A Model for Understanding, Using, and Connecting Representations[1]

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Scenario 1

Teacher: Can you solve this problem? (Gives student paper with $4 - \frac{1}{8}$ written at the top.)

Student: (Writes $\frac{3}{8}$ as answer). Three-eighths. I subtracted 1 from 4, and then kept the denominator, 8, the same.

Teacher: Suppose you had four large brownies and you ate one eighth of one brownie. How many brownies would you have left?

Student: (Pauses, then draws four rectangles, partitions one of the rectangles into eight pieces, and shades one of the pieces.) Three and seven eighths (writes $3\frac{7}{8}$).

Scenario 2

Teacher: I will say a number, and you write it. Please write for me the number one half.

Student: (Writes $1\frac{1}{2}$.)

I have observed both of these scenarios, and I suspect that most teachers have seen something similar. Each situation can be explained in many ways. For example, the first scenario captures the fact that real-life situations may be more comprehensible to students than written symbols; the second scenario exemplifies the fact that the way children hear and use language to represent mathematics may differ from the way adults use the same language. These explanations capture important insights about mathematical thinking. However, these explanations may fail to capture the underlying important mathematical ideas embedded in each scenario. My colleagues and I have found one model (see Figure 1) that subsumes what for us is the essence of each of these scenarios, and this model has been helpful in our teaching and in the teaching of many teachers with whom we have shared and discussed it.

In this paper I present the model, use these two scenarios to explain how teachers have found the model useful, and provide an instructional sequence that is guided by the use of the model. The representations model provides a lens for making sense of students' responses to tasks. It also helps when planning lessons; one can consider whether to include all or most of the representations in a set of lessons. The possibilities presented in the model are also helpful for selecting an order in which to introduce the representations.

Description of the Representations Model

The model is a slight adaptation from one first introduced by Lesh, Post, and Behr (1987). Each oval in the model corresponds to one way to represent a mathematical idea. The *Principles and Standards* for School Mathematics stated

> Representations should be treated as essential elements in supporting students' understanding of mathematical concepts and relationships; in communicating mathematical approaches, arguments, and understandings to one's self and to others; in recognizing connections among related mathematical concepts; and in applying mathematics to realistic problem situations through modeling. (National Council of Teachers of Mathematics, p. 67)

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The five representations presented here are a subset of those that a teacher or student might draw upon when examining mathematical ideas.

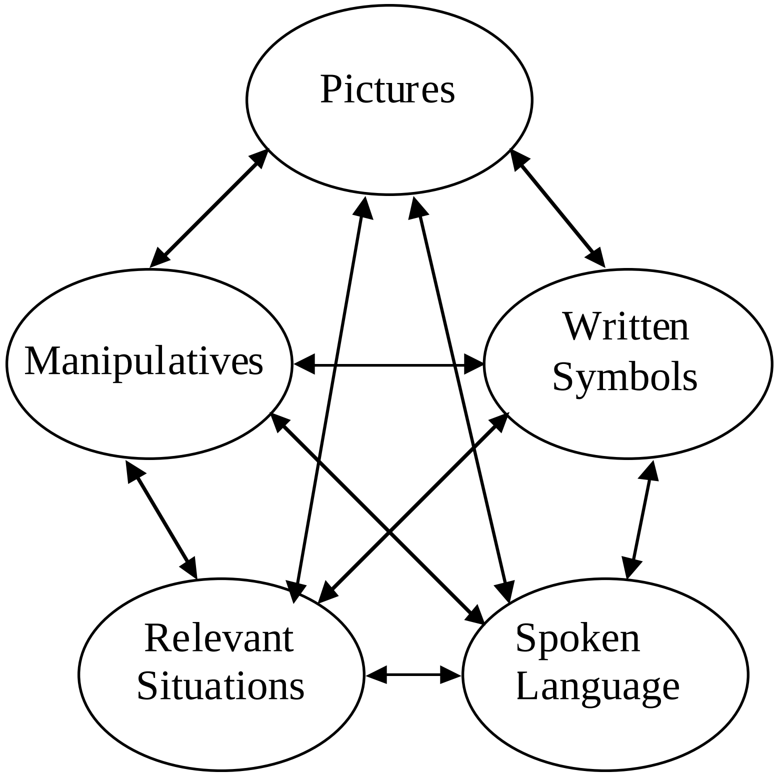

Figure 1. Representations Model adapted from Richard Lesh, Tom Post, and Merlyn Behr.

Pictures. Although the *pictures* representation can refer to the pictures of mathematical ideas that teachers draw or that one finds in textbooks (such as rectangles that are partitioned into four equal pieces with one piece shaded), I tend to think of pictures as those that children draw themselves. When children draw pictures (spontaneously or at a teacher's request), the teacher can learn more about what they understand about a particular mathematical idea and can use the different pictures that children create to provoke a discussion about mathematical ideas. For example, when children are asked to draw one fourth of a brownie, teachers can determine which children understand that every one-fourth piece should have the same area as the others or that they should use the entire brownie when cutting it into fourths. If some children use equal parts and others do not, the teacher can lead a discussion about these differences. Constructing their own pictures can be a powerful learning experience for children because they must consider several aspects of mathematical ideas (such as equal parts) that are often assumed when pictures are predrawn for students.

Manipulatives. The *manipulatives* representation refers to the unifix cubes, base-ten blocks, fraction circles, and the like, that a child might use to solve a problem. Manipulatives are objects that can be touched, moved, and, often, stacked. Because children can physically manipulate these objects, when used appropriately, they provide opportunities to compare relative sizes of objects (that represent mathematical ideas such as fractions or place value), to identify patterns, to put together representations of numbers in multiple ways, and so on.

Spoken Language. I distinguish two uses of *spoken language:* as a way for students to report their answers and as a way for students to express their reasoning. Traditionally, *teachers* often used the spoken language of mathematics but rarely gave *students* opportunities to grapple with it. Yet, when students do have opportunities to express their mathematical reasoning aloud, they may be able to make explicit some knowledge that was previously implicit for them.

Written Symbols. *Written symbols* refer to both the mathematical symbols and the written words that are associated with them (for example, 1/4, 2/8, .25, 25%, and one fourth, two eighths, twenty-five hundredths, twenty-five percent). For students, written symbols tend to be more abstract than the other representations. I tend to introduce symbols *after* students have had opportunities to make connections among the other representations, so that the students have multiple ways to connect the symbols to mathematical ideas, thus increasing the likelihood that the symbols will be comprehensible to students.

Relevant Situations. The representation *relevant situations* particularly merits further comment. The fraction 1/4 can be represented in a variety of relevant situations, for example, Juan ate 1/4 of the pepperoni pizza, or Alicia and her three best friends shared a large chocolate bar fairly. How much of the candy bar did each person get? A relevant situation can be any context that involves appropriate mathematical ideas and holds interest for children; it is often, but not necessarily, connected to a real-life situation (for example, one teacher created a story about leprechauns that, although not a real-life situation, held great interest for her students; because her story held meaning for her students, this context is considered a relevant situation).

Look at the model again in Figure 1 and ask yourself, What is most important about this model? Did you select relevant situations? Or perhaps you selected written symbols? As important as the ovals are in this model, another feature of the model is even more important than the representations themselves: The arrows! The arrows are important because they represent the connections students make between the representations. When students make these connections, they may be better able to access information about a mathematical idea, because they have multiple ways to represent it and, thus, many points of access. Individuals enhance or modify their knowledge by building on what they already know, so the greater the number of representations with which students have opportunities to engage, the more likely the teacher is to tap into a student's prior knowledge. This "tapping in" can then be used to connect students' experiences to those representations that are more abstract in nature (such as written symbols). Because not all students have the same set of prior experiences and knowledge, teachers can introduce multiple representations in a meaningful way so that students' opportunities to grapple with mathematical ideas are greater than if their teachers used only one or two representations.

Similarly, teachers may find that students appear to understand a particular mathematical idea using one representation (say pictures) but struggle when trying to translate that picture into a different representation (for example, symbols). When exploring students' understandings, teachers can consider whether they are garnering information about a student's understanding of a mathematical idea, one representation of a mathematical idea, or a student's ability to transform a representation of a mathematical idea into another representation.

Drawing on Connections Between Representations to Understand Children's Thinking

When teachers use the representations model in Figure 1 to focus their instruction on the representations that may be most familiar to children—relevant situations, manipulatives, pictures, or spoken language—they may provide children with greater access to and enjoyment of mathematics. Teachers can help students make connections from these representations to written symbols, but only after they have begun to connect the former representations with one another, because, for most children, the symbols are more abstract (and thus more difficult to comprehend) than the other representations. For example, Wilcox and Herron (2002, unpublished MA project) found that most 3rd graders incorrectly answered 2 when given the following problem in written form: *There are 6 students on a team; 1/2 of them are girls. How many are girls?* The students who responded with an answer of 2 seemed to focus more on the numeral 2 in the number 1/2 than on the meaning of *one half* in the story. Interestingly, when the teacher *read aloud* the same problem to the students in a one-on-one setting, all answered correctly. In this study, the written symbol 1/2 had little (appropriate) meaning for these students. Had this teacher not asked the question again verbally, she might have thought that her students had no understanding of the concept *one half* when working with set models. Instead, she found that quite the opposite was true. Her students had fairly rich knowledge of the meaning of 1/2 in real-life (relevant) situations but had not yet made meaning for the written symbol 1/2.

Similarly, when teachers verbally prompt students to write the mathematical symbol for *one half*, children often incorrectly write 1 1/2 (as in Scenario 2). Attending to aspects of the model (the connections among relevant situations, spoken language, and written symbols) helped us to consider one possible explanation for this occurrence. For example, children's everyday spoken language used for sharing things equally is half (not *one* half); for example, one child may ask another, "Can I have half your brownie?" Considering children's use of the spoken language *half* and teacher's asking a student to write *one* half, we are less surprised that many students write $1\frac{1}{2}$. They write the symbol *1* to represent the 1 in one half and then write the symbol $\frac{1}{2}$ to represent the *half* that is a familiar term to them. Using the model as a lens for examining this type of student response, a teacher may be poised to search for connections, or lack thereof, that students make among representations in the model.

Additionally, the use of the model helps one to consider which representation would be most meaningful to students when mathematical ideas are first introduced. For example, when students are asked to solve the problem 4 –1/8, many will answer incorrectly with responses such as 3/8 (subtracting 1 from 4 and leaving the denominator the same, as in Scenario 1) or 3/4 (subtracting 1 from 4, and then subtracting 4 from 8). The students who make these errors tend to think of the numerator and denominator as separate numbers to be operated on, instead of treating the fraction, in this case 1/8, as a single number. However, educators have found that these same students are successful when they are asked to solve a problem that corresponds to 4 – 1/8 but is set in a relevant context, for example, "Maria has 4 big brownies. She eats 1/8 of one brownie and saves the rest for later. How much does she have now?" When the students are asked to solve such a problem, they are more likely to have an image upon which they can draw when deciding how to solve it. In this example, even students who are unsure of the final answer almost always know that Maria still has 3 brownies and some part of another brownie. This situation also invites students to draw a picture (or use manipulatives if available) to represent their thinking, whereas the symbols alone rarely prompt students to draw a picture, because, at least initially, the symbols alone tend to have little meaning for most students.

The Model in Action—A Lesson on Decimal Fractions

When I recently had the opportunity to tutor two 5th-grade students to help them compare decimal fractions, I used the model to design and carry out the lesson. I knew from working with students in the past that comparing decimal fractions is challenging for many students. Many students compare decimal fractions using a whole number approach. That is, they ignore the decimal point and treat the numbers as if they are whole numbers. For example, students who use the whole number approach would compare .73 and .8 incorrectly, saying that .73 is larger because 73 is larger than 8. Other students answer correctly by adding zeroes to the decimal fraction with fewer places until both numbers have the same number of places to the right of the decimal point. For example, a student who uses this approach would append a 0 to the right of .8, to get .80. Then, when comparing the numbers, this student could say that .73 is smaller than .80. Although in using this approach students get correct answers, they can bypass any reasoning about numbers less than 1 as being parts of wholes. That is, although they could still be thinking of them as whole numbers, after making both decimal fractions the same number of places, they can get correct answers.

In a preassessment, the students compared pairs of decimal fractions in written, symbolic form (e.g., .73 vs. .8); neither student could successfully compare the numbers. I wanted to begin instruction by using a real-world context (relevant situation) that I thought might be familiar to them, instead of beginning with the written symbols. I also wanted them to use base-ten blocks (flats, longs, and singles) as a manipulative model to support the context (along with spoken language), so that they would have multiple access points for thinking about the relative sizes of decimal fractions. I thus began the lesson by describing the community in which I live; it has many apartment buildings and very little land for things like gardens. I then introduced the flat (a manipulative) as a plot of community land that is cut into small sections so that neighbors could have a place to garden, if they so chose. I asked the students to identify the number of pieces into which the garden was cut (100); I asked them to determine the size of each small piece (one hundredth of the garden) and the size of a row (one tenth of the garden) and to explain why each was so named.

At this point, the instruction was focused on helping the students think about decimal fractions by using a relevant situation, manipulatives, and spoken language and by helping them make connections among the three. Once the students could comfortably shift among the situation, the manipulatives, and the spoken language, I attempted to support their thinking about multiple ways to name the same section of garden. I wanted the students to understand not only the connections *among* the situation, manipulatives, and spoken language but also the connections to be made *within* the same representation; in this case, I wanted them to see that the same part of land can be named in multiple ways (e.g., as three tenths and thirty hundredths). Of course, this naming requires more than a student’s ability to vocalize the correct words; the student must view and think about the same parcel of land using different sized pieces (in this case, tenths and hundredths).

After working similar problems for garden parts of other sizes, naming the garden part in two ways and explaining why both names made sense, the students could build a fraction of the garden when the fraction was provided orally, name the parts of the garden in two ways, and explain why names were appropriate for the particular fraction of the garden. I next asked the students to write the symbols for the fractions with which they were working. At this point in the lesson, they had built with manipulatives both seventy-three hundredths of one garden and eight tenths of another garden. One student knew how to write fractions in decimal form, and so he was able to represent the same fraction in written symbols in multiple ways (as 73/100, 7/10 and 3/100, .73, and .7 and .03); however, he named 73/100 in fraction form as “seventy-three hundredths” but named the decimal representation .73 as “point seventy-three” (instead of as “seventy-three hundredths”). Many students use this “point” language. They may do so because fractions written symbolically in decimal form do not include symbols for denominators, such as /10, /100, or /1000 (in the example of .73, the written symbol does not include the denominator 100), that might help students recognize that decimal fractions are tenths, hundredths, or thousandths, and so on. I thus suggested that the student use the same language for .73 that he used for saying 73/100, because the language might help him to remember that the decimal fractions are fractions rather than whole numbers.

In designing the lesson, I first chose a context that was relevant to the student and that was further supported by use of manipulatives and spoken language by both the teacher and the students. The delayed introduction of written symbols was purposeful and was based on the preassessment that indicated that the students had not yet made appropriate meaning of the symbols. I decided to postpone the introduction of the symbols until the students had made some appropriate connections within and among representations (relevant situations, manipulatives, and spoken language), connections to which they could connect their understanding of the symbols. In a postassessment, these students were able to appropriately compare several pairs of decimal fractions (all less than 1) in symbolic form and to explain, with an explanation that went beyond appending a zero, which number in the pair was larger.

Conclusions

The representations model provides both a lens for making sense of students' responses to tasks and a guide for lesson planning. One can first consider whether to include all or most of the representations in a set of lessons. The model is also helpful for selecting an order in which to introduce the representations: Begin with representations that are most meaningful for students! When analyzing your students' work, you may want to consider these questions to help you make sense of the results.

- What representations did I use when I posed the task?
- What representations did students use to undertake the task?
- In what ways did the students connect the representations to one another and to mathematical ideas?
- Were some translations among representations particularly challenging or helpful for students?
- How did the representations and connections among them support the students' mathematical reasoning and flexibility in their thinking?

When considering an important mathematical idea you plan to teach, you may want to consider these following questions.

- What representation or representations will be most meaningful for my students?
- What other representations will serve to support my students' thinking?
- What order seems most sensible for introducing (or inviting the use of) the different representations?
- Which representations or translations will promote more powerful mathematical thinking for my students?

Look at Figure 1 and think back for a moment to your own elementary school experience. If your experience was like mine, then mathematics was comprised primarily of written symbols, with some occasional pictures from textbooks. I do not remember using manipulatives in elementary school, and although there was spoken language, generally it was the teacher who used language to explain; students used language mainly to report answers. Regarding relevant situations, I recall that story problems were presented at the ends of chapters, but these seemed to be designed to provide students practice on the procedures in the chapter instead of to promote students' mathematical understanding. If the major representations students used were written symbols and textbook pictures, then only one of the connecting arrows was addressed. But that left nine connecting arrows untapped! If we can help students increase the number of connections they have available when grappling with mathematical concepts, we may be able to increase the number of students who report good experiences with mathematics and to increase students' mathematical understanding.

References

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