lesson is presented to help teachers understand why one can invert and multiply to divide fractions. I chose the topic not because I argue for the value of teaching fractions, but because it is a topic that most teachers understand only instrumentally and because understanding the relationships among fractional quantities requires multiplicative reasoning, the cornerstone to proportional reasoning. Fractions are introduced via pattern blocks, and ways to connect the physical models to symbolic representations in combining and separating fractional quantities are explored, with special attention to common confusions between multiplication and subtraction situations. Lesson excerpts and vignettes contrasting a first grader's beginning conceptual understanding of fractions with a fifth grader's procedural view that illustrates a lack of such understanding serve to illustrate three principles for teaching for conceptual understanding of mathematics: the importance of identifying and attending to the main mathematical concepts in a topic, building on existing knowledge, and introducing symbols and procedures *after* introducing the concepts they represent.

Unpacking A Conceptual Lesson: The Case of Dividing Fractions

Why, when dividing fractions, do we invert and multiply? If you are not able to provide a clear explanation, you are not alone. Secondary and elementary preservice teachers have difficulty explaining this procedure (Ball, 1990; Borko et al., 1992) I have found that virtually none of my preservice elementary and secondary school teachers can provide a conceptually oriented explanation,[1] and after experiencing the lesson outlined below, many of them wonder why they were never provided an opportunity to make sense of fractions when they were students.

I first outline a conceptual lesson designed to help students understand division of fractions, culminating with an explanation for why, when dividing fractions, we invert and multiply. I recognize that many teachers reading this article will never teach fraction division, and many may not particularly care why the invert-and-multiply rule works. I chose this topic not for its intrinsic importance but because it is one that most teachers understand only instrumentally; that is, they know how to apply the procedure without understanding why the procedure works (Skemp, 1978). People at all levels can work with fractions, but probably few people could provide a conceptual explanation for many of the procedures associated with fractions. The topic of fractions, then, is an example of the notion that given almost any topic, everyone understands something and no one understands everything, and therefore a key to effective instruction is to find out what knowledge students possess and to build on that knowledge. Furthermore, although learning how to divide fractions may not be critical, learning about the relationships among fractional quantities (e.g., Which is larger, 1/6 or 1/8? How many times as large?) is part of multiplicative reasoning, which in turn is the cornerstone to learning to reason proportionally[2].

The lesson described here is detailed, because it is in the details that one can see the importance of focusing on the conceptual ideas of the mathematics and on the students' thinking about the mathematics. Finally, when I unpack the lesson and consider key components, I hope that readers will see implications for mathematical topics they teach.

From Pieces to Fractions

For this lesson with preservice teachers, I generally begin with pattern blocks: a hexagon; a trapezoid, equal in area to half a hexagon; a rhombus, equal in area to one third of a hexagon; and a triangle, equal in area to one sixth of a hexagon (see Figure 1). I first ask students to consider the yellow hexagon to be one whole, and I ask them to construct one whole in as many ways as they can, provided that they use the same-sized pieces for each whole (see Figure 2). We discuss the fact that because two red trapezoids make one whole, each red trapezoid is one half of a whole. Similarly, each blue rhombus is one third of a whole, and each green triangle is one sixth of a whole. We discuss several key aspects of this area model[3] of fractions at this point. First, to create fractional parts, we must partition the whole so that the pieces within each partitioning are of equal size. Second, the entire whole must be "used up." Third, the more equal-sized parts into which the whole is partitioned, the smaller each part is. Finally, we are able to connect notation to these pieces, so that, for example, each of the six green triangles can be represented as 1/6 of the whole.

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[1] Some of my secondary students attempt to explain division of fractions with an algebraic approach: They show that a/b ÷ c/d can be thought of as a complex fraction (a/b) / (c/d), and then they multiply the complex fraction by 1 in the form of (d/c) / (d/c) resulting in (a/b x d/c) / 1. I am always left with the question: "Of what is this a proof, and for whom?"

[2] By *reason proportionally,* I do not refer to ability to set up a proportion a/b = c/d and solve for the unknown. Instead I refer to ability to, for example, compare the growth of two snakes, one that grew from 4' to 7' and the second that grew from 5' to 8', by recognizing that the growth of the smaller is 3/4 of its original size whereas the growth of the larger is only 3/5 of its original size.

[3] In a set model not all parts of the whole are of equal size. In finding 1/4 of a set of 12 people, the *number* of people, not their *sizes,* is the important factor.

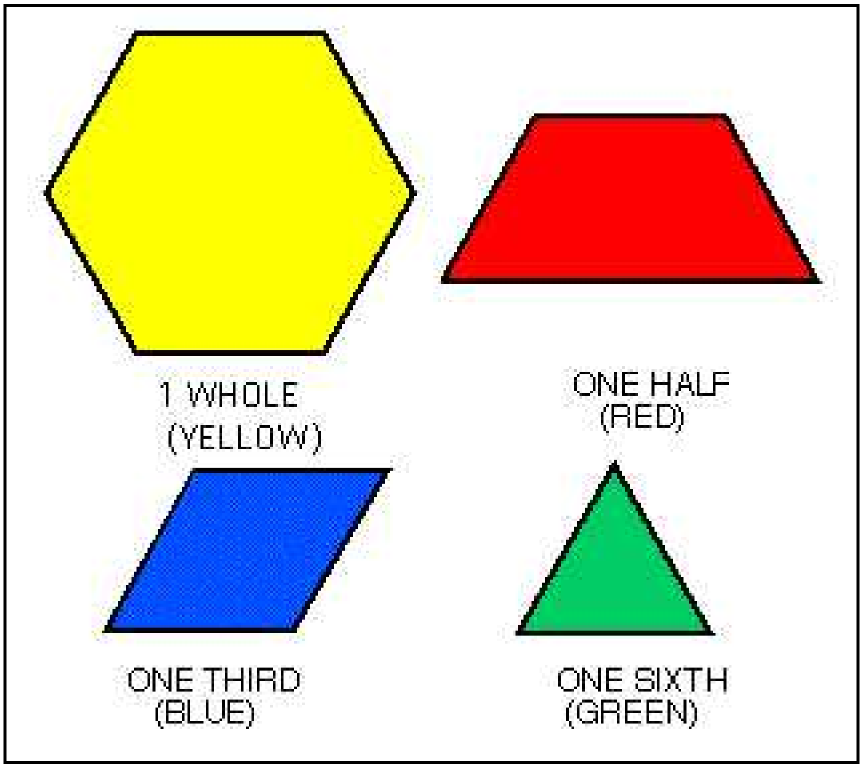

Figure 1. Pattern blocks.

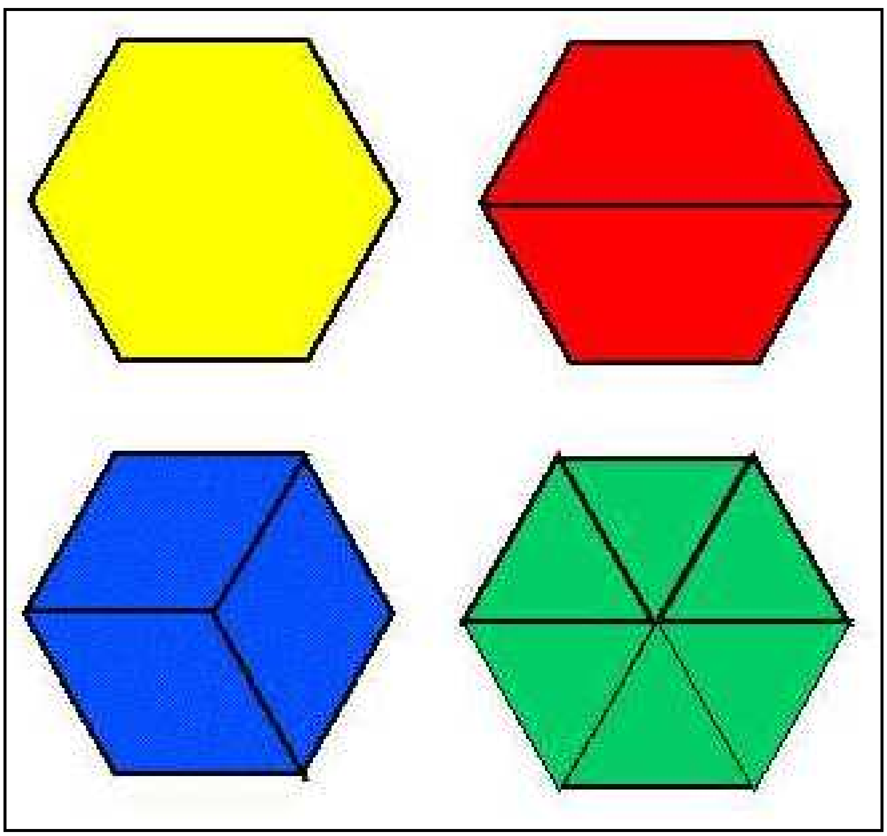

Figure 2. Making one whole with same-sized pieces.

Combining and Separating Portions

At this point students should connect the physical model to the symbolic representation (Wearne & Hiebert, 1988). That is, when students see one red trapezoid, they hear in their mind's ear "one half" and can picture in their mind's eye the symbol 1/2. After they make these connections, students are able to represent one whole without using the same-sized pieces (e.g., see Figure 3 for three common representations), and they can comfortably move among the physical, verbal, and symbolic representations. We discuss the fact that we can construct many equivalences without resorting to determining common denominators (see Figure 4).

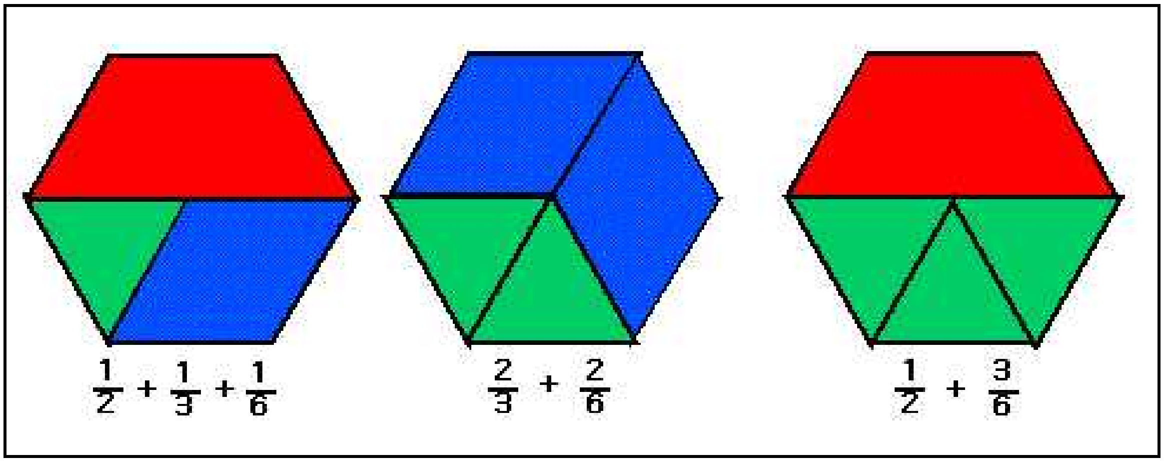

Figure 3. Representations for one whole composed of different-sized pieces.

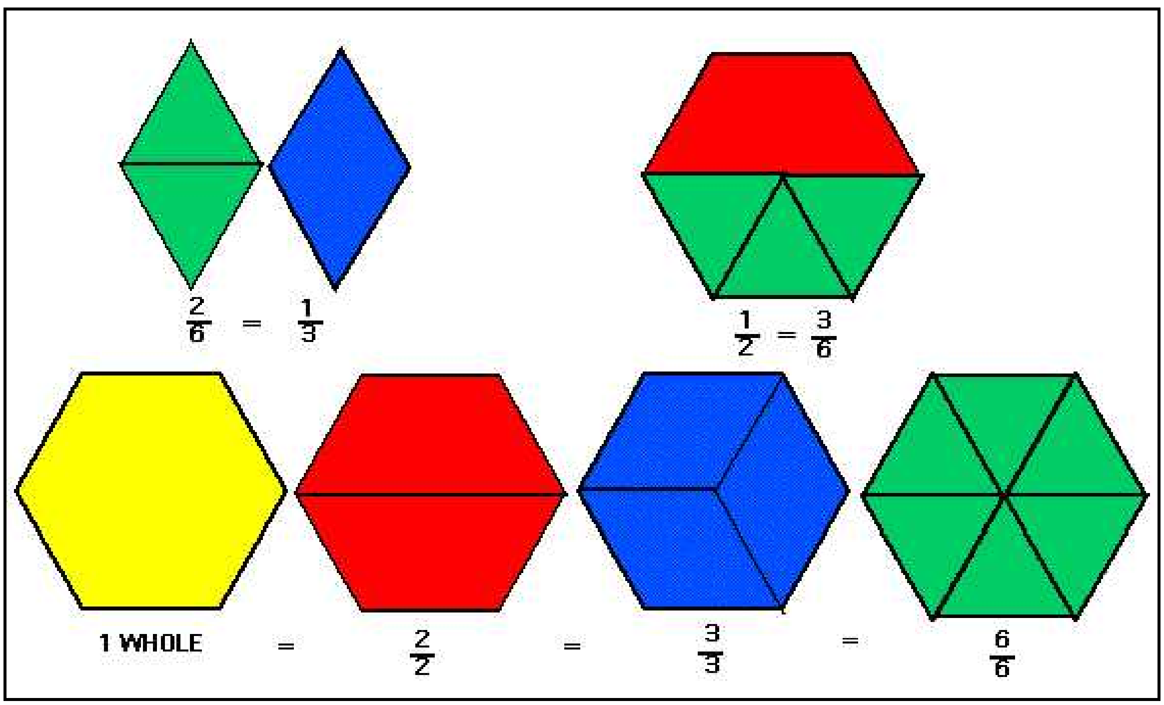

Figure 4. Equivalencies.

When students become comfortable connecting the physical model with the symbolic representation, I begin posing exercises for them. For example, I ask them to show me 1/2 + 1/3. Most students solve this by placing the half (red trapezoid) next to the third (blue rhombus, see Figure 5.1), arriving at an answer of five sixths either by noting that this amount is equivalent to five greens, or 5 sixths (see Figure 5.2), or by noting that inasmuch as the sum is one green (or 1/6) less than one whole, it equals 5/6 of a whole.

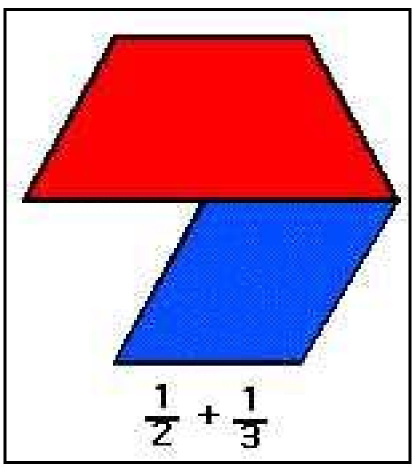

Figure 5.1. Show $\frac{1}{2} + \frac{1}{3}$.

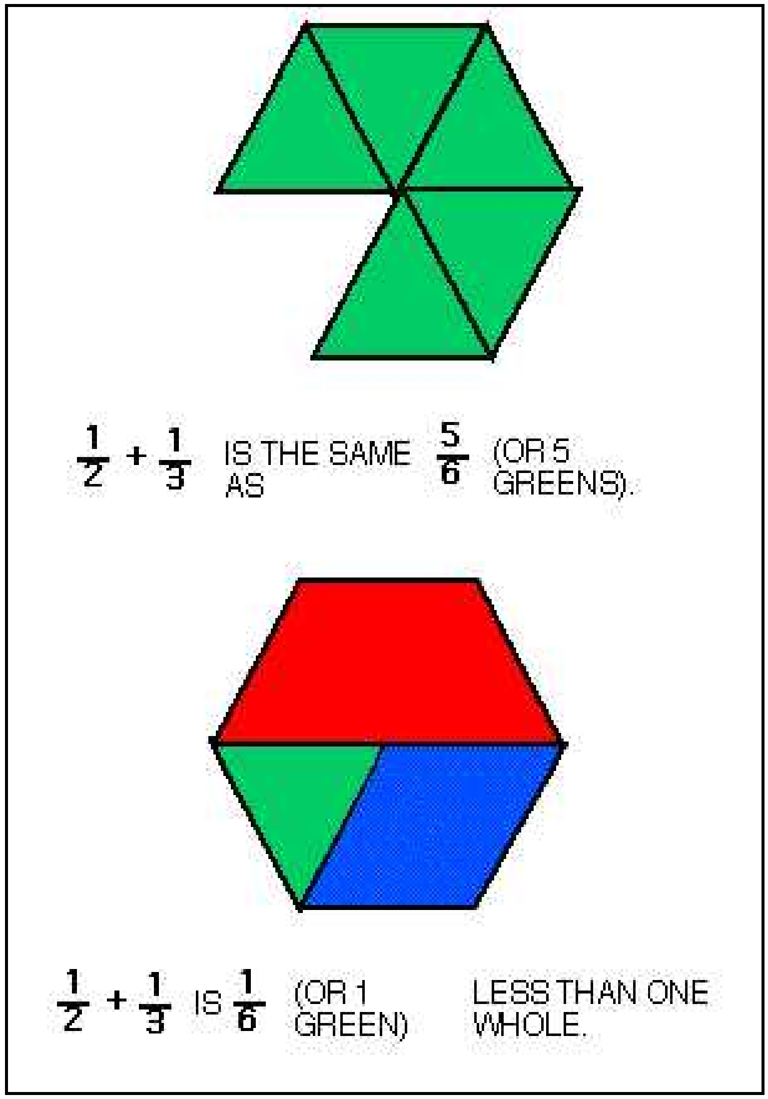

Figure 5.2. Students' strategies for adding $\frac{1}{2} + \frac{1}{3}$.

After posing another addition problem, I ask students to show me 1/2 – 1/3. Students generally solve this problem by representing 1/2 with one red trapezoid and noticing that one green (1/6) plus one blue (1/3) is equivalent to one red (1/2). Students then remove 1/3 by removing the blue rhombus, leaving one green triangle, or 1/6.

Pie, Anyone?

After the preservice teachers work on the task of finding 1/2 – 1/3 using pattern blocks, I ask them to construct a real-life story problem that might be modeled with the expression 1/2 – 1/3. I find that many preservice teachers have trouble constructing a real-life problem, so after the students have thought for a few minutes I might provide them a hint by suggesting that they consider pies. I then ask students to share the problems they have written. I most recently did this activity with elementary school teachers who experienced difficulty constructing a problem within 5 minutes, so I told them that recently someone had suggested the following problem[4]:

> Pat has one half of a pie left in the refrigerator. For lunch Pat eats one third of the pie. How much pie does Pat have left?

We have a nice discussion about this problem, because some students think that this story problem can be represented by 1/2 – 1/3, whereas others are concerned that it is not clear whether Pat eats one third of the remaining pie or one third of a whole pie. This is precisely the point I intend to raise with this problem, and we discuss the difference. Some students struggle with attending to the unit, and I find that they are helped by modeling the situation using the pattern blocks. If they represent half of a pie with one red trapezoid, then how much pie did Pat eat? Did Pat eat one third of half of a pie (one green, which is 1/6 of a whole), or did Pat eat one third of a whole pie (one blue, which is 1/3 of a whole)? They come to recognize that to be more exact, I should have written the following problem:

> Pat has one half of a pie left in the refrigerator. For lunch Pat eats one third of one (whole) pie. How much pie does Pat have left?

The discussion at this point turns to the reason the expression 1/2 – 1/3 was relatively easy to work with, whereas this story problem was confusing. We discuss the fact that, on the one hand, implicit in addition or subtraction is the fact the unit is the same for each term. That is, in the expression 1/2 – 1/3, the 1/2 and the 1/3 each refer to the same unit. On the other hand, in the first story problem I wrote, the 1/2 and the 1/3 do *not* refer to the same unit; the 1/2 refers to one half of one *whole* pie, but the 1/3 refers to one third of one *half* of a pie. I emphasize that this is the critical distinction that sets addition and subtraction apart from multiplication. We may think of 4 x 3 as representing 4 groups of 3. In this case, the 3 refers to 3 whole units, but the 4 refers to 4 groups of 3 whole units. In addition and subtraction, however, we generally do not state the unit because it is implicit that the unit is the same for all addends: 4 + 3 refers to “4 wholes plus 3 wholes of the same thing.” I sometimes tell my students that multiplication is "gossip math," whereas addition and subtraction are not, because in 4 x 3, the 4 is talking about the 3—that is, the 4 refers to the number of 3s—whereas in 4 + 3, neither number is referring to the other, but instead both refer to some inferred whole.

Next I ask the students to show me 1/3 of 2 1/2. Figure 6 shows one approach students take to this task. By representing 2 1/2 with greens, or sixths, they show the 15 sixths in 2 1/2, and one third of 15 sixths is 5 sixths. After some students explain their solutions to this task, I ask them to show me 2 1/2 of 1/3. This is very difficult for the majority of my elementary preservice teachers, and moderately difficult for some secondary preservice teachers. The most popular solution provided for this problem involves noticing that 2 1/2 thirds is two thirds plus half of a third (See Figure 7.)

When discussing these two tasks, we note that although 1/3 x 2 1/2 and 2 1/2 x 1/3 are equal by the commutative property of multiplication, people generally think of “1/3 of 2 1/2” very differently from the way they think of “2 1/2 thirds” We conclude that commutativity of conceptual situations does not even exist.

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[4] The following story problem is mathematically equivalent to the “subtraction” problems written by most of my elementary and many of my secondary preservice teachers. Some of my students continue to write this type of problem, even after spending several lessons discussing fractions.

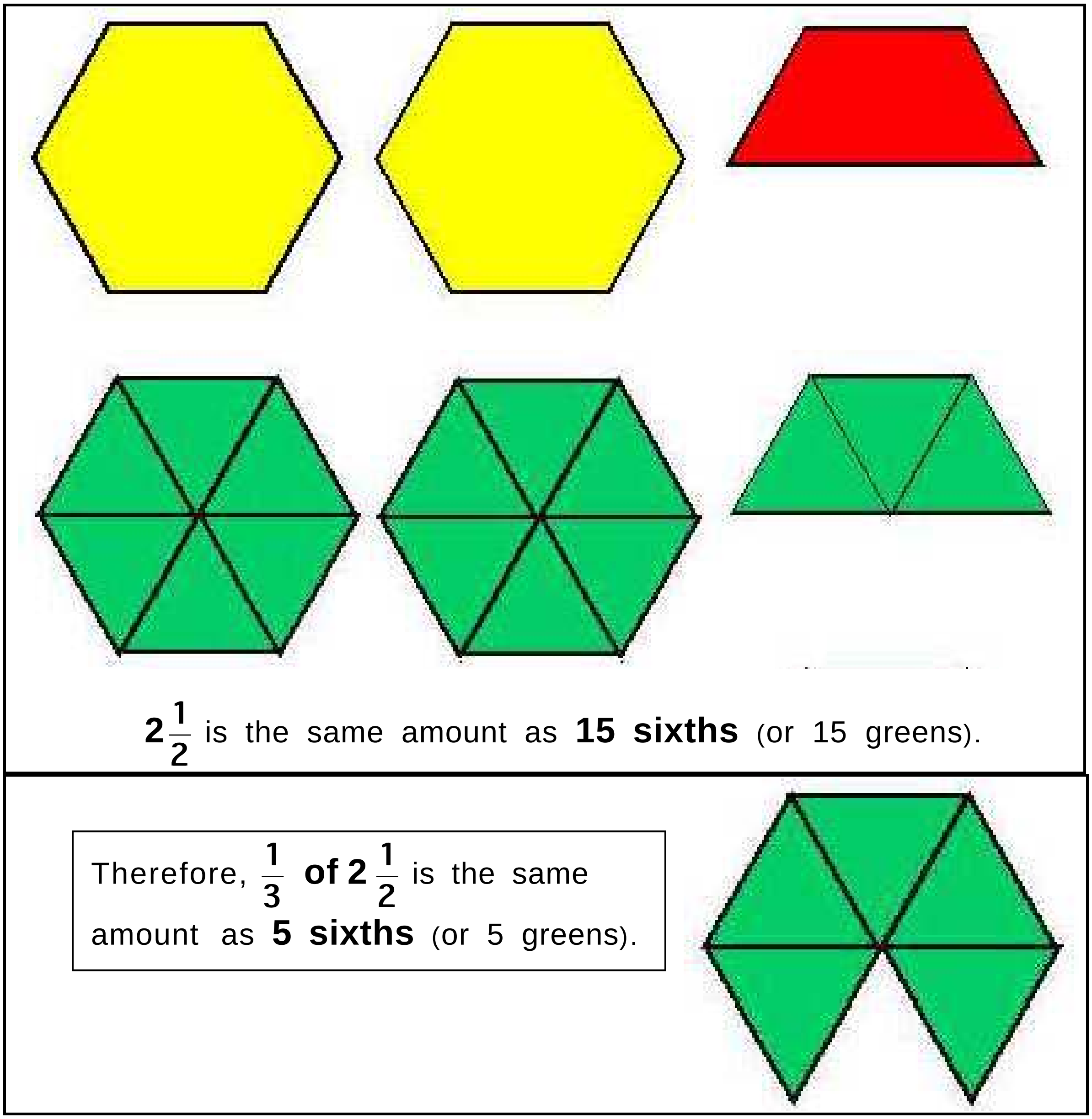

Figure 6. A student's solution for $\frac{1}{3}$ of $2\frac{1}{2}$.

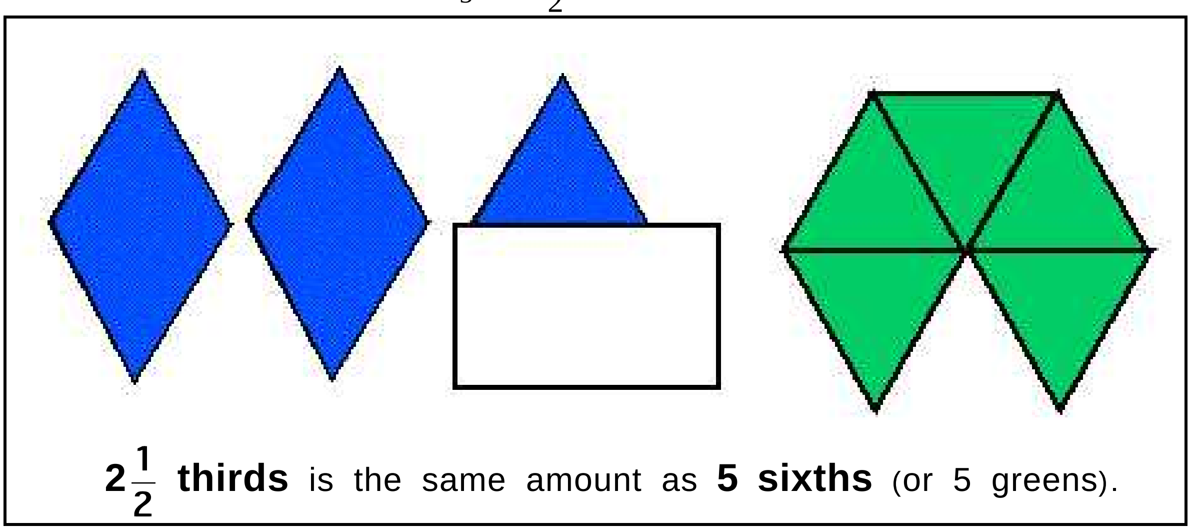

Figure 7.1. A student's representation for $2\frac{1}{2}$ thirds.

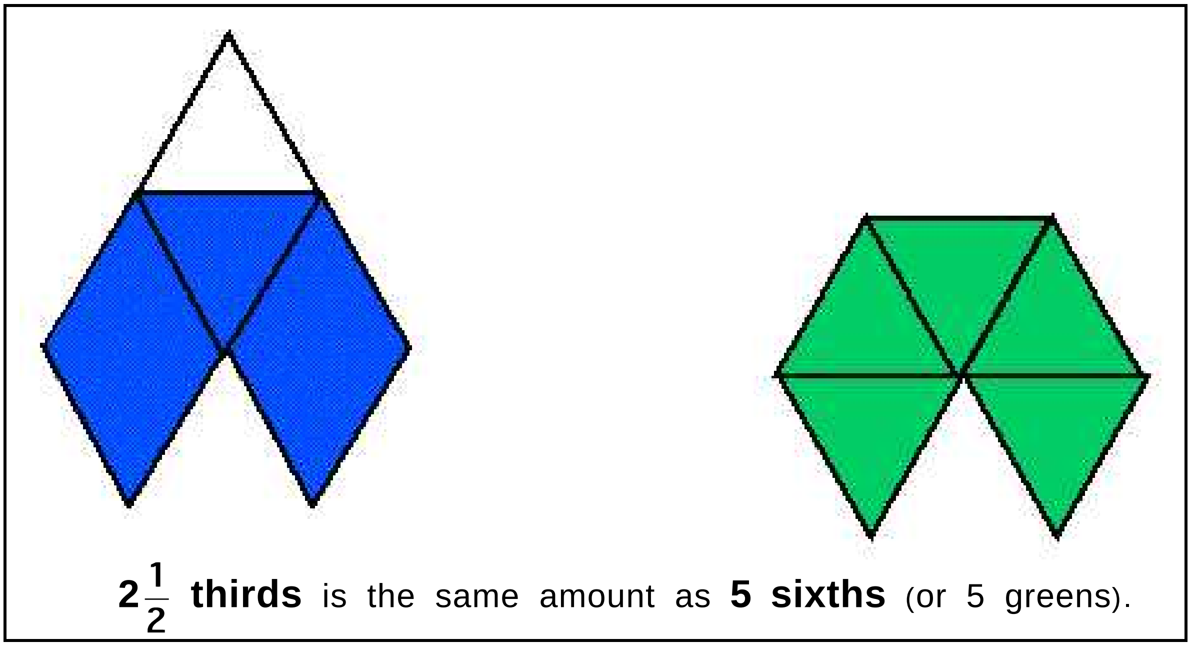

Figure 7.2. Another student's representation for $2\frac{1}{2}$ thirds.

Time to Cook

Next I tell the students that we are going to do some cooking. I ask them to consider a situation:

> A recipe calls for 1/2 cup of sugar, and you have 1 1/2 cups of sugar in the pantry. If you have unlimited supplies of the other ingredients, how many recipes could you bake?

Most students immediately see this as three recipes, because there are 3 halves in 1 1/2. Next I might ask them to consider the same problem situation, but now the recipe calls for 1/3 cup of sugar, and they have 2 1/3 cups. Again they notice that there are 7 thirds in 2 1/3. I then ask them to imagine that the recipe again calls for 1/3 cup of sugar, but this time they have 1 1/2 cups of sugar. This problem is identical in structure to the previous two problems, but changing the values in the problems enables one to focus on a key issue: How do we treat the remainder? Some students respond that they could make 4 recipes and would have 1/6 cup of sugar left over. This answer is mathematically correct, but if I leave it there, the students would not think through a critical idea: that 1/6 cup of sugar is, simultaneously, 1/2 a recipe. We discuss the fact that 1/6 cup of sugar can be viewed as 1/6 of something and 1/2 of something else (see Figure 8).

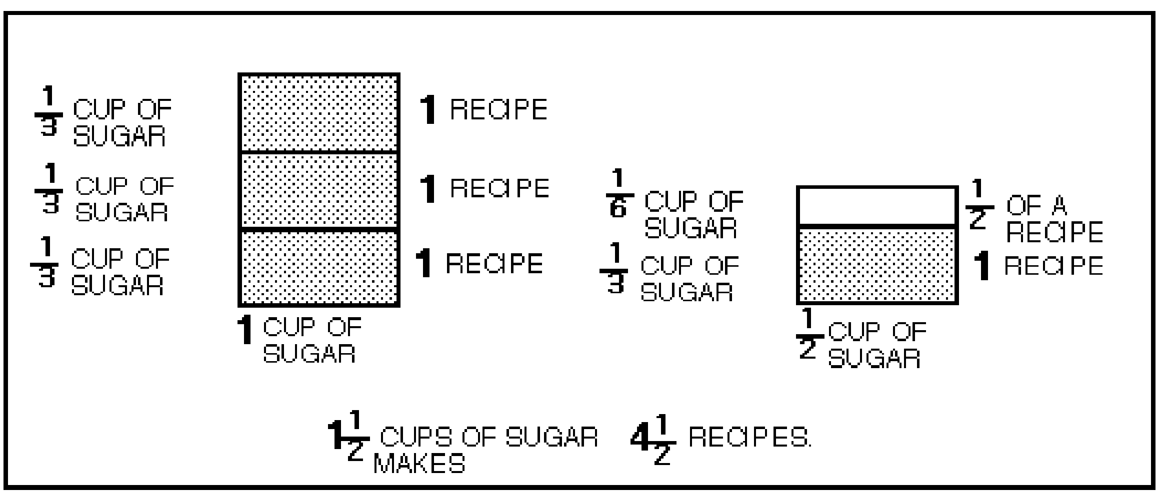

Figure 8. The remaining sugar can be viewed as $\frac{1}{6}$ cup or $\frac{1}{2}$ recipe.

I follow this problem with the following:

Cynthia would like to bake cookies. She has one cup of sugar in the pantry. How many batches of cookies can she make using the one cup if each batch requires
A) 1/2 cup B) 1/3 cup C) 2/3 cup D) 3/4 cup E) 4/5 cup.

Students have no problems with the first two situations, which have no remainder, but they have to think harder about the last three. Problem C can be represented using the pattern blocks, and the students need to recognize that the 1/3 cup leftover is half a recipe, so Cynthia can bake 1 1/2 recipes. I suggest that the students consider an alternate representation for Parts D and E, and I draw a picture of a rectangle to represent the cup of sugar. When considering how many recipes could be baked if each required 3/4 of a cup of sugar, we discuss the picture in Figure 9. The students who struggle with this have trouble simultaneously viewing one segment as 1/4 cup of sugar and 1/3 of a recipe. When the students work Problem E, they again have to consider the fact that 4/5 cup of sugar is equivalent to 1 recipe, and the part left over is simultaneously 1/5 cup of sugar and 1/4 of a recipe (see Figure 10).

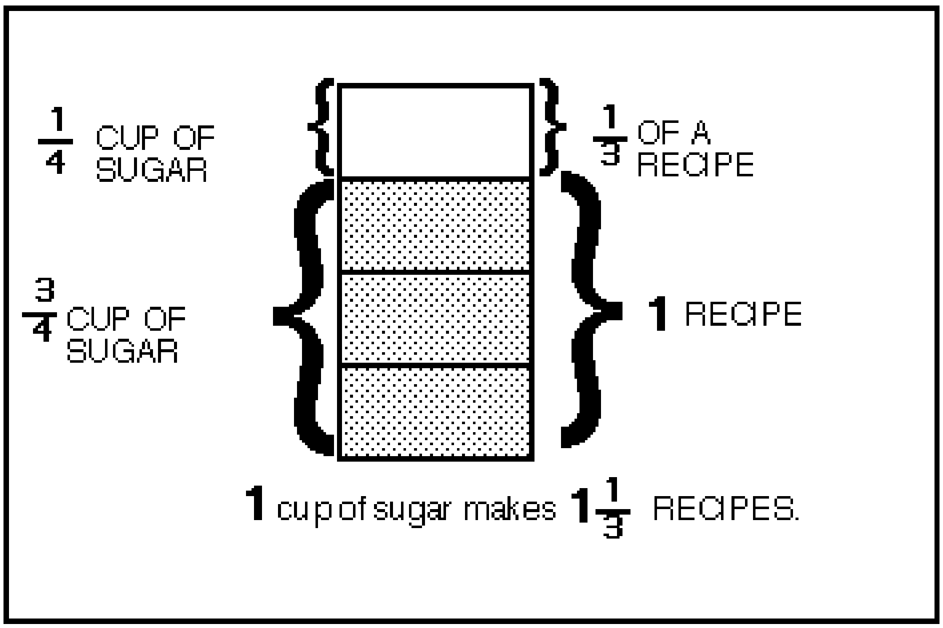

Figure 9. The remaining sugar can be viewed as $\frac{1}{4}$ cup or $\frac{1}{3}$ recipe.

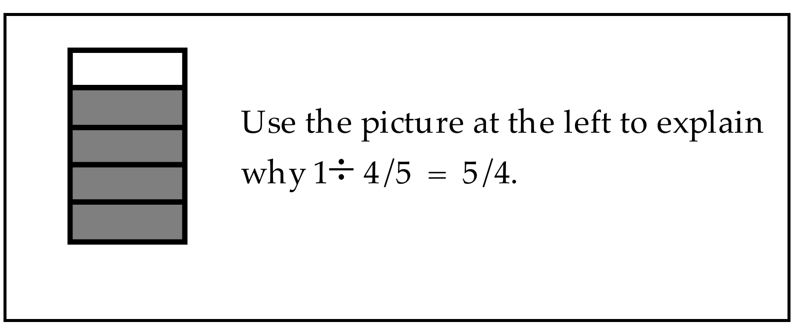

Figure 10. Four-fifths cup of sugar is equivalent to 1 recipe, and the part left over is simultaneously 1/5 cup of sugar and 1/4 of a recipe.

Adapted from Thompson, P. W. (1995). Notation, convention, and quantity in elementary mathematics. In J. T. Sowder & B. P. Schappelle (Eds.), *Providing a foundation for teaching mathematics in the middle grades* (pp. 199–221). Albany, NY: State University of New York Press.

I next ask the students how they would represent Cynthia's baking situations using symbolic notation. Are they addition, subtraction, multiplication, or division problems? After some discussion, the students recognize these as division problems, and I point out that these division problems are *measurement* or *quotitive* division problems, which are different from *partitive* or *sharing* division problems. To use a whole number example, 12 ÷ 3 can be conceptualized as "How many groups of 3 are there in 12?"—a measurement-division problem. Compare that to "If we divide 12 into 3 equal groups, how many are in each group?"—a partitive-division problem. It has been found that children in kindergarten can successfully solve these problems, but their solution strategies for the two types of problems differ (Carpenter, Ansell, Frank, Fennema, & Weisbeck, 1993). In the first case, young children often directly model the problem by counting out 12 objects and then taking out groups of 3. The answer is the *number* of groups. In the second problem, young children may directly model by counting out 12 objects, then sharing the 12 objects among three fictitious people, often "dealing out" the 12 objects one-by-one until they are all shared. The answer is not the *number* of groups, but instead is the *size* of each group.

If you think about these two cases, you may see that conceptualizing 1 ÷ 4/5 using a partitive model is difficult. How might we share 1 cup of sugar among 4/5 people? For fraction situations, using a measurement model is usually more meaningful: If we had 1 cup of sugar and each recipe called for 4/5 cup of sugar, then how many recipes could we bake?

Our thinking might be "One whole has one 4/5 with 1/5 of a whole left over. But that 1/5 of a whole is 1/4 of a 4/5, so one whole has 1 1/4 four fifths."

Here I like to point out a pattern in the problems we have already solved:

1 ÷ 1/2 = 2/1
1 ÷ 1/3 = 3/1
1 ÷ 2/3 = 3/2
1 ÷ 3/4 = 4/3
1 ÷ 4/5 = 5/4

We notice that the answer to "How many A/Bs are there in 1?" is always "B/A." We conceptualize additional tasks, like 1 ÷ 5/6 and 1 ÷ 3/7, until everyone seems comfortable with the notion that the reciprocal of a number tells how many of the original numbers are in 1.

The final step in the derivation for the fractions-division rule is to consider dividing something like 3 ÷ 4/5, which can be interpreted as "How many 4/5 are in 3?" Well, we know how many 4/5 are in 1, so 3 times as many 4/5 are in 3 as are in 1? When faced with 3 ÷ 4/5, we might think of approaching this in two steps: First, note that there are 5/4 four fifths in 1. Second, there are 3 times as many 4/5 in 3 as in 1, or 3 x 5/4.

Three Principles of Teaching for Conceptual Understanding of Mathematics

From the preceding lesson, I highlight three principles of teaching for conceptual understanding of mathematics. After stating each principle, I relate the principle to the specific example of fractions.

Principle 1: Teachers' abilities to identify the important mathematical concepts are critical if their students are to learn and to understand these concepts.

Two main mathematical concepts are demonstrated in the fractions-division example. First, A ÷ B can be thought of as "How many Bs are there in one A?" This measurement approach to division is applicable to division situations that would be much more difficult to think about using only the partitive model of division.

Second, when multiplying or dividing fractions, one must attend to how the unit changes. Students must think beyond 1/2 ÷ 1/3 as 1 with 1/6 left over. Before students can understand fraction division, they must also be able to think of the 1/6 left over as 1/2 of 1/3.

If teachers teaching division of fractions are aware of the role played by these two main mathematical concepts, they can direct their students' attention toward these ideas during the lesson. If teachers do not make these ideas explicit, some students, but probably a small number, may still grapple with the ideas. In such a case, neither the teacher nor the students will be aware that the difference between those who understand and those who do not may lie in their understanding of these ideas.

Principle 2: Teachers must provide opportunities for students to build on their existing knowledge.

Teachers who understand the mathematics are in a position to think about their students' understanding of the mathematics. I ask myself two questions when I consider ways to tap into my students' knowledge. First, what conceptual understanding do the students bring to the task? In the example provided in this paper, I assume the adult students I am teaching can apply procedures for operating on fractions and possess basic understanding of fraction concepts. I do not assume that the students have a good understanding of the two main concepts mentioned above: measurement division and the role of the unit. The lesson was designed to provide teachers opportunities to consider those two concepts.

The other question I ask myself when I consider ways to tap into my students' knowledge is What real-life situations might be used that would be relevant to the students? After posing 1/2 – 1/3, I suggest that the students think about pies. In the past I have suggested that they consider pizzas, but I have found that many students view pizzas as precut into eighths, and this image seems to interfere with learning. Given that pies are generally purchased uncut, students are more comfortable imagining slicing a pie in any manner they choose.

Later in the lesson we dealt with recipes and cups of sugar. This example was chosen because it provided a natural way for students to think about the changing of the unit. When considering how many recipes requiring 1/3 cup of sugar might be made with 1/2 cup of sugar, students can be asked to consider the remaining sugar in terms of cups (1/6 cup of sugar) and in terms of recipes (1/2 recipe). In so doing, the students have a context that makes explicit the different units. Use of the context facilitated the following conversation with a student who had been asked how many recipes she could bake if each recipe required 1/3 cup of sugar and she had 1/2 cup of sugar (the situational equivalent to "How many thirds are in one half?"). (S is the student. I is the interviewer.)

S: I have 1, with 1/6 left over.
I: Consider the 1. What is the 1 a number of?
S: One recipe.
I: And the 1/6? What is it a number of?
S: One-sixth recipe. No, 1/6 cup of sugar.
I: So we know that you have enough sugar for 1 recipe, and you have some sugar left over.
S: Yes.
I: Could the leftover sugar be used in another recipe?
S: Um, I think so. Let's see, it takes 1/3 cup of sugar for one recipe, and I have 1/6 cup of sugar left, so I could make another half.
I: Half what? Half of a cup of sugar?
S: No, half of a recipe.
I: How can 1/6 of something also be 1/2 of something?
S: One-sixth cup can be 1/2 of a recipe.

Consider a conversation without benefit of a context. The student had been asked, "How many thirds are in one half?"

St: There is 1 third in 1/2, and there is 1/6 left.
I: Okay. Now the 1/6 left over, what is it 1/6 of?
St: What do you mean? It is 1/6.
I: Well, is it 1/6 of a whole, or a half, or a third?
St: It is 1/6 of a whole.
I: And the 1 third, when you write that as an answer, don't you write that as 1?
St: What do you mean?
I: Well, suppose you wrote this problem as a division problem with 1/3 as the divisor and 1/2 as the dividend, wouldn't the quotient be 1?
St: Yes.
I: That 1 is 1 what?
St: One third.

I: And the left over 1/6. What is that?
St: That is the remainder.
I: Could you write it as part of the quotient?
St: Yes, I suppose I could divide it by the divisor, the way I do with whole numbers. (She does the calculations and gets 1/2).
I: So what is your answer?
St: One and a half.
I: One and a half what?
St: One and a half thirds.

Of course, the conversation may not go this way, but I have had similar conversations, and I find that they are awkward because the units (the wholes and the thirds) keep getting lost during the discussion. In fact, the student in the second vignette may not have conceptualized 1/6 as half of 1/3 but, instead, to get the 1/2, she may simply have applied a procedure that she remembered having used with whole numbers. A student who possesses a solid understanding of the concepts ought to be able to make sense of these ideas whether or not they are embedded in a context. I see using the context as a means of supporting students whose understanding is still fragile.

When considering situations that would be relevant to students, one must not, of course, assume that a situation relevant to one student will be relevant to another. As a child, I was very interested in baseball and teachers could have used a batter’s slump or hitting streak as a way to help me think about proportional reasoning. This context would not have worked for someone who had no interest in baseball. On one level, this seems obvious. Yet on another level, I think teachers often gloss over the fact that when they use contexts, some students are a step behind their peers because they do not understand the context well enough to be able to move around flexibly within it. Elementary school teachers tend to create contexts in class—but secondary school mathematics teachers often feel that they lack the time for developing meaningful contexts.

Principle 3: Introduce symbols and procedures *after* students understand the concepts these symbols and procedures are meant to represent.

A great deal of discussion is taking place in the United States today about whether the reform documents, including the *NCTM Standards* (1989) and the *California Framework* (1992), have gone too far in de-emphasizing the teaching of skills and procedures.[5] I believe that the discussion has gone in the wrong direction. The issue should not be whether to teach concepts or procedures. We need both. The issue should be whether the order in which these are taught makes a difference. Does it matter whether we teach the procedures or the concepts first? Research has shown that the order is important. If we teach procedures before we teach the underlying concepts, students are less likely to learn the concepts (Wearne & Hiebert, 1988). I will address this issue with contrasting examples of a first grader and a fifth grader.

Once a week I teach a mathematics lesson in a first-grade classroom at a local public school. In October I introduced fractions to the students by having them work with pattern blocks, constructing wholes using red trapezoids (halves), blue rhombi (thirds), and green triangles (sixths). By the end of the lesson, some students were constructing wholes using different-sized pieces. At this time we did not represent any of the fractions symbolically; instead we just spoke about the fractions.

The next day at my son’s soccer practice, a couple of mothers asked me what I was doing in the first-grade class. Alyson, a student from the class, happened to be standing close by, so I called her over. Alyson and I talked:

I: Alyson, is a half and a half more than a whole, less than a whole, or equal to a whole?
A: What does *equal* mean.
I: Equal means “is the same as.”
A: It is equal to a whole.

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[5] It turns out that these documents have not had an important influence in affecting how teachers teach. Jim Stigler, who coordinated the video study of the Third International Mathematics and Science Study, found that although most of the American eighth-grade mathematics teachers videotaped knew about and had read portions of reform documents, most were teaching lessons that emphasized procedures and definitions, not conceptual learning.

I: Is a third and a third more than a whole, less than a whole, or equal to a whole?
A: Less than a whole.
I: How do you know?
A: Because you need another third to make it a whole.
I: Is a half and a third more than a whole, less than a whole, or equal to a whole?
A: (After about 5 seconds): Less than a whole.
I: How do you know?
A: Because you need another sixth to make it a whole.
I: Alyson, how did you think about that question? Are you picturing something in your mind?
A: A red and a green.
I: You mean you know a red is a half and a green is a sixth?
A: Yes.

Notice how quickly Alyson was able to make the link between the physical model and the fractional names. Our schools are filled with first graders who, if given the chance, are ready to reason as Alyson did.

Recently I was being escorted to a class by a fifth grader who told me that his class had been studying fractions, so I posed the same questions I had asked the first grader. Our conversation went as follows:

I: Jim, is a half and a half more than a whole, less than a whole, or equal to a whole?
J: Equal to a whole.
I: Is a third and a third more than a whole, less than a whole. . . ?
J: (Interrupting) It is two sixths.
I: Hm. That's interesting. When I asked you what one half and one half was, you said one whole, but when I asked about one third and one third, you said two sixths.
J: Hm. Oh, okay, I see now.
I: What is it that you see?
J: One half and one half is two fourths.

Unlike the first grader who had a way of picturing halves and thirds in her mind, Jim's primary way of thinking about these fractions was to consider the symbols and procedures. As a result, he would manipulate the symbols in his head, but he had little experience in developing number sense to support his thinking.

We Teach What We Know

Thompson, Philipp, Thompson, and Boyd (1994) suggested that teachers' orientations toward mathematics and toward teaching mathematics are reflected in the kinds of questions they pose and the kinds of conversations they strive to hold. Two teachers who value problem solving may approach the same tasks differently because one teacher may possess a *calculational orientation* whereas the other teacher may possess a *conceptual orientation*.

> The actions of a teacher with a calculational orientation are driven by a fundamental image of mathematics as the application of calculations and procedures for deriving numerical results. This does not mean that such a teacher is focused only on computational procedures. Rather, his view of mathematics is more inclusive but still one focused on procedures—computational or otherwise—for "getting answers." (Thompson et al., p. 86)

A teacher who possesses a calculational orientation may view problem solving as an occasion to solve problems, whereas a teacher who possesses a conceptual orientation will view problem solving as an occasion for students to reason and to reflect on their reasoning.

> A teacher with a conceptual orientation is one whose actions are driven by
> ° an image of a system of ideas and ways of thinking that she intends the students to develop;
> ° an image of how these ideas and ways of thinking can develop;

> ° ideas about features of materials, activities, expositions, and students' engagement with them that can orient the students' attention in productive ways . . . ;
> ° an expectation and insistence that students be intellectually engaged in tasks and activities. (Thompson et al., p. 86)

How does one develop a more conceptual orientation? Alba Thompson once suggested that to help people think differently about mathematics, we should consider changing its name to Mathematical Arts. Teaching Mathematical Arts is fundamentally about getting students to *reason* in particular ways and not about getting students to *do* particular things. If teachers and students internalized this new view of mathematics, then the mathematics classroom would become a place where *doing* would take the back seat to *understanding* and *calculating* would take the back seat to *sense making*. Although a teacher possessing such a view might not deeply understand everything she teaches, she would recognize that there is something to be known. When we come to understand mathematics as a discipline about which we reason, we are poised to recognize when we do not understand something deeply; such recognition is the first step toward developing a deeper understanding.

Lee Shulman (1986) wrote, "Those who can, do. Those who understand, teach." Alba Thompson's research had gone a long way toward showing that we teach what we know and do not teach what we do not know. So what are teachers to do when they find themselves in the position of having to teach a topic they do not understand deeply? This question has no easy answer. Sowder, Philipp, Armstrong, and Schappelle (1998) found that after they provided middle school teachers in-services during which the teachers had opportunities to conceptualize the mathematics they were teaching, the teachers spoke critically of mathematics in-services that did not focus on mathematics. I'll end with one general suggestion: Let's put the mathematics back into teacher in-services.

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