

# IMAP Content Instrument Manual



**IMAP**

Integrating Mathematics  
and Pedagogy



**IMAP**

**Integrating Mathematics  
and Pedagogy**

## **About IMAP**

**Integrating Mathematics and Pedagogy (IMAP) is a federally funded, 3-year project designed to integrate information about children's thinking about mathematics into mathematics content courses for college students intending to become elementary school teachers.**

Principal Investigators

Randolph Philipp, San Diego State University

Judith Sowder, San Diego State University

Directors

Randolph Philipp

Bonnie Schappelle

Content-Instrument Team

Larry Sowder

Eva Thanheiser

Copyright © 2004 by the San Diego State University Foundation.

The production of these materials was undertaken as part of a larger project, Integrating Mathematics and Pedagogy (IMAP), in which we studied the effectiveness of early field experiences in mathematics. The project was supported by a grant from the National Science Foundation (NSF) (REC-9979902). The views expressed are those of the authors and do not necessarily reflect the views of NSF.

## **Contact Information**

IMAP

6475 Alvarado Road Suite 206  
San Diego, CA 92120

Fax (619) 594-1581

E-Mail [rphilipp@mail.sdsu.edu](mailto:rphilipp@mail.sdsu.edu)

# The IMAP Content Instrument

Larry Sowder  
San Diego State University

## *Purpose and Rationale*

Researchers associated with a large-scale project, Integrating Mathematics and Pedagogy<sup>1</sup> (IMAP), designed and administered a content instrument to determine whether Children's Mathematical Thinking Experiences (CMTEs) have measurable effects on college students' performances on items that address the content of the accompanying mathematics course (Math 210). (An IMAP CMTE has been described by Philipp, Thanheiser, and Clement, 2002). Instructors have noticed that too many college students seem to regard the mathematics courses designed for prospective elementary school teachers (PSTs) merely as a requirement, an obstacle to be overcome, with little relevance to their later teaching; they often, therefore, approach the courses in a perfunctory manner.

The IMAP theme of leading the college students to changed views of mathematics through a focus on children's thinking seemed particularly promising. Harel (1998) has offered his *necessity principle* as a guide to instruction: Seek tasks that address an intellectual need of the student. Identifying the immediate, real, mathematical conceptions of a child in an interview situation offers the college student a setting demanding action, preferably informed action. Hence, IMAP's approach of focusing on mathematics through working with children seemed quite promising: The necessity of responding to a child during an interview about mathematics might lead not only to a change in the college student's beliefs about children and mathematics but also to his or her greater seriousness about the mathematics content courses, especially when the college student recognizes that one should know the subject well to respond appropriately to a child. Indeed, several students in past CMTE-like encounters had, as a result of those experiences, expressed a new-found appreciation for the mathematics courses they had taken, so IMAP researchers were optimistic that an early CMTE would enhance the PSTs' efforts and interests in these mathematics courses. Greater effort and higher levels of interest should translate into better performance on the content of the courses. Hence, the IMAP mathematics-via-concern-for-children theoretical

---

<sup>1</sup> This work was supported by the National Science Foundation under Grant #9979902. Any opinions, findings, and conclusions or recommendations expressed herein are those of the author and do not necessarily reflect the views of the National Science Foundation.

position, supported by anecdotal reports and the clear enthusiasm of the students in earlier pilot work, were so promising that IMAP examined the influence of concurrent experiences with children on the college students' performance not only on the IMAP Web-Based Beliefs Survey but also on a test of mathematical content.

### *Test Composition and Piloting*

Because of the pretest/posttest design of the study, the final examination for the MATH 210 course would not be useful. Accordingly, a content-test team was charged with preparing a special paper-pencil content instrument, with focus primarily on the MATH 210 content, which also represented the main topics of the CMTEs: place value and rational numbers (both fractions and decimals). (MATH 210 topics like integers and number theory were not treated in the CMTEs; thus, no items were written for those topics.) The content-test team, a faculty member from the mathematics department who was quite familiar with MATH 210 and a doctoral student in mathematics education, also solicited items and asked for reactions from others involved in IMAP. Items for this special content instrument could well have been used for the final examination, and indeed a few of the items were selected from previous MATH 210 final examinations. In the large, the focus of the items was on conceptual understanding rather than computational skill. A few items were solely objective in nature (i.e., multiple choice or correct/incorrect as choices), but many of the items also called for explanations. Two items are given in Figure 1; the full test is included with scoring information in Appendix A.

IMAP piloted preliminary versions of the Content Instrument, including some pre/post work, during Spring 2001. These results were used in two ways. One use was to gauge the clarity and likely value of the items, as judged by the students' written responses and by an occasional debriefing of a student who had completed the test. The second use was as a basis for designing rubrics with which to score the items that call for explanations. These rubrics were composed by the content-test team during the spring and early summer of 2001. A sample rubric is given in Figure 2. Each rubric has the range of possible scores for the item and for each possible score, a general description of the type of response and unedited examples of responses awarded that score. Appendix A contains the rubrics for all the items. As a result of the piloting, some items were rejected, others were refined, and three items designed to address pedagogical content knowledge were added to the test. The test was designed for completion in one hour, but students were to be allowed to work as long as they wished. A few items from the Beliefs Survey were strictly content in nature (they were used to assess a student's later responses on the Beliefs Survey in early version of that survey); results on these items were also added to the Content Instrument results and, ultimately, were made part of the content instrument itself.

4. For Fiesha's solution to  $32 \times 54$ , (a) first evaluate her mathematical reasoning. Check whether the steps are mathematically correct or flawed, or indicate that you cannot tell; (b) next mark the work with "doesn't appear to understand multiplication," "may or may not understand multiplication," or "shows good understanding of multiplication"; (c) finally, if Fiesha's steps are mathematically correct, use her way of thinking to solve  $24 \times 53$ . If they are not, explain how Fiesha's reasoning is flawed.

**Fiesha:**

$$\begin{array}{r} 54 \\ \underline{32} \\ 1500 \\ 120 \\ 100 \\ \underline{8} \\ 1728 \end{array}$$

- a) Mathematical steps

Fiesha's steps are mathematically correct.	Fiesha's steps are mathematically flawed.	I cannot tell if Fiesha's steps are mathematically correct or flawed.
--	---	---

- b) Understanding of multiplication

Fiesha doesn't appear to understand multiplication.	Fiesha may or may not understand multiplication.	Fiesha shows good understanding of multiplication.
---	--	--

- c) If Fiesha's steps are mathematically correct, use her way of thinking to solve  $24 \times 53$ . If they are not, explain how Fiesha's reasoning is flawed.

5. Antonio asks, "When I multiply [for example,  $49 \times 23$ , shown to the right], why do I have to put in the 0 [points to the zero in 980]?"

What would you say to Antonio?

$$\begin{array}{r} 49 \\ \underline{23} \\ 147 \\ \underline{980} \\ 1127 \end{array}$$

Figure 1. Two content-test items.

---

**QUESTION 5. Antonio asks, “Why is the 0 in multiplication algorithm?”**

**Explanation score (0–2 scale)**

**Explanation score 2.** (Algorithmic with very good understanding OR simply good understanding; uses place-value language with good explanation; indicates that one is multiplying 20 times 49 and not 2 times 49; states, and gives good reason, that the 98 is really 980.)

- Σ When you got the 980 you were multiplying 20 times 49—however it’s easier to think of  $2 * 49$  and just add zero to make it 20.
- Σ I would say you have to put the zero in because otherwise the numbers wouldn’t be in the correct place. When you are multiplying 49 times 2, the 2 is actually 20 so the zero needs to go in for correct place value.
- Σ The 2 is in the tens place, the 3 is in the ones place. You first multiply  $49 * 3$ , that gets you 147. Now you do the same for the 2 but since the 2 is in the tens place it’s the same as  $49 * 20$  that’s why you bring down the 0.

**Explanation score 1.** (Uses place-value language or names the places mechanically; does NOT clearly state that 2 times is really 20 times OR states 20 but then “messes up.” In this category the 0 is seen as a place holder for the one’s place; a reference is made to consider the 3 in 23; states that the 98 is really 980 without giving a “good” reason.)

- Σ Because you need something to replace the ones place.
- Σ Because that zero [referring to a zero that he drew over the 3 in 23] is taking place of the three.
- Σ You put the zero there to show that you are not adding any numbers in the ones place, the zero allows you to know that even though there are no ones, that place value is still there, acting like there is so you don’t put numbers in their wrong place values.
- Σ You have to take into consideration the 3 in 23.
- Σ Ok, the reason you put a 0, is because you are moving from ones place (1) to the tens place (10.) the difference between 1 and 10 is the zero.
- Σ Now we are working the tenths [*sic*] place rather than the ones place therefore you need to move the value over 1 and add a 0.
- Σ Because the #s we are multiplying are in the 10<sup>th</sup> [*sic*] place (the #2) and we have to account for that by adding the zero so we know the 98 is from the 10<sup>th</sup> [*sic*] place.

**Explanation score 0.** (No understanding shown; nonsense; solely algorithmic; refers only to *place holder* [e.g., “0 is a place-holder”].)

- Σ Because every time you finish one row of numbers and move to the next you have to put a 0 first because then the next number can be under that row of numbers.
- 

Figure 2. The rubric for Item 5 in Figure 1.

### *Administrations and Scoring*

For the main study, participants completed the Content Instrument as the pretest during August–September 2001 and as the posttest during December 2001. Students were scheduled individually, in the same way as they were scheduled to complete the Beliefs Survey. Students were paid to complete the Content Instrument and were allowed as much time as needed to complete the test. The same form was used as both the pretest and the posttest.

Test proctors stamped each page of a completed test booklet with a unique random number for that student and that pre/post administration. Once the cover pages with the students' names were removed, these numbers allowed a blind, randomly ordered scoring with respect to treatment, MATH 210 instructor, and test-administration time (pre or post).

The pretests and posttests were scored during January 2002 by four pairs of paid scorers. Scorers were familiar with the content of MATH 210, mathematics education issues, or both and hence were qualified to exercise the judgments needed for scoring with rubrics. None was familiar with the thrusts of IMAP before the scoring sessions, beyond knowledge that IMAP had something to do with MATH 210.

Each day, one of the content-test team trained each pair of scorers on the rubrics for one page of the Content Instrument by explaining the rubrics and then through a mutual scoring of five tests. The scorers next further refined their training through scoring another 20 tests to reach a high degree of cross-scorer reliability. Each scorer then marked a set of 30 tests, with a random 10 in common with the set of the other scorer. The two scorings of the common 10 were then compared, with a target of at least 80% agreement as an index of reliability. Most agreement percentages were in the 90%–100% range; when an agreement percentage fell below an acceptable rate, further discussion of some responses to the troublesome item was used to restore a higher percentage of agreement. One of the content-test team was available to help with any problematic scoring of an unanticipated response that did not seem to fit any rubric category and to determine whether or how to modify the rubric to accommodate such cases.

For the MATH 313 study, administration and scoring followed the same procedures. Pretests for this study were administered in January–February 2002 and posttests in May–June 2002. Four of the eight content-test scorers from the main study scored these tests in June 2002. MATH 313 students are nearing the end of their undergraduate programs and have very demanding schedules. Consequently, the scale of the MATH 313 study was much smaller: 10 CMTE students and 23 control students, with an attrition of 24 students who completed the pretest. The content of MATH 313 is also less directly related to that tested by the Content Instrument than is the content of the first course. Furthermore,

the backgrounds of the MATH 313 students are quite mixed, with many students transferring from community colleges. Accordingly any statistical inferences would have dubious value.

*Weighting the items.* The rubric scores are probably at best ordinal scores, with a higher score representing a judgment of a higher degree of conceptual focus in the respondent's answer. Rubrics for some items are on a 3-point (0–2) scale, with those for others on 4-point (0–3) or 5-point (0–4) scales. Furthermore, the items are not considered to be equally important, just as items on an examination might not be weighted equally. Hence, it was necessary (a) to rescale the rubrics to create a more uniform scale across rubrics and (b) to weight the items so that scores could be summed in a meaningful fashion, much as one does in weighting items on examinations. Accordingly a panel of four mathematics educators met several times to rescale the rubrics for uniformity and to assign weights to each item. Appendix B has the panel's final decisions on rescaling the rubrics and weighting the items. Objectively scored items were usually weighted 1; rubric-scored items were usually weighted 2; some objectively scored items were weighted 2 because correct responses were deemed to require considerable reasoning; and three items (one dealing with importance of the unit in fraction work and two with the ability to write a story problem that could be represented by a given mathematical expression) were adjudged by the panel to be particularly important for teachers and were weighted 3. These rescalings and weightings gave 82 as the maximum possible score for the Content Instrument (as augmented by the content items from the Beliefs Survey).

### Analyses and Results

*Main study.* Complete data were available for 159 college students. All 159 were concurrently enrolled in the first mathematics course (MATH 210) for prospective elementary school teachers, with 77 in the Children’s Mathematical Thinking Experience (CMTE) experimental group, and 82 in nonexperimental groups (48 making organized visits to classrooms and 34 with no special extra-course experiences). The strong theoretical and pilot-based rationale lead to a one-directional hypothesis to be tested:

$$H_1: \text{CMTE} \leq \text{non-experimental}$$

$$H_2: \text{CMTE} > \text{non-experimental}$$

with the means referring to the means in the content instrument change scores. The combined CMTE treatment subgroups showed higher average pretest-to-posttest changes for the Content Instrument than did the combined nonexperimental classroom-visitation and control subgroups (see Table 1); a *t*-test of these change scores for the two groups gave a significance level of 0.0497,  $t(157) = 1.66$ . Hence, the hypothesis  $\text{CMTE} \leq \text{non-experimental}$  was rejected at the  $\alpha = 0.05$  level in favor of the alternative,  $\text{CMTE} > \text{non-experimental}$ .

Table 1.

#### Mean Change Scores, by Treatment, Main Study

Treatment	<i>n</i>	Pretest mean	Posttest mean	Mean change score	Variance of change scores
<b>CMTE</b>	<b>77</b>	<b>37.6</b>	<b>51.9</b>	<b>14.3</b>	<b>67.5</b>
CMTE-L	50	36.8	51.5	14.7	58.4
CMTE-V	27	39.1	52.6	13.5	86.2
<b>Non-CMTE</b>	<b>82</b>	<b>36.3</b>	<b>48.5</b>	<b>12.2</b>	<b>61.9</b>
MORE-T	25	32.3	44.6	12.3	59.2
MORE-R	23	38.7	49.9	11.2	63.7
Control	34	37.6	50.4	12.8	65.3
All	159	36.9	50.1	13.2	65.3

Notes. Maximum score possible = 82. CMTE = Children's Mathematical Thinking Experience, L = Live, V = Vicarious; MORE = Mathematical Observation and Reflection Experience, T = in a Traditional Classroom, R = in a Reform Classroom.

*MATH 313 study.* Any statistical analyses of the MATH 313 data would be suspect, for the reasons given above. Nonetheless, the means on the pretest (for all students who completed the pretest), the posttest (for students completing the study), and the change scores (for students completing the study) are offered in Table 2. The means do indicate that the CMTE might have a more positive effect on performance on the Content Instrument than does MATH 313 alone.

Table 2. *Means, MATH 313 Study, by Treatment*

Group	Completed pretest ( <i>n</i> = 57)	Completed pre & post tests ( <i>n</i> = 33)	Control ( <i>n</i> = 23)	CMTE ( <i>n</i> = 10)
Pretest	46.1	46.0	45.7	46.8
Posttest		50.4	49.1	53.4
Change		4.4	3.4	6.7

#### *Closing Comments and Speculations*

*Conclusion.* College students concurrently enrolled in a first mathematics course for prospective elementary school teachers and participating in an experience in which the focus is on children’s mathematical thinking, especially as encountered in an interview setting, will, on a test of content for the mathematics of the mathematics course, out-perform other students who do not have such an experience. This latter group of students includes those who visit elementary school classrooms but have neither an explicit focus on children’s thinking nor an interview experience.

*Practical significance?* For the main study, the statistical results alone disguise the fact that the mean change score for the experimental group, 14.3, was only 2.1 points higher than the mean change score for the nonexperimental group, 12.2, on an 82-point test. Although that difference translates into a change score for the experimental group 17% better than that for the nonexperimental group, IMAP investigators were expecting a greater difference. An effect size only on the order of 0.25 (Cohen’s  $d = 0.2645$ ) was unexpected. By ignoring the very positive results from the Beliefs Survey, one might even argue that such a small improvement in the content score is not worth the time and effort of a CMTE.

*Authentic teaching/assessment?* Possibly there are no greater practically significant treatment effects on content learning than the study found. Or possibly the Content Instrument, made up as it was of relatively standard test questions, was not sensitive enough to detect greater differences. Consider the latter possibility.

Although the CMTE was somewhat beneficial in terms of content knowledge, it may be that a greater benefit of the CMTE was to help broaden students' abilities to navigate the mathematical landscape—Where is the child on this topic? What should one do next? What might one do if something unanticipated happens? How might one follow up a correct solution? Might a different representation be helpful? That is, the CMTE students grew in their knowledge of content in the interface with the child. (Hypothesized) growth in such areas would be shown in responses to different questions from most of those in the Content Instrument.

The IMAP situation may also be analogous to the (predicted) different responses by a would-be physician to certain facets of his or her preparation, as in Table 3. Physicians have finished or nearly finished their preparation by the time they are performing particular surgeries on real patients, but unfortunately teachers often have had only their own experiences as children and book exposure to the mathematical topics that they do teach.

IMAP's position is that an early interaction with children should make the teacher-in-preparation more attentive and reflective about all the topics in the college mathematics classes, with the result's being a more effective teacher of all such topics. One problem that affects some students' performance in the college mathematics class, however, is that they feel that they "already know the material," because they have already seen most of the topics during earlier schooling, although often at a level beneath that desirable in a teacher (cf. Ma, 1999; Skemp, 1978) and often filtered by a view of school mathematics focused on computational techniques. Such an attitude would naturally militate against a greater attention to much of the focus on conceptual understanding in the mathematics class. One such student might regard his or her ability to calculate  $17 \times 43$ , for example, as satisfactory evidence of *understanding* multiplication, with resistance to questions about what  $17 \times 43$  might model or about when  $17 \times 43$  might be applicable in a story problem. A physician-in-training, on the other hand, no doubt recognizes that a surgical technique requires more than the skills of cutting and sewing or stapling alone and, furthermore, that an appendectomy, for example, demands more knowledge than removing an ingrown toenail.

Table 3. *A Possible Analogy of Preparation of a Surgeon and a Teacher*

Preparation for surgical technique X	Preparation for teaching topic X (IMAP)
Classes on anatomy, physiology, surgical techniques, including X	Mathematics classes that include topic X (as with the control group in the study)
Watching an <i>average</i> surgeon in action, but not necessarily with technique X	Watching a <i>traditional</i> teacher in action, but not necessarily with topic X (MORE-T)
Watching an <i>expert</i> surgeon in action, but not necessarily with technique X	Watching a <i>reform</i> teacher in action, but not necessarily with topic X (MORE-R)
Studying videos of expert surgeons with technique X, within a class	Studying class videos of expert teachers/ interviewers with topic X (CMTE-V)
Performing technique X on a real patient, with feedback from an expert	Interviewing a child on topic X (CMTE-L), with feedback from an expert
Performing technique X in an emergency room or triage situation	Teaching topic X to a larger group

Unlike the situation with physicians in training, perhaps the college students' fascination with the children they are interviewing (or the children in videos of interviews) overrode any thoughts about their own needs for a sound knowledge of the mathematics. Some students concurrently enrolled in Math 210 and a CMTE have offered statements that such is NOT the case, as in the following:

A lot of the time in [the Math 210] class, like, people get mad and so frustrated as to why they're learning what they're learning—and then you come here [to the CMTE] and you see a kid doing exactly what you're learning in class. It just makes sense. It eliminates that whole frustration of like, “Why am I learning this? Where am I going to use it?” So by taking this class, you see how they actually—the children actually apply what you're learning. ... You're not wasting your time in Math 210.

Because of such reactions, we do not wish to claim that the CMTE students are losing sight of the mathematics. Nonetheless, the tremendous variation in change scores (shown in Figure 3) indicates that among students in the content course, reactions to the mathematics differ vastly. Change scores for individual students ranged from more than +39 to -3 (i.e., one student performed 3 points worse on the posttest than on the

pretest, and the student was not among the high scorers on the pretest). But perhaps the allure of interviewing a child—something novel and closer to *teaching*—blinds the student to the fact that he or she will eventually be *teaching some particular content*. Early assessments that show the need for strong content knowledge during teaching might better pave the way to a greater improvement in content knowledge.

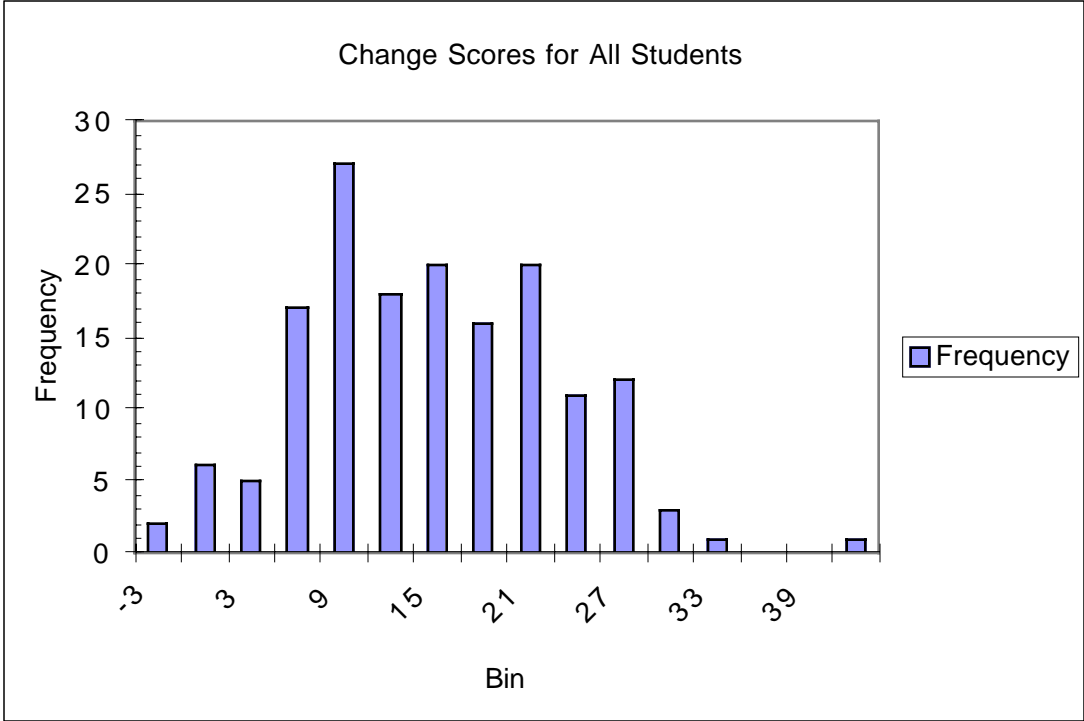


Figure 3. The distribution of change scores for the content instrument (for all students who completed the study).

The possibility that the particular items on the Content Instrument were not sensitive enough to reveal actual treatment effects should also be considered. Pre- and post-performances for the items on the Content Instrument are included in Appendix C. Surprisingly, some items showed virtually no improvement from pretest to posttest, and performance on one item (Jan, in Item 1) actually declined! We are led to the following recommendation.

*Study the contributions of a mathematics course more carefully.* The content score for the control group students, the ones who had only the mathematics course and nothing else, grew from 37.6 on the pretest to 50.4 on the posttest, a gain of 12.8 points out of 82. Is an improvement from about 46% to about 61% the best that such a course can accomplish? Again, such a mild improvement may reflect only a limitation of the particular items and the mode of testing or even the particular course offered, but it may also be the case that such improvement is typical of such courses. It is, of course, natural not to have or to find pretest/posttest designs for college courses, so we have no bases for comparison. Although the Math 313 data are limited, note that the entering Math 313 students, with an additional two mathematics courses (geometry/measurement and probability/statistics) beyond the first content course, scored only 46.1. (Control students beginning the first course averaged 37.6 and finished the course with a mean of 50.4.) In these studies, the relatively low improvement indicates that perhaps educators should more closely examine what such courses do accomplish, so that the community can compare results. A nationwide project in which pretest and posttest data are collected and the results studied would be valuable. At a minimum, statewide or regional meetings could address the issue *What do courses for prospective elementary school teachers accomplish in the way of increasing content knowledge?*

*Post-study result at SDSU.* The all-university committee that oversees the SDSU major in which the prospective teachers are enrolled has studied IMAP's impressive results on the Beliefs Survey, along with the less impressive but somewhat promising results on the Content Instrument. That committee was grateful to have data on which to base a decision, rather than to be guided by the usual opinion, anecdote, or face validity as guides for program changes. Consequently the committee decided, with the strong concurrence of the mathematics department, to require a Children's Mathematical Thinking Experience (CMTE) course to accompany the first mathematics course for the prospective elementary school teachers. In addition, an interview possibility has been set up in the other mathematics courses for prospective teachers.

In an ideal world, the content course and the CMTE would likely be integrated into one course, but the usual SDSU instructors for the first mathematics course often do not have a sufficiently rich backgrounds in work with children or with practices and curricula in elementary schools to offer a CMTE. Instructors for the content courses on other campuses may well have the expertise to incorporate CMTE-like work into their courses, and we IMAP researchers hope to hear about their assessments of such an integration.

## References

- Harel, G. (1998). Two dual assertions: The first on learning and the second on teaching (or vice versa). *The American Mathematical Monthly*, 105, 497–507.
- Ma, L. (1999). *Knowing and teaching elementary mathematics*. Mahwah, NJ: Erlbaum.
- Philipp, R. A., Thanheiser, E., & Clement, L. (2002). The role of a children's mathematical thinking experience in the preparation of prospective elementary school teachers. *International Journal of Educational Research*, 37, 195–210.
- Skemp, R. R. (1978). Relational understanding and instrumental understanding. *Arithmetic Teacher*, 7, 9–15.

citation on effect size (still none added. Needed?)

Appendix A1. Content Instrument Items

Appendix A2. Content Instrument Scoring Rubrics

Appendix B. Rescaling the rubrics for content-test items.

Appendix C. Item statistics, pretest and posttest.

---

Starting time \_\_\_\_\_ Ending time \_\_\_\_\_ Date \_\_\_\_\_

Last name \_\_\_\_\_ First name \_\_\_\_\_

MATH 210 class (days, time) \_\_\_\_\_

*San Diego State University faculty regard teacher preparation as very important. Your answers to the following items will help us to plan the work in the mathematics part of your program of study.*

Experienced mathematics teachers are able to understand children's thinking, whether the thinking is correct, partially correct, or incorrect. One purpose for this survey is to assess your ability to understand children's thinking. In order for us to develop the clearest sense of your understanding, please answer all of the items. (A test proctor may check to see whether you have answered all the items.) Please work carefully.

**This survey was designed to be completed WITHOUT use of calculators. Show all your work on the pages of questions.**

1. A teacher gave her class the challenge to find how many ways the number **423.1** could be thought about. Following are four children's answers. For each answer, mark whether it is correct or incorrect. If it is incorrect, please explain.

**Dale's answer:** 423.1 could be thought about as 42,310 hundredths

- a) Is Dale's answer correct or incorrect? Correct \_\_\_ Incorrect \_\_\_  
b) If Dale's answer is incorrect, please explain the error.

**Pat's answer:** 423.1 could be thought about as 400 ones and 23.1 tenths

- a) Is Pat's answer correct or incorrect? Correct \_\_\_ Incorrect \_\_\_  
b) If Pat's answer is incorrect, please explain the error.

**Lesley's answer:** 423.1 could be thought about as 41 tens, 12 ones, and 11 tenths

- a) Is Lesley's answer correct or incorrect? Correct \_\_\_ Incorrect \_\_\_  
b) If Lesley's answer is incorrect, please explain the error.

**Jan's answer:** 423.1 could be thought about as 420 tens and 31 tenths

- a) Is Jan's answer correct or incorrect? Correct \_\_\_ Incorrect \_\_\_  
b) If Jan's answer is incorrect, please explain the error.

2. Following is an example of a child's work. You are to study the work and then to judge the student's understanding.

**Hiro** was asked to divide 4240 by 6. His work is shown below.

Hiro's work:  $\underline{76}R4$

$$\begin{array}{r} 6 \overline{)4240} \\ \underline{42} \phantom{0} \\ 040 \\ \underline{36} \phantom{0} \\ 4 \phantom{0} \end{array}$$

- a) Is Hiro's work correct or incorrect? Correct \_\_\_ Incorrect \_\_\_
- b) If the work is incorrect, please explain how.

3. Following is an example of a child's work. You are to study the work and then to judge the student's understanding.

**Rona** was asked to subtract  $2\frac{5}{8}$  from  $4\frac{1}{8}$ . Her work is shown below.

Rona's work:  $4\frac{1}{8} = 3\frac{11}{8}$

$$\begin{array}{r} -2\frac{5}{8} = -2\frac{5}{8} \\ \hline 1\frac{6}{8} \end{array}$$

- a) Is Rona's work correct or incorrect? Correct \_\_\_ Incorrect \_\_\_
- b) If the work is incorrect, please explain how.

4. Following are **two** students' solutions to  $32 \times 54$ .

- a) For each student, first evaluate the student's mathematical reasoning. Check whether the steps are mathematically correct or flawed, or indicate that you cannot tell.
- b) Next mark each student's work with "doesn't appear to understand multiplication," "may or may not understand multiplication," or "shows good understanding of multiplication."

**Fiesha:** 
$$\begin{array}{r} 54 \\ \times 32 \\ \hline 1500 \\ 120 \\ 100 \\ \hline 8 \\ \hline 1728 \end{array}$$

a) Mathematical steps

<input type="checkbox"/> Fiesha's steps are mathematically correct.	<input type="checkbox"/> Fiesha's steps are mathematically flawed.	<input type="checkbox"/> I cannot tell if Fiesha's steps are mathematically correct or flawed.
---	--	--

b) Understanding of multiplication

<input type="checkbox"/> Fiesha doesn't appear to understand multiplication.	<input type="checkbox"/> Fiesha may or may not understand multiplication.	<input type="checkbox"/> Fiesha shows good understanding of multiplication.
--	---	---

- c) If Fiesha's steps are mathematically correct, use her way of thinking to solve  $24 \times 53$ . If they are not, explain how Fiesha's reasoning is flawed.

**Amy:** 54 is 4 more than 50, so find  $32 \times 50$  and add 4 back to get 1728.

a) Mathematical steps

<input type="checkbox"/> Amy's steps are mathematically correct.	<input type="checkbox"/> Amy's steps are mathematically flawed.	<input type="checkbox"/> I cannot tell if Amy's steps are mathematically correct or flawed.
--	---	---

b) Understanding of multiplication

<input type="checkbox"/> Amy doesn't appear to understand multiplication.	<input type="checkbox"/> Amy may or may not understand multiplication.	<input type="checkbox"/> Amy shows good understanding of multiplication.
---	--	--

- c) If Amy's steps are mathematically correct, use her way of thinking to solve  $24 \times 53$ . If they are not, explain how Amy's reasoning is flawed.

5. Antonio asks, "When I multiply [for example,  $49 \times 23$ , shown to the right], why do I have to put in the 0 [points to the zero in 980]?"

What would you say to Antonio?

$$\begin{array}{r} 49 \\ \times 23 \\ \hline 147 \\ 980 \\ \hline 1127 \end{array}$$

6. a) Circle the larger number:      0.4              0.36
- b) Grady thinks that 0.36 is bigger than 0.4 because 36 is bigger than 4. Comment on Grady's reasoning.
- c) Judy says, "Well, hundredths are smaller than tenths. So 0.36 is smaller than 0.4." Comment on Judy's reasoning.

7. Donny is asked to solve the following problem: If cheese is \$1.89 per pound, how much does 0.67 pound cost?

Circle the expression that correctly represents the problem.

- A.  $1.89 + 0.67$       B.  $1.89 - 0.67$       C.  $0.67 - 1.89$   
D.  $0.67 \times 1.89$       E.  $1.89 \div 0.67$       F.  $0.67 \div 1.89$   
G. None of A–F

8. Is Ardis's reasoning correct? Explain.

Shannon: "I still have half my spelling words to learn and  $\frac{3}{4}$  of my vocabulary words to learn."

Ardis: "Well,  $\frac{3}{4}$  is more than a half because  $\frac{1}{2} = \frac{2}{4}$ . So, you have more vocabulary words than spelling words still to learn."

9. Write a real-life story problem that could be represented by the expression  $\frac{1}{2} - \frac{1}{3}$ .

10. Following are **three** students' solutions to  $\frac{9}{16}$  of 48.

- a) For each student, first evaluate the student's mathematical reasoning. Check whether the steps are mathematically correct or flawed or indicate that you cannot tell.
- b) Next mark each student's work with "doesn't appear to understand," "may or may not understand," or "shows good understanding of" multiplication of fractions.

**Jessica:**  $\frac{9}{16}$  is  $\frac{1}{16}$  more than  $\frac{1}{2}$ , and half of 48 is 24 and  $\frac{1}{16}$  of 48 is 3, so 27.

a) Mathematical steps

<input type="checkbox"/> Jessica's steps are mathematically correct.	<input type="checkbox"/> Jessica's steps are mathematically flawed.	<input type="checkbox"/> I cannot tell if Jessica's steps are mathematically correct or flawed.
--	---	---

b) Understanding of multiplication of fractions

<input type="checkbox"/> Jessica doesn't appear to understand multiplication of fractions.	<input type="checkbox"/> Jessica may or may not understand multiplication of fractions.	<input type="checkbox"/> Jessica shows good understanding of multiplication of fractions.
--	---	---

- c) If Jessica's steps are mathematically correct, use her way of thinking to solve  $\frac{3}{8} \times 32$ . If they are not, explain how Jessica's reasoning is flawed.

**Justin:**  $\frac{9}{16} \times 48 = \frac{9}{\cancel{16}^3} \times \frac{48}{1} = \frac{27}{1} = 27$

a) Mathematical steps

<input type="checkbox"/> Justin's steps are mathematically correct.	<input type="checkbox"/> Justin's steps are mathematically flawed.	<input type="checkbox"/> I cannot tell if Justin's steps are mathematically correct or flawed.
---	--	--

b) Understanding of multiplication of fractions

<input type="checkbox"/> Justin doesn't appear to understand multiplication of fractions.	<input type="checkbox"/> Justin may or may not understand multiplication of fractions.	<input type="checkbox"/> Justin shows good understanding of multiplication of fractions.
---	--	--

**Stacy:**  $\frac{9}{16}$  of 48?  $9 \times 16 = 144$ ;  $144 \div 48 = 3$ ;  $9 \times 3 = 27$ . So  $\frac{9}{16}$  of 48 is 27.

a) Mathematical steps

<input type="checkbox"/> Stacy's steps are mathematically correct.	<input type="checkbox"/> Stacy's steps are mathematically flawed.	<input type="checkbox"/> I cannot tell if Stacy's steps are mathematically correct or flawed.
--	---	---

b) Understanding of multiplication of fractions

<input type="checkbox"/> Stacy doesn't appear to understand multiplication of fractions.	<input type="checkbox"/> Stacy may or may not understand multiplication of fractions.	<input type="checkbox"/> Stacy shows good understanding of multiplication of fractions.
--	---	---

11. Betty was asked to give her best estimate for 25% of 7991.8.

She estimates by taking  $\frac{1}{4}$  of 8000, which is 2000, and taking  $\frac{1}{4}$  of .8, which is .2.  
Therefore, her estimate is 2000.2

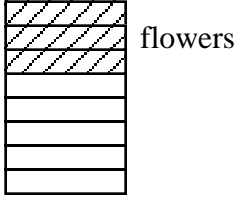

Comment on Betty's reasoning.

12. a) Give two decimals between 2.3456 and 2.3457. If it is not possible, explain why not.

b) Give two fractions between  $\frac{2}{5}$  and  $\frac{3}{5}$ . If it is not possible, explain why not.

13. Finish the following story problem so that your question could be answered by the calculation  $18 \div \frac{1}{2}$ : "You buy 18 muffins for an after-school faculty meeting . . . ."

14. Bill and Tom have pieces of land that are the same size. Each plants flowers on part of his land, as shown in the sketches.

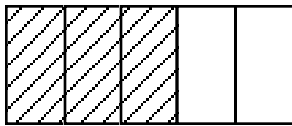
The shaded part shows Bill's flowers.	The shaded part shows Tom's flowers.
	

Who has more land in flowers?

- A. Bill and Tom have the same amount of land in flowers.
- B. Bill has more land in flowers.
- C. Tom has more land in flowers.
- D. One cannot determine who has more land in flowers.

**Explain your choice.**

15. Pat and Dana like to argue with each other about mathematics problems. They discuss the figure below:

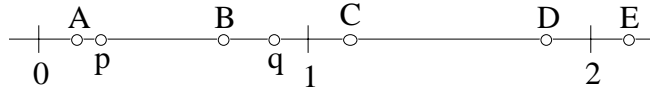


*Pat:* The shaded region is one-and-a-half times as much as the unshaded region.

*Dana:* Wait! I think that the unshaded region is  $\frac{2}{3}$  of the shaded region.

Who is correct? Why?

16. For this number line



- a) the point for  $p + q$  would be closest to A? B? C? D? E? (circle)
- b) the point for  $p \times q$  would be closest to A? B? C? D? E? (circle)

17. Here is Ben's work:

$$\begin{array}{r} 2 \\ \cancel{3} \ 0 \overset{1}{2} \\ - \quad \underline{9} \\ 2 \ 0 \ 3 \end{array}$$

$$\begin{array}{r} 3 \\ \cancel{4} \ 0 \overset{1}{7} \\ - \quad \underline{1 \ 0 \ 8} \\ 2 \ 0 \ 9 \end{array}$$

- a) Make up another subtraction problem that would lead Ben to apply his same method. Then finish the calculation as Ben would. Show the work Ben would do.
- b) What might you do next on this topic, with this child? (Suggest specific possible options.)

18. A child says, "My mom says to put = between  $\frac{3}{4}$  and  $\frac{6}{8}$ , but I think  $\frac{6}{8}$  is bigger."

- a) How might the child be thinking?
- b) What might you do next on this topic, with this child? (Suggest specific possible options.)

19. You ask a child to find the value of  $8 - .5$ . The child's answer is 3.

a) How might the child be thinking?

b) What might you do next on this topic, with this child? (Suggest specific possible options.)

20. Which of A–E gives the order of the following decimals, from smallest to largest? \_\_\_\_\_

0.008      0.01      0.06387      0.00589346

A. (smallest) 0.00589346    0.008            0.01            0.06387    (largest)

B. (smallest) 0.008            0.00589346    0.01            0.06387    (largest)

C. (smallest) 0.00589346    0.008            0.06387    0.01            (largest)

D. (smallest) 0.01            0.06387            0.008            0.00589346 (largest)

E. None of A–D. The order should be \_\_\_\_\_.

21. A soap factory packs 100 bars of soap in each box for shipment. If the factory makes 15,287 bars of soap, how many **full** boxes will they have for shipment? \_\_\_\_\_

22. Which of A–E gives the order of the following numbers, from smallest to largest? \_\_\_\_\_

0.156       $\frac{1}{10}$       0.5      0.009       $\frac{5}{8}$

A. (smallest) 0.009            0.156            0.5             $\frac{1}{10}$              $\frac{5}{8}$     (largest)

B. (smallest) 0.009             $\frac{1}{10}$             0.156            0.5             $\frac{5}{8}$  (largest)

C. (smallest) 0.009             $\frac{1}{10}$             0.156             $\frac{5}{8}$             0.5 (largest)

D. (smallest)  $\frac{1}{10}$              $\frac{5}{8}$             0.009            0.156            0.5 (largest)

E. None of A–D. The order should be \_\_\_\_\_.

23. Without actually doing the calculations, put the decimal points in the answers. Briefly explain your thinking.

a)  $77.5 \times 2.84 = 2201$  Explanation:

b)  $100.26 \div 3.6 = 2785$  Explanation:

24.  $234_{\text{five}} + 321_{\text{five}} = \underline{\hspace{2cm}}$  in base five.

25. **Estimate** the sum of  $\frac{12}{13} + \frac{7}{8}$ . \_\_\_\_\_

26. For each of the following numbers, **circle the larger** or write = if they are equal.

a)  $\frac{5}{7}$                        $\frac{5}{9}$

b)  $1$                                $\frac{5}{4}$

c)  $\frac{3}{7}$                                $\frac{2}{7}$

d)  $\frac{3}{6}$                                $\frac{1}{2}$

27. Here is the approach that Ariana used to solve the problem  $635 - 482$ .

$$\begin{array}{r} 635 - 400 = 235 \\ 235 - 30 = 205 \\ 205 - 50 = 155 \\ 155 - \underline{2} = 153 \\ \hline 482 \end{array}$$

Ariana explains, "First I subtracted 400 and got 235. Then I subtracted 30 and got 205, and I subtracted 50 more and got 155. I needed to subtract 2 more and ended up with 153."

Describe how Ariana would solve this item:  $700 - 573$ .

28. Below is the work of Terry, a student who solved one addition and one subtraction problem.

$$\begin{array}{r} \text{A)} \quad 2 \overset{1}{5} 9 \\ + \quad 38 \\ \hline 297 \end{array}$$

$$\begin{array}{r} \text{B)} \quad \overset{3}{4} \overset{1}{2} 9 \\ - \quad 34 \\ \hline 395 \end{array}$$

In the addition problem, Terry wrote a 1 higher than the 5. In the subtraction problem, Terry wrote a 1 next to and slightly higher than the 2.

- a) Does each 1 in these problems represent the same thing? Please explain your answer.
- b) Terry knows that when adding in Problem A, he adds 1 to the 5 and when subtracting in Problem B, he adds 10 to the 2, but he doesn't know why. Can you explain the reasons to Terry?

## Appendix A2: SCORING OF THE CONTENT INSTRUMENT

For each correct/ incorrect item, we give 1 point for the correct answer and 0 points for an incorrect answer. No answer for correct/incorrect is coded as *nd* (no data). For all other items, rubrics are listed below—often with sample answers (**unedited**); *nid* for no identifiable data is used occasionally.

### **ITEM 1 Four ways to express 423.1**

**(a) Content score** Jan's work and Pat's work are incorrect. Each part is scored as a correct/incorrect answer.

Dale: 42.310 hundredths	Dale correct = 1, incorrect = 0;	C=1
Pat: 400 ones and 23.1 tenths	Pat correct = 0, incorrect = 1;	I=1
Les: 41 tens, 12 ones, and 11 tenths	Lesley correct = 1, incorrect = 0;	C=1
Jan: 420 tens and 31 tenths	Jan correct = 0, incorrect = 1;	I=1

Explanations are scored **only** for Pat and Jan

### **(b) Explanation scores for Pat (0–1 scale)**

**Explanation score 1** (Uses place value explicitly and correctly in explanation; explanation correct but seems rule-based; gives correct explanation but adds something incorrect.)

- 23.1 tenths is only 2.31 ones
- There would be 231 tenths.

**Explanation score 0** (Thought original work was correct; explanation literally okay but does not address error; nonsense.)

- It would have to be 423.1 tenths.
- 423 ones and 1 tenth.

### **(b) Explanation scores for Jan (0–1 scale)**

**Explanation score 1** (Uses place value explicitly and correctly in explanation; explanation correct but seems rule-based.)

- 420 tens = 4200, incorrect place value.
- Only 42 tens
- There are not 420 tens in 423; there would be 420 ones.

**Explanation score 0** (Thought original work was correct; vague; nonsense.)

- 420 is not the tens place and 31 has a decimal between the 3.1 so it is not the 31 tenth place value.
- There is only one tenth.
- This is too much.

**ITEM 2 Hiro's division of 4240 by 6, with the answer 76 R4**

**(a) Content score** (Hiro is incorrect = 1; Hiro is correct = 0) **If 0, then (b) Explanation score = 0.**

**(b) Explanation score (0–3 scale)**

**Explanation score 3** (Explanation shows understanding beyond algorithmic; estimates or uses multiplication to check; explanation uses number sense, even if additional information is not completely correct.)

- Although he is using the algorithm that is common he really isn't using common sense b/c lets say we round up 7 really high to 100 ( $6 \times 100$ ) is nowhere near 4240.
- Calculates  $6 \times 76$
- (algorithm( & He may have realized (the answer was incorrect) if he looked at the number in the equation and his answer.

**Explanation score 2** (Does, or refers to, usual algorithm only.)

- He forgot to put a zero after the 7 when 6 wouldn't divide into 04.
- He forgot to put a zero after the seven because since 4 could fit in 6, he had to carry down the zero.
- Brought down 0 with the 4
- He forgot to write down a zero. The answer should have been 706 R4.

**Explanation score 1** (Seems to accept given incorrect answer.)

- He just isn't placing his values (76 R4) correctly on the problem
- He miss placed the 7 and put it in the wrong place making it a different #. He know what he is doing but it just a little confusing.
- Respondent's only criticism is "not lined up."

**Explanation score 0** (Imitates the incorrect solution or thinks original is correct.)

- Does the division and makes the same mistake

**ITEM 3 Rona incorrectly subtracts  $2\frac{5}{8}$  from  $4\frac{1}{8}$ .**

**(a) Content score** (incorrect = 1; correct = 0) **If 0, then score for (b) = 0.**

**(b) Explanation score (0–2 scale)**

**Explanation score 2** (Recognizes error in thinking and realizes how error could have come about; links renaming [borrowing, converting,...] in whole numbers to  $11/8$  error, as opposed to arguing that child added  $8/8 + 1/8$  incorrectly.)

- Because the  $5/8$  is bigger than the  $1/8$  and you are subtracting it. So you borrow from the 4 and whole 1 or  $8/8$ . When you add  $1/8 + 8/8 = 9/8$ , not  $11/8$ . Her miss calculation set the problem off by a whole fourth.
- Her thinking is correct but the  $4\frac{1}{8}$  would change to  $3\frac{9}{8}$  not  $3\frac{11}{8}$  because you would add  $8/8$  or 1.
- $1 = 8/8$ , not  $10/8$
- She knew she had to borrow, but instead of getting  $9/8$ ths she got  $3\frac{11}{8}$ ths.
- Recognizes via comparing improper fractions that  $4\frac{1}{8} \neq 3\frac{11}{8}$  (as opposed to just noting  $4\frac{1}{8} = 3\frac{9}{8}$ ).
- Attempt is logical, but  $33/8 \neq 35/8$ .

**Explanation score 1** (Uses different algorithm, or does problem and just compares answer with Rona's [answer seems to be the focus]; recognizes error in thinking but gives only some indication of the nature of the error; or not clear that recognizes  $10/8$  error.)

- $4\frac{1}{8}$  is  $3\frac{9}{8}$  because  $4 - 1 + 8/8 = 3\frac{1}{8} + 8/8 = 3\frac{9}{8}$ .
- Because this fraction (referring to  $11/8$ ) should have been  $9/8$ .
- $4\frac{1}{8} = 3\frac{9}{8}$  not  $3\frac{11}{8}$
- She should of converted the problem into improper fractions – PST also does that to get right answer.

**Explanation score 0** (Knows [explicitly or implicitly] that it is wrong but cannot solve correctly; does not indicate nature of the error; or nonsense.)

- It should look like  $3\frac{8}{8} - 2\frac{5}{8}$ , when she borrowed she did not correctly change the fraction.
- What you must do to one fraction you must do to the other.
- He changed  $4\frac{1}{8}$  into  $3\frac{11}{8}$  I'm not sure why.
- He borrowed from the 4 to get  $11/8$  but he didn't put enough onto the fraction. It's not like regular subtraction.
- You can't have a mixed fraction that is improper.
- You cannot borrow from the whole number to subtract a fraction.

**ITEM 4 Fiesha’s and Amy’s computations for 32 x 54 (not using standard algorithm)**

**Fiesha**

**(a, b) Content score** (See below for scoring if one part omitted.)

<b>a)</b>	<b>C</b> reasoning correct	<b>F</b> reasoning is flawed	<b>CT</b> cannot tell
<b>b)</b>	<b>N</b> doesn’t appear to understand	<b>M</b> may or may not understand	<b>U</b> shows good understanding

**Scoring if Part a or Part b omitted**

Fiesha a. C only = 1            F,CT only = 0  
 b. U only = 1            F,M only = 0

**Scoring for Fiesha**

a)	b)	SCORE
C	U	4
C	M	3
CT	M	2
CT	N	1
F	N	1
F	M	1
C	N	0
CT	U	0
F	U	0

**(c) Explanation score (0–3 scale) for Fiesha**

**Explanation score 3** (All four partial products [1000, 200, 60, 12] & final answer [1272] okay; okay if order of partial products is different from Fiesha’s order; a minor calculational error such as  $4 \times 3 = 11$  is o.k. as long as conceptually wanting to multiply the correct numbers.)

**Explanation score 2** (Some but not all four partial products okay, but final answers okay—some partial products combined.)

**Explanation score 1** (Unable to solve as Fiesha did; shows some understanding of algorithm; can explain where at least one of the numbers came from, such as  $50 \times 30$ .)

**Explanation score 0** (Explanation shows misunderstanding, or rejects algorithm because it is not the same sequence as the standard algorithm; or comes up with algorithm that “puts zeros in the number”)

- I do not understand where she got those numbers.
- She is forgetting to use place values and add 0 to the multiplication therefore she is coming up with a different answer

• “wrong algorithm used”

1000	(5 x 2 and 2 zeros)
600	(2 x 3 and 2 zeros)
200	(4 x 5 and 1 zero)
12	(3 x 4 and no zeros)
1812	

## Amy

**(a, b) Content score** (See below for scoring if one part omitted.)

**Scoring for Amy**

a)	b)	SCORE
F	N	4
F	M	3
CT	M	2
C	N	0
C	M	0
CT	N	0
C	U	0
F	U	0
CT	U	0

### **Scoring if Part a or Part b omitted**

Amy a. F only = 1            C, CT only = 0  
b. N only = 1            M,U only = 0

### **(c) Explanation score (0–3 scale) for Amy**

**Explanation score 3** (Sees defect in Amy’s method, knows how to correct)

- Amy needs to add  $32 \times 4$ , not just 4.
- $32 \times 50$  is okay, but then need  $32 \times 4$ .
- IF CLEAR, INTERPRET “add 4 back” to mean “add 4 32s back,” then rescore a, b as with Fiesha.

**Explanation score 2** (Good argument but states that you need 4 grps of 50 instead of 4 grps of 34; asks 4 what? somehow indicates that it is 4 of something that needs to be added back.)

**Explanation score 1** (Calculates and recognizes that Amy’s answer is off but does not clearly explain that it is  $34 \times 4$ ; states something like “You cannot add in multiplication”; is too general.)

**Explanation score 0** (Thinks okay, gets  $1200 + 3 = 1203$ .)

## **ITEM 5 Antonio asks, “Why is the 0 in multiplication algorithm?”**

### **No content score!**

### **Explanation score (0–2 scale)**

**Explanation score 2** (Algorithmic with very good understanding OR simply good understanding; uses place-value language with good explanation; suggests that one is multiplying 20 times 49 and not 2 times 49; states and gives good reason that the 98 is really 980.)

- When you got the 980 you were multiplying 20 times 49—however it’s easier to think of  $2 \times 49$  and just add zero to make it 20.
- I would say you have to put the zero in because otherwise the numbers wouldn’t be in the correct place. When you are multiplying 49 times 2, the 2 is actually 20 so the zero needs to go in for correct place value.
- The 2 is in the tens place, the 3 is in the ones place. You first multiply  $49 \times 3$ ; that gets you 147. Now you do the same for the 2 but since the 2 is in the tens place it’s the same as  $49 \times 20$  that’s why you bring down the 0.

**Explanation score 1** (Uses place-value language or names the places mechanically; does NOT clearly state that 2 times is really 20 times, OR states 20 but then makes an error. In this category the 0 is seen as a place holder for the one’s place; a reference made to consider the 3 in 23; states that the 98 is really 980, without giving good reason.)

- Because you need something to replace the ones place.
- Because that zero (ref. to a zero that he drew over the 3 in 23) is taking place of the three.
- You put the zero there to show that you are not adding any numbers in the ones place, the zero allows you to know that even though there are no ones, that place value is still there, acting like there is so you don’t put numbers in their wrong place values.
- You have to take into consideration the 3 in 23.
- Ok, the reason you put a 0, is because you are moving from ones place (1) to the tens place (10.) the difference between 1 and 10 is the zero.
- Now we are working the tenths place rather than the ones place therefore you need to move the value over 1 and add a 0.
- Because the #s we are multiplying are in the 10<sup>th</sup> place (the #2) and we have to account for that by adding the zero so we know the 98 is from the 10<sup>th</sup> place.

**Explanation score 0** (Shows no understanding; nonsense answer; solely algorithmic; says only that 0 is a place holder.)

- Because every time you finish one row of numbers and move to the next you have to put a 0 first because then the next number can be under that row of numbers.
- Just refers to “place holder,” as in “0 is a place holder.”

## **ITEM 6 Which is larger, 0.36 or 0.4?**

### **(a) Content score was recorded but was not included in final scoring.**

(0.36 is scored 0; 0.4 is scored 1)

### **(b) Explanation score Grady (0–2 scale)**

**Explanation score 2** (Compares 4 tenths and 3.6 tenths OR compares 40 and 36 hundredths; changes to fractions; compares the two numbers conceptually.)

- I understand his reasoning at first glance but the .4 is 4 tenths where the .36 is 3.6 tenths.
- Grady thinks that .36 is bigger than .4. He isn't understanding that there is a decimal there. The decimal shows that the number is less than 1; 4 is closer to 10 than 3.6 is.

**Explanation score 1** (Attempts conceptual explanation but does not quite carry it out; uses whole number reasoning; algorithmic without reference to correct place values—states only something to the effect that the first digit after the decimal should be compared; suggests appending a zero to make the numbers the same length; mentions tenths but not more, compares the two numbers algorithmically; suggests that the value changes if a number is “behind” the decimal place.)

- 36 is bigger than 4 in whole numbers; tenths is bigger than 36 hundredths because they are both pieces of something bigger.
- Grady doesn't understand the place value of numbers that appear before the decimal. He simply assumes that more digits will create a larger number.
- Children need to be taught that when there is a decimal you need to look at the number next to the decimal and see which is the largest.
- I would say that sometimes in decimals like 0.4 its actually .40 and we don't need that extra 0 so we just say .4 but in money sense .36 cents is smaller than .40 cents [true, but respondent probably intended to refer to cents, not hundredths of cents].
- Put in a 0 to make .4 into .40
- She doesn't understand that in decimals the larger # in the 10<sup>th</sup> place will make a larger value regardless of how many #s are in the other #.
- You need to break the numbers Down first and then compare them 0.4 & 0.36. which is larger 3 or 4?
- When he sees the # he looks at the value as it would in tens, hundreds, ones. He doesn't understand what the decimal signifies.
- It is very common. Most kids can't imagine the decimal in terms of tens, hundreds etc. They see “2” #s compared to “1” + they automatically think it is bigger.
- The value of a number changes when you put a decimal point.

**Explanation score 0** (Thinks that Grady's reasoning is okay; does not address the question; or is too general; OR gives wrong reason for Grady's being incorrect.)

- He is wrong but I understand where he would see that. He needs to know that numbers that get longer after the decimal point get smaller.
- ... when decimal is there the number is actually smaller because it is farther from 0 than 4 is.
- Respondent's comment is too general

**(c) Explanation score Judy (0–3 scale)**

**Explanation score 3** (Recognizes that J's reasoning is insufficient; acknowledges importance of place value and total value. Gives counterexample of decimal with digit in hundredth's place but bigger or two decimals with hundredth's places and asks, "What would she do now?")

- I don't know if she understands entirely if she knows that 40 hundredths is more hundredths than 36, then its o.k.
- Hundredths are smaller than tenths, but it depends on how many hundredths you have vs how many tenths.
- .4 has more tenths than .36
- She is wrong, what if she would compare .6 and .69. The second one has a hundredths place but is bigger!
- What about .36 and .37 how would she compare those?

**Explanation score 2** (Recognizes that Judy's reasoning is insufficient; recognizes significance of larger place value [talks about .40; may or may not mention .36]; states that tenths need to be compared; makes .4 into .40; seems algorithmic.)

- Judy has good reasoning but may or may not see that .4 is 40 hundredths which is the same as 4 tenths.
- It is interesting. But it's not bigger b/c of that. The tenths are all that need to be compared in this instance.
- 4 has a hundredths place also. It is a zero.
- Judy is partly correct. She doesn't understand that the digits in the tenths place are more important than the digits in the hundreds place.
- 36 is smaller than .4 but not because hundredths or tenths. The numbers .36 is smaller than .4(0).
- What would she do with .40?

**Explanation score 1** (Recognizes that Judy's reasoning is incorrect, but explanation is lacking; does not explain *why* .36 is smaller than .4; describes how Judy might be thinking, but without comment on how she should be thinking.)

- Judy's reasoning that hundredths are smaller than tenths is correct. However I don't know if she really understands the reason for .36 being smaller than .4
- She has the correct answer but in the wrong place value. Sort of getting the idea though.
- Judy thinks .36 is smaller because it has a hundredths place

**Explanation score 0** (Does not recognize that Judy's reasoning is incorrect.)

- Judy understands place value and realizes how decimals are looked at.
- She knows this and understands place value.
- She is right – hundreds in decimals are smaller than tenths

**ITEM 7 How much does 0.67 lb of cheese cost if 1 lb costs \$1.89?**

**Content score** D is the correct choice (D = 1; all others = 0).

**ITEM 8 Ardis has half of her spelling words and  $\frac{3}{4}$  of her vocabulary words to learn.**

**Content score** Agrees with the statement (yes) = 0; disagrees (no) = 1.

**Explanation score (0–3 scale)**

**Explanation score 3** (Recognizes importance of the unit; states that reasoning is correct if the numbers of spelling and vocabulary words are the same; gives counterexample with numbers to show that reasoning is not correct generally.)

- This is correct if the number of spelling words and vocab words are equal. The reasoning to it is completely correct.
- You could have 100 spelling – so  $\frac{1}{2}$  is 50 to learn. But you could have 12 vocab words so you'd only need to learn 9 words – that's more spelling. The fractions do not have the same units.

**Explanation score 2** (Seems aware of the unit, but explanation is not precise.)

- This is like adding apples and oranges. Spelling and vocabulary cannot be added like that.

**Explanation score 1** (Thinks that the reasoning is correct, focuses on equivalent fractions without referring to the unit; compares  $\frac{2}{4}$  and  $\frac{1}{2}$  with reasonable drawings; thinks *word* is the unit; mentions more generally conversion to same denominator.)

- Because he/she changed it to fourths to see the larger #.
- $\frac{1}{2} < \frac{3}{4}$  yes she is right, she knew that half of 2 = 1 and 4 = 2
- Yes it is because the kid is adding to halves to see how much it gives him. He sees that  $\frac{2}{4}$  is less than  $\frac{3}{4}$  so he has more vocabulary words to learn.
- It is correct b/c they use common denominators to find out and  $\frac{3}{4}$  is more than  $\frac{2}{4}$ . They have a good understanding.
- An easy way for a child to understand fractions is to convert them to have the same denominator & compare.
- He understands to convert to same denominators to make things the same and easier to compare.
- This is a good way to evaluate the problem by changing it to the same term you are then able to see exactly the comparison.

**Explanation score 0** (Shows no awareness of importance of the unit; thinks reasoning is okay; says nothing about equivalent fractions.)

- Yes I think his reasoning is correct.
- Yes, he compared fractions correctly.

**ITEM 9** Write a story problem for  $\frac{1}{2} - \frac{1}{3}$ .

**Content score** NONE GIVEN

**Explanation score (0–3 scale)**

**Explanation score 3** (Okay—story relates to  $\frac{1}{2}$  of something, and  $\frac{1}{3}$  of that same something or size being removed [e.g., eaten]—first example below (although the  $\frac{1}{2}$  of a whole pizza from which the brother’s  $\frac{1}{3}$  of a whole pizza is removed is implied as the half left after the friends ate half of the original pizza); or the  $\frac{1}{2}$  and  $\frac{1}{3}$  refer to separate but equal units and the two amounts are being compared [how much more or less ...]—second example below.)

- Jane got a pizza for her B-day party. Her brother said to save him  $\frac{1}{3}$  of it. At the party her friends ate  $\frac{1}{2}$  of it, she put her brother’s  $\frac{1}{3}$  in the fridge—how much was left for herself?
- If Bridgette has  $\frac{1}{2}$  a pie, and Dylan has  $\frac{1}{3}$  of a pie, how much more pie does Bridgette have than Dylan?
- Last night I left a half a pie in the refrigerator. This morning I found someone had eaten another third of the pie. How much of the pie was left for me?

**Explanation score 2** (Phrasing could be interpreted as correct; comparison subtraction but reversed (first example); correct but states  $\frac{3}{6}$  and  $\frac{2}{6}$ ; no question stated but solves it and implies good question.)

- Robby has  $\frac{1}{2}$  a pizza and Tony has  $\frac{1}{3}$  of a pizza, how much more does Tony have?
- Erin has  $\frac{1}{2}$  a whole cake in the fridge. When she got back from work  $\frac{1}{3}$  was missing. How much is left?
- I have a half a bag of chips. If I want to give  $\frac{1}{3}$  of that bag to Sally how much do I have left?

**Explanation score 1** (Incorrect, but story does involve subtraction; includes subtraction or intent is subtraction; no question asked but otherwise good; setup is good but “odd” question [last example]; puts the situation into a concrete setting [slices of pizza, pieces of pie] in which the result is not  $\frac{1}{6}$  but something like 1 slice, etc. Setup requires algebra.)

- The Smith family bought a cake and the parents ate half of it. Then Kelly, their daughter, ate  $\frac{1}{3}$  of what was left. How much is left for Rob, their son, to eat? ( $1 - \frac{1}{2} - \frac{1}{2} \times \frac{1}{3} = \frac{2}{3}$  *Incorrect but involves subtraction.*)
- A candy bar that was half eaten was given to me. I had 3 friends with me and we decided to share. How much of the whole candy bar that I was given was left? (*Gives the correct answer with wrong operation  $\frac{1}{2} \div 3$  instead of  $\frac{1}{2} - \frac{1}{3}$ . “Left” in question shows trying to involve subtraction.*)
- Pam found a box of cookies that had  $\frac{1}{2}$  the amount of cookies that it originally had in it. After she found it she ate  $\frac{1}{3}$  of the cookies she found. How many were left? ( $\frac{1}{2} - \frac{1}{2} \times \frac{1}{3}$  *uses subtraction.*)
- I started with  $\frac{1}{2}$  a pie. I took  $\frac{1}{3}$  of the half. How much pie was left? ( $\frac{1}{2} - \frac{1}{2} \times \frac{1}{3}$  *uses subtraction.*)
- Jane had 12 apples she gave  $\frac{1}{2}$  away. Jane then ate  $\frac{1}{3}$  of the remaining apples. Using fractions in an equation how many apples does she have now
- Chris had a apple pie. he ate  $\frac{1}{2}$  of it and his girlfriend ate  $\frac{1}{3}$  of it. who had more?

**Explanation score 0** (Omitted, or story unrelated to subtraction or includes no question but implies multiplication [e.g., by incorrect drawing].)

12 people went on the field trip to the museum.  $\frac{1}{2}$  of the kids brought their lunch to eat.  $\frac{1}{3}$  of those kids that brought their lunch had sodas to drink. How many kids brought sodas? ( $12 \times \frac{1}{2} \times \frac{1}{3}$  *Incorrect answer with incorrect operations.*)

**ITEM 10 Three students' solutions to  $\frac{9}{16}$  of 48**

**(a, b) Content score**

a)	C reasoning correct	F reasoning is flawed	CT cannot tell
b)	N doesn't appear to understand	M may or may not	U shows good understanding.

See below for cases in which one of Part a and Part b has been omitted.

**Scoring for Jessica**

a)	b)	SCORE
C	U	4
C	M	3
CT	M	2
CT	N	1
F	N	1
F	M	1
C	N	0
CT	U	0
F	U	0

If either Part a or Part b is omitted for **Jessica**,

a. C only = 1 F, CT only = 0.

b. U only = 1 M, F only = 0

**(c) Explanation score Jessica (0–3 scale)** Part c) is scored, for Jessica only, on a scale from 0–3 (if the respondent performs Jessica's algorithm correctly, he or she scores 3).

**Explanation score 3** (Uses idea of building up from  $\frac{1}{4}$  or coming down from  $\frac{1}{2}$ ; correct answer.)

- $\frac{3}{8}$  is  $\frac{1}{8}$  less than  $\frac{1}{2}$ ;  $\frac{1}{2}$  of 32 is 16;  $\frac{1}{8}$  of 32 is 4  $\Rightarrow 16 - 4 = 12$ .
- $\frac{3}{8}$  is  $\frac{1}{8}$  more than  $\frac{1}{4}$ ;  $\frac{1}{4}$  of 32 is 8;  $\frac{1}{8}$  of 32 is 4;  $8 + 4 = 12$ .

**Explanation score 2** (Says *less* but adds in the solution, or says *more* but subtracts.)

- $\frac{3}{8}$  is  $\frac{1}{8}$  less than  $\frac{1}{2}$ ;  $\frac{1}{2}$  of 32 is 16;  $\frac{1}{8}$  of 32 is 4  $\Rightarrow 16 + 4 = 20$ .

**Explanation score 1** (Provides incomplete solution; starts out correctly but stops; says does not understand but uses standard algorithm to determine that answer is correct; starts well but uses  $32 - 4 = 28$ ; calculates  $\frac{1}{8}$  of 32, then triples that answer.)

- $\frac{3}{8}$  is  $\frac{1}{8}$  less than  $\frac{1}{2}$ ;  $\frac{1}{2}$  of 32 is 16.
- $3 \times (\frac{1}{8} \times 32)$

**Explanation score 0** (Incorrect, or does not explain Jessica's method at all [cf. Explanation score 1].)

- She needs to look more closely at the numbers she is comparing and the numbers she is breaking down.
- Uses usual multiplication algorithm for  $\frac{3}{8}$  of 32 via  $\frac{3}{8} \times \frac{32}{1}$  or canceling.

### Scoring for Justin

a)	b)	SCORE
C	M	4
C	U	3
CT	M	2
CT	N	1
F	N	1
F	M	1
C	N	0
CT	U	0
F	U	0

If either Part a or Part b is omitted for **Justin**,

a. C only = 1 F, CT only = 0;

b. M only = 1 F, U only = 0.

### Scoring for Stacy

First row	Second row	SCORE
F	N	4
F	M	3
CT	M	2
C	N	0
C	M	0
CT	N	0
C	U	0
F	U	0
CT	U	0

If either Part a or Part b is omitted for Stacy,

a. F = 1 C, CT = 0;

b. N = 1 M, U = 0.

## **ITEM 11 Evaluating Betty's estimate of 25% of 7991.8 as 2000.2**

### **Content score NONE GIVEN**

### **Explanation score (0–2 scale)**

**Explanation score 2** (Explicitly expresses concern; realizes that the .2 is not needed; .8 is already accounted for in rounding to 2000; treats 7991.8 as a single number; realizes that the .2 is not needed, but still thinks that Betty's reasoning is good.)

- She didn't estimate – all she needed to do was 1/4 of 8000
- The problem here is that she is adding an extra .2 that doesn't need to be added because she already estimated 8000.
- The .8 can just round the entire number up to 8000. Then take 25% of that. But I do see why she got that.
- Betty's reasoning is very good but she didn't need to take 1/4 of .8. 2000 would have been a good estimate, plus she already rounded up so the .2 didn't matter.
- Looks good to me, except when she rounded she should have rounded up to 8000 which would have included the .8 in it.
- She reasons well by rounding up to 8000 and dividing by 4. However, she must remember that she rounded up, so the actual answer is going to be a bit less than 2000.
- Her work is correct. But if she wanted to be more accurate, instead of taking 1/4 of .8 she should have taken 1/4 of 8. Since  $8000 - 7991.8 = 8.2$ , it would have been a closer estimate.
- She forgot that anything past the decimal when rounding up such a large # is useless. She doesn't understand digit placement.

**Explanation score 1** (Shows some number sense, but explanation is lacking; makes egregious error.)

- Very logical Easy to follow & estimates the whole problem. I would think that the child would drop the decimal and just round to 8000.
- I would just forget about the .2 and just put 2000 but the answer is correct.
- Great estimate (although I think the .2 is unnecessary).
- Her reasoning is pretty good. I believe she didn't need the .8 because she's estimating. Overall great reasoning.
- Estimating should be 1/4 of 8000 is 2000. Why worry about .8 when you skipped the other 9. (1101).
- She reasons well and notes actual answer < 2000.

**Explanation score 0** (Says, "Good"; attempts to explain but does not finish; thinks using .2 makes the estimate more specific; thinks the estimate is okay.)

- She knew that  $1/4 = 25\%$  and that 7991 rounded up to 8000. She also understood how to divide.
- Good estimation although I don't think it needs to be so specific.
- It is pretty smart and it shows good understanding.
- She is using good number sense and estimation. She is thinking logically.
- That is a good estimate.
- Spells out exactly the steps Betty is doing but doesn't comment.
- ... but should have rounded to nearest 10 not 1000.
- Very good estimation, she did all the steps clearly, even estimation 1/4 of the .8

**ITEM 12 Give 2 decimal fractions between 2 given decimal fractions and 2 fractions between 2 given fractions.**

**Content score**

**Part a)** Scoring (0, 1, 2) is # of correct decimal fractions, OR

**1 Point** (Student knows that finding such numbers is possible but cannot do it; says infinitely many but gives none; typography incorrect but may have okay idea.)

- It is possible; I just have forgotten how to get the numbers.

**0 Points**

- Not possible since the numbers are so close together, as close as one number. 1 number is possible, but not 2 numbers.
- Without changing the fractions two fractions cannot be given between  $\frac{2}{5}$  and  $\frac{3}{5}$ .
- There probably is a number somewhere in between but I cannot find it.
- (Gives decimals rather than fractions).

**Part b)** Scoring (0, 1, 2) is # of correct fractions, OR see Part a.

The fractions must have different values (e.g.,  $\frac{1}{2}$  and  $\frac{3}{6}$  would get score 1) but can be in ANY FORM OF a correct FRACTION (e.g.,  $\frac{25}{5}$ ).

**Explanation score NONE GIVEN**

**ITEM 13** Finish the following story problem for  $18 \div \frac{1}{2}$ : You buy 18 muffins for an after-school faculty meeting ... .

**Content score NONE GIVEN**

**Explanation score (0–3 scale) Be sure to discuss ambiguous responses.**

**Explanation score 3** (Okay—clearly means How many  $\frac{1}{2}$ s are in or make 18? with the  $\frac{1}{2}$  referring to  $\frac{1}{2}$  muffin [not  $\frac{1}{2}$  of the muffins] as in the first example, or  $\frac{1}{2}$  of some amount is 18 as in the second example; okay but question slightly problematic.)

- and you divide them in halves so everyone can have a half of a muffin. How many people are at the meeting?
- but it's only half as many as you need. How much muffins will you need in all?

**Explanation score 2** (Recognizes 36 as answer and writes story problem involving 36 in the body of the story, or perhaps  $18 \div 36$  (not 36 as addition); okay. but not finished or not in question form; okay. but not in question form; okay but question really messed up; okay question but lead-in messed up, perhaps with collateral work unsupportive of division.)

- and you must split them between thirty [read her 36 as 30 I believe] people what will you have to do so everyone gets a piece.
- and give  $\frac{1}{2}$  to each person. How much did each person get?
- And you need to cut those muffins in  $\frac{1}{2}$  to have enough for everyone.
- You divide each muffin to two. How many halves of muffins do you have? (collateral:  $18 + 18 = 36$ ).
- You slice each muffin into 2 pieces. At the end of the meeting there are no muffins left over. Each person only ate 1 piece. How many people were at the meeting?

**Explanation score 1** (Instead of  $\div \frac{1}{2}$ , relates to  $\frac{1}{2}$  times or  $\div 2$ .)

- and you have to put an equal # on 2 tables in the room – how many per table [*would be  $18/2$  not  $18 / \frac{1}{2}$* ]
- and you decide to keep  $\frac{1}{2}$  for your family for breakfast. How many muffins are left?
- $\frac{1}{2}$  of the muffins were eaten at the meeting. How many muffins were left?
- $\frac{1}{2}$  of the muffins are eaten. How many muffins were eaten?
- And give  $\frac{1}{2}$  of them to Mary. How many muffins do you have left?

**Explanation score 0** (Neither multiplication nor division is represented by the problem.)

- And 9 are eaten, how many are left behind? [*would be  $18-9$* ].

**ITEM 14** Bill and Tom each have some land in flowers (picture shows that Bill's and Tom's original land areas are equal, that Bill has  $\frac{3}{8}$  of his land in flowers, and that Tom has  $\frac{3}{7}$  of his land in flowers). Who has more land in flowers?

**Content score** Correct (Tom) = 1; all others = 0.

**Explanation score (0–3 scale).** *Note.* The examples refer to an earlier version about eating cake, but they can be used to indicate the scoring for similar types of explanations for the land version.

**Explanation score 3** (Explicitly refers to sizes; explanation is good but focused on unshaded; may use  $\frac{1}{2}$  as benchmark.)

- Bill ate  $\frac{3}{8}$ , Tom ate  $\frac{3}{7}$  a 7<sup>th</sup> is larger than an 8<sup>th</sup> so Tom ate more – meaning Bill has more left.
- Bill has more cake left because they both ate the same amount of pieces but Bill's were smaller so he has more left.
- Tom's cake was cut into sevenths, Bill's eighths. That means Tom's cake has bigger slices because there aren't as many pieces cut. IF they both have 3 slices left but Tom had less pieces cut than Bill's cake, Tom has more left.
- They both have the same number of pieces, but sevenths are larger than eighths so Tom has more cake in each piece.
- Because Bill ate more than Tom, Tom has more cake left. Bill ate  $\frac{1}{8}$  more than half while Tom only ate  $\frac{1}{14}$  more than half.

**Explanation score 2** (Asserts answer without explicitly stating the sizes of the pieces; simply compares fractions without using common denominators.)

- $\frac{3}{7}$  is larger than  $\frac{3}{8}$
- $\frac{3}{7}$  is a larger amount of cake than  $\frac{3}{8}$  so Tom has more left than Bill.
- 3 pieces of cake out of seven is more than 3 pieces of cake out of 8.
- Tom has less pieces so more cake for him.
- makes picture that shows  $\frac{3}{7} > \frac{3}{8}$

**Explanation score 1** (Algorithmic using common denominators.)

- Compares  $\frac{3}{7}$  to  $\frac{3}{8}$  using common denominators.

**Explanation score 0** (Says amounts are the same because each has 3 pieces left.)

**ITEM 15 Pat and Dana use different reasoning in comparing three shaded to two unshaded rectangles of the same size.**

**Content score** Both are correct = 2; only one of Pat or Dana correct = 1; neither correct = 0.

**Explanation score (0–4 scale)**

**Explanation score 4** (Recognizes and explains both; recognizes that both are correct, but gives “skimpy” explanation; somehow states that the two people look at the problem from different perspectives but are both correct.)

- Pat is looking at the shaded compared to the unshaded while Dana is looking at the unshaded compared to the shaded
- Pat—3 pieces is  $\frac{1}{2} \times$  of two pieces – unit is 2 unshaded, Dana – the two pieces are  $\frac{2}{3}$  of the 3 that are shaded if the unit is the shaded area.
- They are both looking at the problems from way different perspectives. They are looking at it from one angle and the other looks at it from a completely different angle.
- Pat is talking about the shaded region and Dana is talking about the unshaded region. The ratios they are comparing are the same.
- It depends on what you’re using as the whole.
- Both are correct—it depends on the unit.
- States ratio is 3:2 and states that Pat and Dana are looking @ different aspects.

**Explanation score 3** (Good explanation for one but skimpy for the other—see last example below.)

- They both are b/c the ratio of shaded to unshaded is 3:2. There are also the same amount of blocks shaded in Pat's as in Dana's but Pat has 1 more & that 1 is  $\frac{1}{2}$  of Dana's. So Pat has  $1 \frac{1}{2}$  times as much as Dana.
- Just states ratio is 3:2

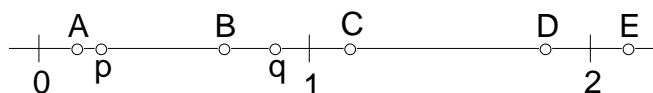
**Explanation score 2** (Justifies either Pat’s or Dana’s reasoning, but not both, correctly; simply restates the problem.)

- Pat is correct 2 of the rectangles symbolize 1 and one rectangle symbolizes  $\frac{1}{2}$ .
- Dana is right b/c the shaded region is not one and  $\frac{1}{2}$  times as much as the unshaded region it is a  $\frac{1}{2}$  bigger.
- Pat is correct because there is  $1 \frac{1}{2}$  times as much shaded then unshaded.
- $\frac{3}{5}$  compared to  $\frac{2}{5}$  is  $1 \frac{1}{2}$  times  $\frac{2}{5}$

**Explanation score 1** (Justifies either Pat’s or Dana’s reasoning vaguely, unclear whether student totally understands.)

**Explanation score 0** (Thinks neither is correct or explanation is faulty; justifies one explanation incorrectly or without understanding.)

**ITEM 16** On this number line, (a)  $p + q$  and (b)  $p \times q$  are closest to which points?



**Content score**

- a) Point C = 1; others = 0
- b) Point A = 1; others = 0

**Explanation score NONE GIVEN**

**ITEM 17** (a) Create a subtraction problem for which Ben would use his method (Ben erroneously renames 100 across a zero in the ten’s place), and (b) explain what you would do next on this topic to help this child.

**a) Make up another problem score (scale 0–1)**

**Score 1** (Numbers involve renaming across a 0 in ten’s place or across two 0s; subtrahend may involve nonzero number of tens.)

**Score 0** (Problem posed does not require renaming across a 0 in the ten’s place.)

**b) What would you do next? score (scale 0–2)**

**Next? score 2** (SENSE MAKING) Would point out answers not sensible; suggests work with manipulatives; sense making is involved.)

- I would point out that his/her answers do not make sense, and ask him/her if there was another way to solve the problem.
- I would suggest working these problems out with base blocks with the child.
- I would help them visualize the problem with blocks.

**Next? score 1** (TEACHING BY TELLING or SYMBOL BASED—Says need explanation about renaming 100 to 10s, but no elaboration; emphasis is on symbols even if suggest checking.)

- Some explanation is needed about borrowing from the 100's to the 10's from 10's to 1's.
- Show him that 0's can't be forgotten they need to be mathematically done as well as all the rest.
- You can ask why the zero just remaining a zero. Ask if he knows what do with it.
- Everything from what the 2, 0, & 7 represent. Ask him to add  $108 + 9$  and see if he gets the right answer. Then show him the correct way for this procedure.
- Use another number beside zero to see how child does. Then use  $407 - 129$ .
- Help him see that the set of 100 was borrowed to put in the tens place too.
- Tell him that he has to count the zero, and make it a ten, because if the zero borrows one from the 8 it would be the same as borrowing 100 from 800, so we put the one in the zero to make it 100.

**Next? score 0** (GENERIC OR NONSENSE ANSWER) Says would go to place value or borrowing but provides no elaboration; generic answer.)

- Next topic would be place value and borrowing.
- I would next show him how to move from the ones column to the tens.

**ITEM 18** Child thinks that  $\frac{6}{8}$  is bigger than  $\frac{3}{4}$ .

### **Child's thinking score (scale 0–1)**

**Thinking score 1** (6, 8 bigger than 3, 4.)

**Thinking score 0** (Thinks student is correct.)

- He/she is thinking that  $\frac{3}{4}$  makes  $\frac{6}{8}$  & doesn't realize that  $\frac{6}{8} = \frac{3}{4}$  therefore they are one in the same or equal.

### **What would you do next? score (scale 0–2)**

**Next? score 2** (SENSE MAKING) Refers to manipulatives or pictures; if there is a drawing, the drawing is correct.)

- I would have the child use blocks to see what  $\frac{3}{4}$  is and what  $\frac{6}{8}$  is.
- Show pics with circles or use pattern blocks.

**Next? score 1** (TEACHING BY TELLING or SYMBOL BASED) Symbolic explanation; or suggests manipulatives but drawing is incorrect; suggests improper manipulative [e.g., base ten].)

- Some explanation about what  $\frac{6}{8}$  is in lowest form or what a common denominator would be could simplify the child's thought.
- I would explain that by multiplying each by 2 they end up with  $\frac{3}{4} = \frac{6}{8}$ .
- Go back and review the rules regarding simplifying fractions.
- Drawing (but incorrect).

**Next? score 0** (GENERIC ANSWER) Merely names "next topic"; ratio argument without support.)  
Next topic

- Equivalents (that's all that is said.)
- value of fractions (that's all that is said.)
- Introduce multiples of numbers.
- Show her with 4 objects  $\frac{3}{4}$  of them and show w/ 8 objects  $\frac{6}{8}$  of them how they = the same amount.

**ITEM 19 Child's work is  $8 - .5 = 3$ .**

**Child's thinking score (scale 0–1)**

**Thinking score 1** (Child does not understand decimals.)

- The child doesn't understand what the decimal point means.
- The child cannot differentiate between 3 and .3.
- The child is ignoring the decimal point.

**Thinking score 0** (Thinks student is correct.)

**What would you do next? score (scale 0–2)**

**Next? score 2** (SENSE MAKINGÑ Recommendation has clear focus on conceptual understanding; “good” use of manipulatives or other materials such as pictures; concerned with child’s current level and suggests good use of materials.)

- I would point out the decimal and see if the child knows what it represents. Then, depending on the age of the child, I would talk about numbers less than 1 using a picture (draws thin rectangle cut into 10 parts) and explaining each segment is 1/10 of 1.

**Next? score 1** (TEACHING BY TELLING or SYMBOL BASEDÑ Concerned with the child’s current level; procedural or vague use of materials [number line and picture considered materials] suggested; put the numbers in context with question.)

- I would probably want to work with fractions (1/2) and then see if the child would understand it better if they knew the relationship of the decimal to fraction. Maybe it would be easier to see.
- He is not ready for decimals. I wouldn't do anything but ask him how he thought then depending on his answer I may explain that .5 actually is half of one.
- Depending on what grade level the child is in I may or may not try to help them understand. If I know that they have worked with decimals before I would ask him what .5 means and work from there.
- I would maybe show the child that .5 is equal to 5/10 or 1/2 and then ask him how many candy bars are left if we only eat 1/2 of 1 and have 8 to begin with.

**Next? score 0** (GENERIC ANSWERÑ Recommendation not very specific or appropriate; procedural focus; too broad; not starting at child’s current level of understanding.)

- I would explain the value of numbers when they are on the RIGHT side of the .
- I would have to go into a whole explanation of fractions and decimals to help.
- Explain how .3 is a small part of 1 and that 3 is much larger.
- I would teach the child how to handle the problems.
- Help them see that .5 is 1/2 and  $8 - .5$  is 7.5.
- He doesn't understand decimals so that would be the next step.

**ITEM 20 Ordering of four decimal numbers (multiple choice)**

**Content score** A = 1; all others = 0.

**Explanation score** NONE GIVEN

**ITEM 21 Number of full 100-bar boxes that can be filled with 15,287 bars**

**Content score**

Answer	Score
152 with no computation	3
152 x 100	2
152.87, or 152 with computation, or 15,200 or 152 by moving decimal point two places	1
Others	0

**Explanation score** NONE GIVEN

**ITEM 22 Ordering 5 decimal and fraction numbers (multiple choice)**

**Content score** B = 1; all others = 0.

**Explanation score** NONE GIVEN

**ITEM 23 Locate decimal points in answer for  $77.5 \times 2.84$  and  $100.26 \div 3.6$ .**

**Content score** NONE GIVEN

**Explanation score- 23a)** 220.1 is correct decimal

**Explanation score 3** (Uses estimation to determine decimal-point placement.)

- i.e.  $75 \times 2 = 150 \Rightarrow$  place value as 220
- $77 \times 2$  is 154 so  $77.5 \times 2.84$  has to be bigger than 154  $\Rightarrow$  220.1

**Explanation score 2** (Suggests counts of decimal places but uses estimation to verify.)

- counts decimal places (assume before the decimal point) & uses estimate  $78 \times 3$
- $75 \times 2 = 150$  plus the add decimal places so we know it has to be 220.1 value.

**Explanation score 1** (Applies usual count-places algorithm—inappropriate here.)

- Respondent counts decimal places **to right** of decimal point

**Explanation score 0** (Uses wrong algorithm or gives no explanation.)

- count decimal places **to left** of decimal point
- U can't do it unless U calculate—I am unable to think that way I have to do the work this is a guess.

**Explanation score-23 b)** 27.85 is correct decimal

**Explanation score 3** (Uses estimation to decide on decimal-point placement.)

- $100/4$  is 25, 25 is close to 27
- $100/3$  is a little over 30 ...
- You know  $100/4 = 25$  therefore it has to be larger than 25 and within that same value.

**Explanation score 2** (Suggests counts of decimal places but uses estimation.)

**Explanation score 1** (Usual algorithm started.)

- move decimal in 3.6 to 36 that means you would change 100.26 to 1002.6 (but no further explanation).

**Explanation score 0** (Wrong algorithm or no explanation; applies usual count-places algorithm—inappropriate here.)

- You don't move the decimal point with a division problem.
- When dividing, the decimal stays in the same place.
- count decimal places behind decimal point

**ITEM 24 Base 5 addition**

**Content score** 1110 = 1 point; any other answer = 0.

**Explanation score** NONE GIVEN

**ITEM 25** Estimate the sum of  $\frac{12}{13} + \frac{7}{8}$ .

**Content score**

1 point for any of the following:
$1\frac{9}{11}$
$1\frac{9}{10}$
$1\frac{10}{11}$
$1\frac{3}{4}$
$1\frac{7}{8}$
1.75
1.8
1.85
1.9
1.95
2

Others = 0.

**Explanation score** NONE GIVEN

**ITEM 26** Which number in each pair is larger?

**Content score Part a)** Correct ( $\frac{5}{7}$ ) = 1; other = 0.

**Content score Part b)** Correct ( $\frac{5}{4}$ ) = 1; other = 0.

**Content score Part c)** Correct ( $\frac{3}{7}$ ) = 1; other = 0.

**Content score Part c)** Correct (numbers are =) = 1; other = 0.

**Explanation score** NONE GIVEN

**ITEM 27** To subtract 482 from 635, Ariana subtracted the hundreds, then the tens (first 30, then 50 more), then the ones. Explain how she would solve  $700 \ominus 573$ .

!!!!

**Content score** Correct = 1; other = 0.

An example of a correct explanation follows: Ariana first subtracts 500 from 700 to get 200; she then subtracts 70 to get 130; finally, she subtracts 3 to get 127. (Okay if the tens are subtracted in two steps [e.g., 50 then 20])

**ITEM 28** Terry uses standard algorithms to add  $259 + 38$  and to subtract 34 from 429. (a)!! Does each of the regrouped 1s in these problems represent the same thing?

**Content score NONE GIVEN**

**Explanation score**

**Explanation score 5** (Explains both that the 1 in A represents 1 ten or 10 ones and that the 1 in B represents 100 ones or 10 tens.)

- A is 1 ten or 10 ones; B is 10 tens or 100 ones.
- No, the 1 in problem A represents a ten being carried over to the tens column. In B the 1 represents 1 one hundred being borrowed by the tens column giving the tens column ten tens.

**Explanation score 4** (Explains that the 1 in A represents 1 ten or ten ones and the 1 in B represents 100 ones OR 10 tens.)

- A is 10; B is 100.
- No, the 1 in A represents a group of tens which was created by addition while in B the 1 represents a group of 100s already present in the problem.
- No, because the one from the addition represents one ten that came from  $9 + 8$ . The one from the subtraction represents 10 tens that came from the 4.

**Explanation score 3** (Names places but is not explicit about number of tens in A or hundreds in B; probably means 1 of each, but does not state so explicitly. Seems to have the place values, but makes erroneous statement.)

- 1 in A is tens; 1 in B is a group of 100s.
- 1 in B represents 12 tens.

**Explanation score 2** (Names the ten's place but not the hundred's.)

- One in A is 1 ten (or 10 ones); in B 1 is a group of ten from place to left

**Explanation score 1** (Says that the 1s are not the same, but does not correctly state their values.)

- No; 1 in A is 1; 1 in B is a group of 10 (or a 10).
- In the addition, the 1 represents a one. So it would be  $1 + 5 + 3$ , but in subtraction, the 1 represents a group of 1 tens, so it would actually be  $12 - 3$ .

**Explanation score 0** (Says that the 1s are the same, or mentions differences only in terms of algorithm [borrowing vs. carrying].)

- Yes. They are both 1s.
- No. One is borrowed to make 12; one is carried.
- No. The 1 in B became the value of 12.
- No. You are adding a ten vs. taking away a ten.
- Refers to tens as tenths

(b)!!Why is the regrouped 1 added to the 5 (in the tens place) in the addition problem but 10 is added to the 2 (in tens place) in the subtraction problem?

**Content score NONE GIVEN**

**Explanation score** Note that respondents sometimes answer Part b as an elaboration of their explanation for Part a; they may not restate in their Part b explanation information that is important but that they have included in the Part a explanation.

**Explanation score 5** (Explains the regrouping in both A and B and, explicitly, the formation of the 12 tens in B.)

- The 1 in Problem A represents the 1 group of 10 that was regrouped. So you need to add the 1 group to the 5 groups of 10. In Problem B you need to add the 10 to the 2 because you regroup the 100 ones and make it 10 groups of 10. You took the 10 groups of 10 from the 400, because you cannot take the 3 from the 2. The  $10 + 2 = 12$ , and 3 can be taken from 12.
- In Problem A, it [1] is 1 group of 10 + 5 groups of 10 + 3 groups of 10 is 9 groups of 10. The 1 in Problem B is 1 group of 100 or 10 groups of 10. Therefore, 10 groups of 10 + 2 groups of 10 is 12 groups of 10. The 3 groups of 10 can be subtracted from the 12 groups of 10 to get 9 groups of 10.
- Because the 1 in A represents adding 1 ten, and the 1 in B represents regrouping of the 4 to 3 and bringing a 100 (10 tens) to the tens place, making 12 tens.

**Explanation score 4** (Explains the regrouping in B and explains explicitly the formation of the 12 tens in B but does not explain the regrouping in A. The respondent may be relying on his or her explanation for Part a to explain the regrouping in A, so promote such explanations to a score of 5 if the regrouping in Problem A is explicitly addressed in the Part a response. OR Explains the formation of the 12 tens in B and explains the regrouping in A, but does not explain the source of the 10 tens in B.)

- In B Terry regrouped the 1 one hundred into 10 tens. Terry would then need to add the new group of 10 tens to the original 2 tens to make 12 tens.
- You can't subtract 2 tens from 3 tens, so you move over a place value and take 1 one hundred and add it to the 2 tens to make 12 tens, minus 3 tens.
- In A, he adds the one set of 10 to the 5 sets of 10 because it's more than the one's place. In B, he adds the 10 sets of 10 to the 2 sets of 10 in order to have a number big enough to subtract from.
- In the subtraction, you're taking a group of something; since it was from the 400, you took 100 and added it to the 2, making it 120, so you're really taking 30 from 120.
- Terry adds 10 to the 2 in Problem B because he borrowed one hundred from the hundreds column giving the tens column ten tens.

**Explanation score 3** (Explains the regrouping but does not explicitly explain the formation of the 12 tens in B or explains only the formation of the 12 but not the origin of the 1s [may have explained these origins in Part a].)

- When subtracting we need to regroup the 1 group of 100 to the tens place. The 1 group of 100 can also be thought of as 10 groups of 10.
- It is because the 1 in B represents 10 tens and the 2 represents 2 tens, so you add ten to the two to get 12.
- While adding 10 to 2, you are really adding 100 to 20 making it possible to subtract.
- You take the one from the four. The four had forty tens, so you can take 10 tens from the four and you can add it to 2.
- {Merely reiterates answer to Part a) Because the 1 represents 1 ten in A while it represents 10 tens in B.

**Explanation score 2** (Attempts to explain in some way, but does not show clear understanding, that in Problem B the 1 represents 100 OR explains the place-value ideas correctly except for naming the places

*hundreds* and *thousands* instead of tens and hundreds. May understand, but explanation, especially for the 1 in Problem B, lacks important details.)

- One in A is 1 ten (or 10 ones); in B 1 is a group of ten from place to left.
- He adds 1 to 5 because it is one set of ten; he adds ten because it is ten sets of ten, 1 set of 100.
- You can't take 3 tens from 2 tens so you must make it 12 tens and you do this by borrowing from the hundreds place.
- He adds the 1 to the 5 because  $8 + 9 = 17$ . He writes the 7 down in the ones column and brings the 10 over to the tens column. In B he adds  $10 + 2$  because there are not enough tens to subtract the 3 or 3 groups of 10 from.
- Because you cannot take 30 from 20, so you borrowed 100 from the column before so that you can subtract 30 from 120.

**Explanation score 1** (Seems to attempt to explain the algorithm with some reference to place value but has errors or ambiguity OR explains that the 1 in A is a group of 10 but does not explain the 1 in B. Thus the explanation is poor, but the respondent seems to recognize that place value is relevant in using the algorithms.)

- You are borrowing a group of 10 so you can subtract.
- You are adding 1 group of 10 in A.
- In Problem B he adds 10 to the 2 because you are borrowing one group of 10 in order to subtract correctly.
- When you are adding 1 to the 5 you are just adding a ten to the place because its left over. When adding 10 to 2 you are really adding 100– 20 to make the problem solvable.

**Explanation score 0** (Explains differences only in terms of **algorithm** [borrowing vs. carrying], gives other **procedural** explanation, or **can't explain**. Or makes **major error** [e.g., calls tens *tenths*].)

- You are carrying instead of borrowing.
- Make the 2 bigger so you can subtract.
- Take 100 and 10; bring back 10 to make a bigger number, 12, so can subtract).
- You add the 10 to the two because you need to borrow that group in order to subtract 3 from two (or now 12).
- You can't take 3 from 2 so have to borrow [or steal] 10 to add to the 12 so can subtract.
- Refers to tens as **tenths**